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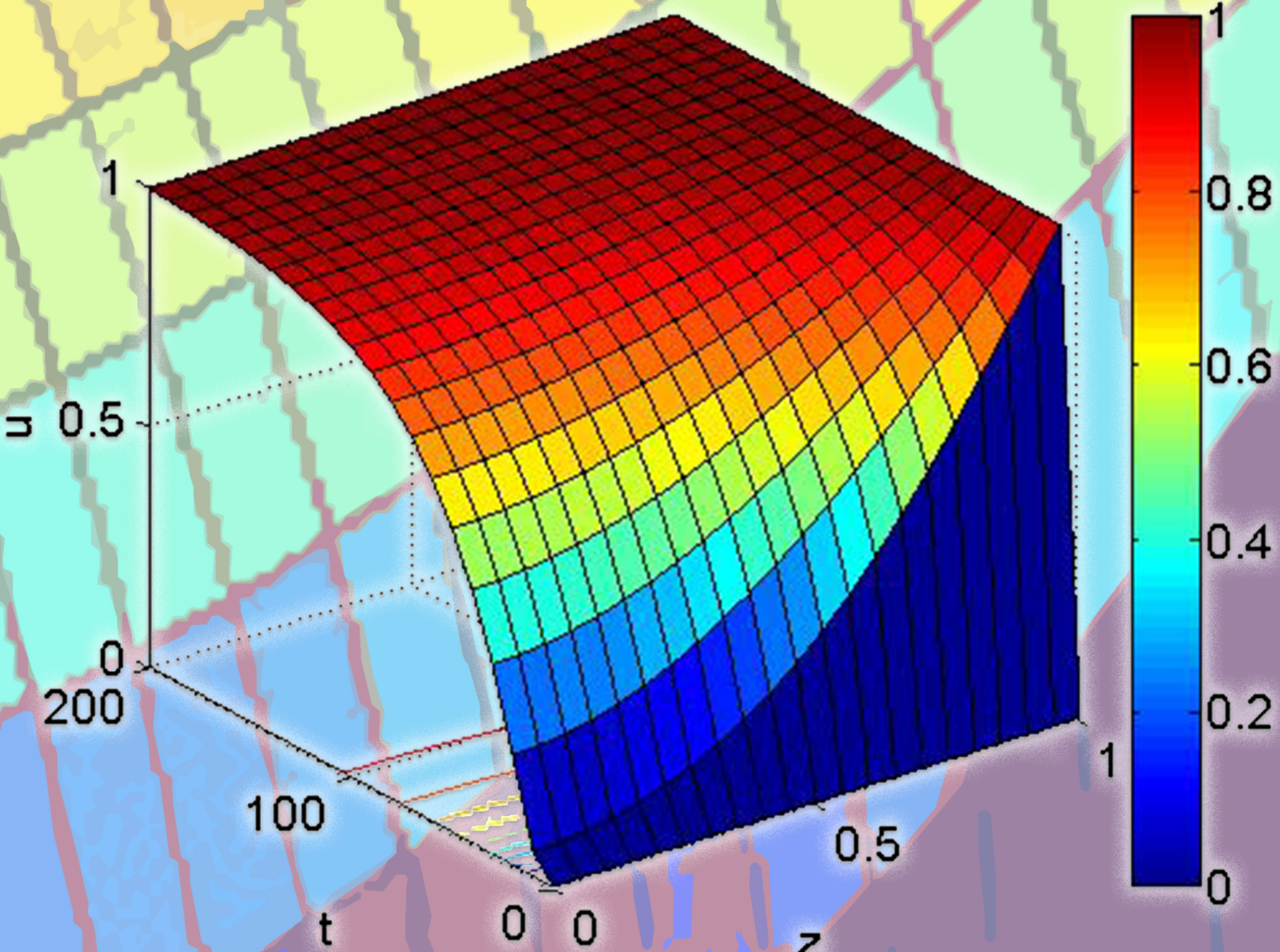
Effective Finite Difference and Conservative Averaging Methods for Solving Problems of Mathematical Physics



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Preface

The solutions of corresponding 1-D initial-boundary value problem and inverse problem for some parabolic, elliptic, and hyperbolic type equations are obtained, using the method of lines for approach the partial differential equations (PDE) the discretization in space applying the finite difference scheme with central differences of second order of approximation (FDS) and the finite difference scheme with the exact spectrum (FDSES) ([2], [14]). The solutions of corresponding 3-D initial-boundary value problems are obtained with conservative averaging method (CAM) using different spline approximations methods: parabolic, hyperbolic, and exponential type spline-functions ([9], [12], [32]). The FDS method in the uniform grid is used for the approximation the differential operator of second order derivatives in the space. The solution in the time is obtained analytically and numerically with continuous and discrete Fourier methods. Using the spectral method are obtained new transcendental equation and algorithms for obtaining the last two eigenvalues and eigenvectors of finite difference operator. For the third kind boundary conditions (BC) this algorithm depends on the value of the special parameter $Q = \frac{L\sigma_1\sigma_2}{\sigma_1+\sigma_2}$; where σ_1, σ_2 are the parameters (heat transfer coefficients) in the boundary conditions and L is the length.
We define the (FDSES), where the finite difference matrix A of N+1-order is represented in the form form $A = WDW^T$, W, D are the matrixes of finite difference eigenvectors and eigenvalues correspondently and the elements of diagonal matrix D are replaced with the first N+1 eigenvalues from the differential operator. If the diagonal matrix D contain the discrete eigenvalues, then the matrix A is of the standard 3-diagonal form.
Are presents the results of analytical and numerical solutions of different typical problems related to considered problems: boundary value problems of ordinary differential equations (ODEs), the problems of heat transfer and of hyperbolic heat conduction equations and others.
Numerical solutions of the **ODE**s for method of lines in the time are obtained analytically and numerically by the MATLAB solvers. For finite difference approximation with central differences strong numerical oscillations are presented, when the initial and boundary conditions are discontinuous. The method of FDSES is without oscillations and this is effective for numerical solutions. The advantages of the method of FDSES are demonstrated via several numerical examples in comparison with some other well-known used

method. This method is more stable as the method of finite difference approximation with central difference.

Finite difference scheme with exact spectrum is built using Fourier series for different mathematical problems. Special heat transfer, wave, Poisson's problems are modelled, also with convection [1]. We can consider corresponding 1-D partial differential equations of these problems in following general form:

$$a_1(u)\frac{\partial^2 u}{\partial t^2} + a_2(u)\frac{\partial u}{\partial t} = \frac{\partial^2 g_1(u)}{\partial x^2} + \frac{\partial g_2(u)}{\partial x} + f(u),$$

where the coefficients a_1, a_2 and functions $g_1(u)$, $g_2(u)$, $f(u)$ can be depending on the solution $u = u(x; t)$ ($x; t$ are the space and time variables). This problem is modelled using method of lines and discrete approximation in x direction and using FDS and FDSES. The convergence and error analysis of the method of lines for solving the initial-boundary value problems of the PDE are given in ([25],[22],[23],[24]). Linear problem is solved analytically and non-linear problem numerically. Are considered also the system of PDE.

The approximations of the nonlinear heat transport problem are based on the finite volume (FV) method. That procedure allows one to reduce the nonlinear 2-D heat transport problem described by a partial differential equation (PDE) to an initial-value problem for a system of two nonlinear ordinary differential equations (ODEs) of the first order. This method and method of finite-difference scheme are compared.

The 3-D problems are reduced to the 1-D problems using the CAM ([13], [31], [32]). The 2-D and 3-D problems are numerically solved with the ADI method by Douglas and Rachford [37].

In the following 12 chapters different mathematical models and methods for solving PDE are considered:

1) algebraical spectral problems for the self-conjugate and non-conjugate finite difference operator with first, third kind BC and periodical BCs,
2) mathematical models for linear and nonlinear heat transfer equations, the dynamic and the shape hysteresis of magnetic droplet in a rotating field, some ill- posed problems for heat transfer equations, the special local and global approximations methods, some aspects for modelling combustion process,
3) mathematical model for linear and nonlinear hyperbolic type PDEs,
4) mathematical model for solving the inverse and direct problems of 1D hyperbolic heat conduction equation,

5) mathematical model and CAM with hyperbolic type integral splines for solving the 3-D problem for hyperbolic heat conduction equation,
6) linear and nonlinear heat transfer equations and the system of parabolic type equations with the periodical BCs,
7) problem for Poisson equation and equations with convections with periodical BCs,
8) diffusion equation with piecewise constant coefficients in multi layered domain, reducing 3-D problems to 2-D initial-boundary value problems,
9) mathematical model in the different coordinates for analytical solutions of the 1-D continuous and discreate problems for Poisson's equations,
10) CAM with the integral splines for simple engineering calculations,
11) some applications of electromagnetic field and force, the forced and free 2-D MHD convection flow around periodically placed cylinders,
12) numerical calculation for the flows field caused by the chain of vortexes, the circular vortexes lines, the spiral vortexes in the cylinder and in the conus.

Kopsavilkums

Attiecīgo 1-D jaukta veida problēmu un dažu paraboliskā, eliptiskā un hiperboliskā tipa vienādojumu problēmu risinājumi tiek iegūti, izmantojot taisnes metodi parciālo diferenciālvienādojumu (PDV) diskretizācijai telpā ar otrās kārtas aproksimācijas galīgo diferenču shēmu ar centrālajām diferencēm (GDS) un galīgo diferenču shēmu ar precīzo spektru (GDSPS) [2], [14].
PVD atbilstošo 3-D jaukta veida problēmu risinājumi tiek iegūti ar konservatīvās viduvēšanas metodi (KVM), izmantojot dažādas splainu aproksimācijas: paraboliskās, hiperboliskās un eksponenciālās funkcijas [9], [12], [32].
GDS metodi vienmērīgā režģī izmanto, lai aproksimētu otrās kārtas atvasinājuma diferenciālo operatoru telpā. Atrisinājumu iegūst analītiski un skaitliski ar nepārtraukto un diskrēto Furjē metodi. Izmantojot spektrālo metodi, tiek iegūti jauni transcendentālie vienādojumi un algoritmi 3. veida robežnosacījumiem atkarībā no parametra $Q = \frac{L\sigma_1\sigma_2}{\sigma_1+\sigma_2}$ vērtībām, kur σ_1, σ_2 ir parametri robežnosacījumos (siltuma pārneses koeficienti) un L ir segmenta garums.
GDSPS metode tiek definēta ar N+1 kārtas galīgo diferenču matricu $A = W D W^T$, kur W, D ir galīgo diferenču īpašvektoru un īpašvērtību matricas un diagonāles matrica D tiek aizstāta ar pirmajām N+1 diferenciālā operatora īpašvērtībām. Ja diagonālajā matricā D ir diskrētas īpašvērtības, tad matrica A ir standarta 3-diagonālā formā (GDS).
Tiek parādīti dažādu tipisku problēmu analītiskie un skaitliskie risinājumi: parasto diferenciālo vienādojumu robežproblēmām, siltuma pārneses un hiperboliskā siltuma vadīšanas (HSV) vienādojumu problēmām un citām. Skaitliskie risinājumi iegūti ar MATLAB paketi.
Aprēķinos ar CDS metodi tiek iegūtas lielas skaitliskas svārstības, ja sākuma un robežnosacījumi nav saskaņoti. GDSPS metode ir bez svārstībām, un tā ir efektīva skaitlisku risinājumu iegūšanai. GDSPS metodes priekšrocības tiek demonstrētas, izmantojot vairākus skaitliskus piemērus, salīdzinot to ar kādu citu labi zināmu metodi. Apskatītā GDSPS metode ir stabilāka par GDS. GDSPS metode tiek arī veidota, izmantojot Furjē metodi.
Tiek modelētas īpašas siltuma pārneses, viļņu izplatīšanās un stacionāras problēmas, ieskaitot arī konvekciju [1].
Iespējams risināt arī atbilstošas 1-D nelineāras problēmas šādā vispārīgā formā:

$$a_1(u)\frac{\partial^2 u}{\partial t^2} + a_2(u)\frac{\partial u}{\partial t} = \frac{\partial^2 g_1(u)}{\partial x^2} + \frac{\partial g_2(u)}{\partial x} + f(u),$$

kur koeficienti a_1, a_2 un funkcijas $g_1(u)$, $g_2(u)$, $f(u)$ var būt atkarīgi no atrisinājuma $u = u(x;\ t)$, ($x;\ t$ ir telpas un laika mainīgie, d – parciālais diferenciāloperators). Šī problēma ir modelēta, izmantojot taišņu metodi un diskrētu aproksimāciju x virzienā ar GDS un GDSPS.

PVD jaukta veida problēmu risināšanai taišņu metodes konverģence un kļūdas analīze ir aprakstīta [25], [22], [23], [24]. Lineārā problēma tiek atrisināta analītiski, bet nelineārā – skaitliski. Tiek risinātas arī PVD sistēmas.

Nelineāro siltuma pārneses problēmu aproksimācijai lieto galīgā tilpuma metodi. Šī procedūra ļauj reducēt nelineāro 2-D PVD problēmu uz sākuma vērtību problēmu sistēmai ar diviem nelineāriem pirmās kārtas parastajiem diferenciālvienādojumiem. Šī metode tiek salīdzināta ar galīgo diferenču shēmu.

Izmantojot KVM, 3-D problēmas tiek reducētas par 1-D problēmām [13], [31], [32]. 2-D un 3-D problēmas risina skaitliski ar alternējošo virzienu metodi (ADI metodi), lietojot Duglasa un Rahforda algoritmu [37].

Turpmākajās 12 nodaļās tiek aplūkoti dažādi matemātiskie modeļi un metodes PVD risināšanai:

1) algebriskās spektrālās problēmas saistītam un nesaistītam galīgo diferenču operatoram ar pirmā, trešā veida un periodiskajiem robežnosacījumiem (RN);
2) matemātiskie modeļi lineāriem un nelineāriem siltuma pārneses vienādojumiem, magnētisko pilienu dinamikai un formas histerēzei rotējošā laukā, dažas inversās problēmas siltuma pārnesei, speciālas lokālās un globālās aproksimācijas metodes, daži aspekti degšanas procesa modelēšanai;
3) matemātiskais modelis lineāriem un nelineāriem hiperboliska tipa PVD;
4) matemātiskais modelis 1-D hiperboliskā siltuma vadīšanas (HSV) vienādojumam inverso un tiešo problēmu risināšanai;
5) matemātiskais modelis un KVM ar hiperboliska veida integrāliem splainiem 3-D (HSV) vienādojuma problēmas risināšanai;
6) lineāri un nelineāri siltuma pārneses vienādojumi un paraboliskā tipa vienādojumu sistēma ar periodiskiem RN;
7) problēmas Puasona vienādojumam un vienādojumiem ar konvekciju – periodiskie RN;
8) difūzijas vienādojums ar daļveida konstantiem koeficientiem daudzslāņu apgabalā. 3-D problēmas redukcija par 2-D jaukta veida problēmām;

9) matemātiskais modelis dažādās koordinātu sistēmās. 1-D nepārtrauktās un diskrētās problēmas analītiskie risinājumi Puasona vienādojumiem;
10) KVM ar integrāliem splainiem vienkāršiem inženiertehniskiem aprēķiniem;
11) daži elektromagnētiskā lauka un spēka lietojumi, uzspiestās un brīvās 2-D magnētiskās hidrodinamikas (MHD) konvekcijas plūsmas ap periodiski izvietotiem cilindriem;
12) skaitliskie aprēķini plūsmu laukam, ko izraisa virpuļu ķēde, apļveida virpuļu līnijas, spirālveida virpuļi cilindrā un konusā.

Contents

Chapter 1
Spectral represention of the matrix: H. Kalis, 2011 [74]

The linear problems of mathematical physics can be approximated with the linear system of algebraical equations in following form

$$Au = f, \tag{1.1}$$

where A, u, f are correponding the quadratic matrix and column-vectors of M order.

1.1 The algebraical spectral problems

The spectral problems for matrix A we can represented in the form

$$Aw^k = \lambda_k w^k, \quad A_* w_*^k = \lambda_k w_*^k, k = \overline{1,M}, \tag{1.2}$$

where w^k, w_*^k are eigenvectors with the elements w_j^k, w_{*j}^k, $j = \overline{1,M}$, A_* is the conjugate (transposed) matrix of A, λ_k is the eigenvalues.
This problems can be writte in the matrixes form

$$AW = WD, \quad A_* W_* = W_* D, \tag{1.3}$$

where W, W_* are the corresponding matrices of eigenvectors in the column with the elements $w_j^k, w_{*j}^k, j = \overline{1,M}, k = \overline{1,M}$, $D = diag(\lambda_k)$, $k = \overline{1,M}$ is the diagonal matrix with the eigenvalues.
We can normed the eigenvectors and obtain the system of the biorthonormed eigenvectors
w^k, w_*^m in the scalar product $(w^k, w_*^m) = \sum_{j=1}^M w_j^k w_{*j}^m = \delta_{k,m}$ for $k, m =$

$\overline{1,M}$, where $\delta_{n,m}$ is the Kroneker's symbol.
For different eigenvalues the eigenvectors are orthogonal $(0 = (w_*^m, Aw^k) - (w^k, A_* w_*^m) = (\lambda_k - \lambda_m)(w_k, w_*^m).)$

1.2 The solution of the algebraical problem

For solving the system of equations (1.1) the column-vector f is represented in the form

$$f = \sum_{k=1}^{M} \beta_k w^k, \tag{1.4}$$

where $\beta_k = (f, w_*^k), k = \overline{1,M}$.
Usind the spectral problem the solution of (1.1) we can obtain in the form

$$u = \sum_{k=1}^{M} \alpha_k w^k, \tag{1.5}$$

and $(w_*^k, Au) = \sum_{m=1}^{M} \alpha_m \lambda_m (w_*^k, w^m) = \alpha_k \lambda_k$,
$(w_*^k, f) = \sum_{m=1}^{M} \beta_m (w_*^k, w^m) = \beta_k, k = \overline{1,M}$.
Therefore $\alpha_k = \frac{\beta_k}{\lambda_k}, k = \overline{1,M}$ and the solution of the (1.1) is

$$u = \sum_{k=1}^{M} \frac{\beta_k}{\lambda_k} w^k. \tag{1.6}$$

For the symetrical matrixes $A = A_*$ follows that $W = W_*$ and we obtain the orthornormed system of eigenvectors $w^k, (w^k, w^m) = \delta_{k,m}$.
From (1.3) follows that

$$A = WDW^{-1}, A_* = W_* D W_*^{-1}, WW_*^T = E, W^{-1} = W_*^T, W_*^{-1} = W^T, \tag{1.7}$$

where W^T, W_*^T are the transposed matrixes, E is the unit matrix.
For the symmetric matrix A the matrix W is also symmetric and $WW = E, A = WDW$.

1.3 The self-conjugate finite difference operator with first kind BC

We consider uniform grid in the space $x_j = jh, j = \overline{0,N}, Nh = L$.
Using the finite differences of second order approximation for second order respect to x $(-u''(x))$ we obtain the 3-diagonal matrix A of $M = N-1$ order. Then the expression Ay can be represented with following way

$$Ay_j = -(y_{j+1} - 2y_j + y_{j-1})/h^2, \quad j = \overline{1,N-1}, \qquad (1.8)$$

where
y is the column-vector of $N-1$ order with elements $y_j, j = \overline{1,N-1}, y_0 = y_N = 0$.
Using two vectors y^1, y^2 scalar product
$[y^1, y^2] = h(\sum_{j=1}^{N-1} y_j^1 y_j^2)$ can prove, that the operator A is symmetrical and $[Ay, y] \geq 0$ [3]. The corresponding discrete spectral problem $Aw^k = \mu_k w^k, k = \overline{1,N-1}$ have following solution
$\mu_k = \frac{4}{h^2}\sin^2\frac{k\pi}{2N}$ (elements of the matrix D), $w_{i,j}=\sqrt{\frac{2}{L}}\sin\frac{\pi ij}{N}, i, j=\overline{1,N-1}$ (elements of the symmetrical matrix W).
Every vector g of M order with the elements $g_j, j = \overline{1,M}$ we can be expanded in the basis of eigenvectors

$$g = \sum_{k=1}^{M} a_k w^k, g_j = \sum_{k=1}^{M} c_k \sin\frac{\pi kj}{N}, \quad a_k = (g, w^k), c_k = \frac{2}{N}\sum_{j=1}^{M} g_j \sin\frac{\pi kj}{N}.$$

The solutions of the finite difference equations

$$-\Lambda w_j = -(w_{j+1} - 2w_j + w_{j-1})/h^2 = \mu w_j, \quad j = \overline{1,N-1}$$

can obtain in the classical form
$1 - \frac{\mu h^2}{2} = \cos(ph)$ or $\mu = \frac{4}{h^2}\sin^2\frac{ph}{2}$. Then $w_j = C\sin(px_j) + B\cos(px_j)$ and the constants $C, B = 0$ are determined from the boundary conditions by $w_0 = 0, w_N = 0$. The values $p_k, k = \overline{1,N-1}$ we obtain from the condition $C \neq 0, \sin(p_k L) = 0, p_k = \frac{k\pi}{L}, k = \overline{1,N-1}$. We can proved the orthogonality $[w^k, w^m] = 0, k \neq m$ in following way $(w_0 = w_N = 0)$:
$[\Lambda w^k, w^m] + \mu_k[w^k, w^m] = 0, [\Lambda w^m, w^k] + \mu_k[w^m, w^k] = 0,$
where
$[\Lambda w^k, w^m] = -\frac{h}{h^2}\sum_{j=1}^{N-1}((w_j^k - w_{j-1}^k) - (w_{j+1}^k - w_j^k))w_j^m) =$

$-\frac{h}{h^2}(\sum_{j=1}^{N}(w_j^k - w_{j-1}^k)w_j^m - \sum_{s=1}^{N}(w_s^k - w_{s-1}^k)w_{s-1}^m) =$
$-\frac{h}{h^2}\sum_{j=1}^{N}(w_j^k - w_{j-1}^k)(w_j^m - w_{j-1}^m) =$
$[\Lambda w^m, w^k], (\mu_k - \mu_m)[w^k, w^m] = 0 (s = j+1).$
Similarly $C_k^2/h = [w^k, w^k]/h = \sum_{j=1}^{N}\sin^2(0.5\alpha_k j) = \frac{1}{2}(N - S_k)$,
where $S_k = \sum_{j=1}^{N}\cos(\alpha_k j) = Re(\sum_{j=1}^{N} q_k^j) = Re(\frac{q_k^N - 1}{q_k - 1}) = 0$, $\alpha_k = \frac{2\pi k}{N}, q_k = \exp(i\alpha_k), i = \sqrt{-1}$
We get $w_j^k = C_k \sin(p_k x_j)$ and from $[w^k, w^k] = 1$ follows that $C_k = \sqrt{\frac{2}{L}}$.
Using the usual scalar product of two vectors for eigenvectors without the step h $(w^k, w^m) = \sum_{j=1}^{N-1} w_j^k w_j^m = \delta_{k,m}$
we get $C_k = \sqrt{\frac{2h}{L}} = \sqrt{\frac{2}{N}}$, and $WW = E, W^{-1} = W, A = WDW$, where $D = diag(\mu_k)$.

Example 1.1. The solution of discrete boundary value problem

$$-(y_{j+1} - 2y_j + y_{j-1})/h^2 = f(x_j), y_0 = y_N = 0, \quad j = \overline{1,M}, M = N-1, \tag{1.9}$$

or of the finite difference scheme with second order of approximation the boundary value problem of differential equation (1-D Poisson equation) $-u''(x) = f(x), u(0) = u(L) = 0$,
we can write in following form

$$y_j = \sum_{k=1}^{M}\alpha_k w_j^k, y = \sum_{k=1}^{M}\alpha_k w^k,$$

or

$$y_j = \sum_{k=1}^{N-1} a_k \sin\frac{k\pi j}{N}, a_k = \frac{b_k}{\mu_k}, b_k = \frac{2}{N}\sum_{j=1}^{M} f(x_j)\sin\frac{k\pi j}{N},$$

where $\alpha_k = \frac{\beta_k}{\mu_k}, \beta = (w^k, f) = \sum_{j=1}^{N-1} w_j^k f(x_j)$.

The solution of the spectral problem for the corresponding differential problem $-w''(x) = \lambda w(x), w(0) = w(L) = 0$ is in following form:

$$w^k(x) = \sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}, \lambda_k = (\frac{k\pi}{L})^2, (w^k, w^m)_* = \int_0^L w^k(x)w^m(x)dx = \delta_{k,m}.$$

For the discrete problem the integral in the scalar product $(w^k, w^m)_*$ is approximated with the trapezoidal formula $h(w^k, w^m)$.
We can constructed the FDSES when in the representation $A = WDW$ the diagonal elements of matrix D are replaced with the eigenvalues λ_k from the differential problem. Then the matrix A is not in the 3-diagonal form but this is full matrix.
Used for analytical solution for the discrete problem $Ay = f$ the transformation $v = Wy$ we have $Dv = Wf$ or $v_i = \frac{1}{d_i}\sum_{k=1}^{M} w_{i,k} f_k$, where $M = N-1, f_k = f(x_k), v_i, i = \overline{1,M}$ are the components of the column-vector v. Then $y = Wv$ or $y_j = \sum_{i=1}^{M} w_{j,i} v_i, j = \overline{1,M}$. If $d_i = \mu_i, i = \overline{1,M}$ then we have the above-mentioned solution of FDS, but for $d_i = \lambda_i$ we obtain the solution of FDSES.

Example 1.2. The solution of the differential problem for $f(x) = \sin(x\pi/L)$ is $u(x) = (\frac{L}{\pi})^2 f(x), y_j = (2/h\sin(0.5\pi h/L))^{-2} f(x_j)$, $j = \overline{0,N}$. The calculations with MATLAB by FDS $(L = 10)$ give following results for maximal error:
0.5376 (N=5), 0.1873 (N=10), 0.06596 (N=20). The FDSES give exact results (the error is 10^{-15}). If $f(x) = \sin(x\pi/L) + x/L(1-x/L)$, then $u(x) = (\frac{L}{\pi})^2\sin(x\pi/L) - \frac{x^3}{6L} + \frac{x^4}{12L^2} + \frac{xL}{12}$ and we have following results for the errors:
1)FDS: 0.6735 (N=5), 0.2353 (N=10), 0.1278 (N=15), 0.0593 (N=25), 0.0293 (N=40),
2)FDSES: 0.00289(N=5), 0.00026(N=10), 0.00006 (N=15),0.00001 (N=25), 2.10^{-6} (N=40).
We have ollowing MATLAB m.file **”PDS1veid” :**

```
%ODE  -U''=f   U(0)=0, U(L)=0
%Analyt.sol., spectral probl.
% Au=lam1^2 *u,A=1/h^2(u_j+1-2u_j +u_j-1);discrete probl.
% 2 example:f(x)=sin(x pi/L),U(x)=(L/pi)^2 sin(x pi/L);
function PDS1veid(N)
N1=N+1;N2=N-1; L=10;x=linspace(0,L,N1)';x=x(2:N);h=L/N;
B1=zeros(N2,N2);
B1=B1+2*diag(ones(N2,1))-diag(ones(N2-1,1),-1)-. . .
diag(ones(N2-1,1),1);
B1=B1/(h^2); %3-diag. matrix
%F=sin(x*pi/L);
%prec=(L/pi)^2*F;
F=sin(x*pi/L)+x/L.*(1-x/L);
prec=(L/pi)^2*sin(x*pi/L)-x.^3/(6*L)+ x.^4/(12*L^2)+L*x/12;
u=B1 / F;$ \% slesh \\
lk=4/(h^2)*(sin(0.5*(1:N2)'*h*pi/L)).^2;
CK1=sqrt(2*h/L);
lk0=((1:N2)'*pi/L).^2;
W=sin(pi*(1:N2)'*x'/L)';
for j=1:N2
W(:,j)=W(:,j).*CK1;end;
%W'*W,A2=W*diag(lk)*W' % control
%U=W*diag(lk.^(-1))*W'*F; %FDS
U=W*diag(lk0.^(-1))*W'*F; %FDSES
%[V1,D1]=eig(B1);lkk=diag(D1);[lk,lkk,lk0] % control
figure,plot(x,prec,'k*',x,u,'ko',x,U,'b*')
legend('prec. atr.', 'Matlab atr.','FDS or FDSES')
title(sprintf('Solution  on x,N= %3.0f',N))
kMat=abs(u-prec); kU=abs(U-prec);norm(kU)
figure,plot(x,kMat,'k*',x,kU,'b*')
legend('kMatlab atr.','kFDS')
title(sprintf('Err-Sol.on x,N=%2.0f,kM=%6.5f,kFDS=%6.5f',. . .
N,norm(kMat),norm(kU)))
```

The results are represented with the operator **"PDS1veid(10)"**.

We have the solution of the differential problem in the form $u(x) = \frac{x}{L}\int_0^L (L-\xi)f(\xi)d\xi - \int_0^x (x-\xi)f(\xi)d\xi$.
Using the expression $g(x) = \sum_{k=1}^{\infty} a_k w^k(x) = \sum_{k=1}^{\infty} c_k \sin\frac{k\pi x}{L}, a_k = (g, w^k)$, $c_k = \frac{2}{L}\int_0^L \sin\frac{k\pi x}{L} dx$, we can the solution write in the form

$$u(x) = \sum_{k=1}^{\infty} a_k w^k(x) = \sum_{k=1}^{\infty} c_k \sin\frac{k\pi x}{L}, a_k = \frac{b_k}{\lambda_k}, c_k = \frac{\bar{b}_k}{\lambda_k}, b_k = (f, w^k),$$

$\bar{b}_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}dx$.
If $f(x) = \sum_{k=1}^{K} a_k \sin\frac{\pi k x}{L}, K \le M$, ($a_k$ are constant coefficients) then FDSES method at least with M summands are exact methods, but FDS is the method of the second order approximation. In this case the exact

solution is $u(x) = \sum_{k=1}^{K} \frac{a_k}{d_k} \sin\frac{\pi k x}{L}$.
By $K = 1, L = 10$ the calculations with FDS using MATLAB give preliminary results.

1.4 The non-conjugate finite difference operator with first kind BC

We consider following boundary value problem

$$u''(x) + au'(x) = -f(x), u(0) = u(L) = 0, \tag{1.10}$$

where a is constant parameter. The analitycal solution is
$u(x) = \frac{1}{a}[\frac{1-\exp(-ax)}{\exp(-aL)-1}\int_0^L(\exp(-a(L-t))-1)f(t)dt +$
$\int_0^x(\exp(-a(x-t))-1)f(t)dt]$. In this case we have corresponding A. Iljin FDS $(M = N-1)$

$$\Lambda y = -(\gamma y_{x\bar{x}} + a y_{\dot{x}}) = f(x), y(0) = y(L) = 0, x = x_j = jh, \quad j = \overline{1,M}, \tag{1.11}$$

where $y_{x\bar{x},j} = (y_{j+1} - 2y_j + y_{j-1})/h^2$, $y_{\dot{x},j} = (y_{j+1} - y_{j-1})/(2h)$ are central finite difference expressions, γ is the grid parameter. For the monotonous approximation $\gamma = \alpha\coth(\alpha)$ (A. Iljin FDS),
$\gamma = |\alpha| + (1+|\alpha|)^{-1}$ (A.Samarsky FDS),
$\gamma = |\alpha| + 1$ (upwind FDS), where $\alpha = 0.5ah$. For the central FDS $\gamma = 1$.
The corresponding spectral problem is $\Lambda w = \mu w$ or in the index form $(w_0 = w_N = 0)$

$$(\gamma+\alpha)w_{j+1} - 2(\gamma - 0.5\mu h^2)w_j + (\gamma-\alpha)w_{j-1} = 0, j = \overline{1,M}.$$

For solving this difference equation we use the transformation
$w_j = \kappa^j z_j$. Then we have

$$z_{j+1} - 2\cos(ph)z_j + z_{j-1} = 0, z_0 = z_N = 0, j = \overline{1,M},$$

where
$\kappa = (\frac{\gamma-\alpha}{\gamma+\alpha})^{0.5}$, $\cos(ph) = (\gamma - 0.5\mu h^2)/\sqrt{\gamma^2 - \alpha^2}$.
Similarly (section 1.3) we obtain $z_j^k = C_k \sin(p_k x_j)$,
$w_j^k = C_k \kappa^j \sin(p_k x_j)$, $p_k = \frac{k\pi}{L}, k = \overline{1,M}$ and the eigenvalues are

$$\mu_k = \frac{2}{h^2}(\gamma - \sqrt{\gamma^2 - \alpha^2}\cos\frac{k\pi h}{L}), \quad k = \overline{1,M}. \tag{1.12}$$

The solution of the conjugate spectral problem (the parameter a is replaced with $-a$) is in the form
$w^k_{*,j} = C^*_k \kappa^{-j}\sin(p_k x_j), \quad k,j = \overline{1,M}.$
Determine the constants C_k, C^*_k from the biorthonormed condition $(w^k_*, w^m) = \delta_{k,m}$ we get $C_k = C^*_k = \sqrt{\frac{2}{N}}$ and

$$w^k(x_j) = \sqrt{\frac{2}{N}}\kappa^j \sin\frac{k\pi x_j}{L}, w^k_*(x_j) = \sqrt{\frac{2}{N}}\kappa^{-j}\sin\frac{k\pi x_j}{L}, j = \overline{0,N}, k = \overline{1,M}. \tag{1.13}$$

If $\gamma = \alpha\coth(\alpha)$ (A. Iljin FDS), then $\kappa = \exp(-\alpha)$ and on the grid points x_j the eigenfunctions (1.13) are coincided with the eigenfunctions of the corresponding differential problems
$w''(x) + aw'(x) + \lambda w(x) = 0, w_*(x) - aw'_*(x) + \lambda w_*(x) = 0,$
$w(0) = w(L) = 0, w_*(0) = w_*(L) = 0:$

$$w^k(x) = \sqrt{\frac{2}{L}}\exp(-0.5ax)\sin\frac{k\pi x}{L}, w^k_*(x) = \sqrt{\frac{2}{L}}\exp(0.5ax)\sin\frac{k\pi x_j}{L}, \tag{1.14}$$

but the eigenvalues are different $(\lambda_k = (\frac{k\pi}{L})^2 + (\frac{a}{2})^2), k = 1,2,....$
This eigenfunctions are biorthonormed in the scalar product $(w^k, w^k_*)_* = \int_0^L w_k(x)w^k_*(x)dx.$
If $\gamma = 1$ (the central FDS), then we have real solution only by the condition $|\alpha| \leq 1$ or $h \leq \frac{2}{|a|}$.

Example 1.3. The solution of discrete boundary value problem (1.11) or of the FDS with second order of approximation for boundary value problem (1.10) we can write in following form

$$y_j = \sum_{k=1}^{M}\delta_k w^k_j,$$

where $\delta_k = \frac{\beta_k}{\mu_k}, \beta_k = (w^k_*, f) = \sum_{j=1}^{M} w^k_{*,j} f(x_j).$

For the FDSES by $\gamma = \alpha\coth(\alpha)$ in the representation $A = WDW_*$ the diagonal elements of matrix D are replaced with the eigenvalues λ_k. Then for the solution of the problem $Ay = f$ we can used the transformation $v = W_* y$. Then $Dv = W_* f$ or $v_i = \frac{1}{d_i}\sum_{k=1}^{M} w^k_{*,i} f_k$ and $y = Wv$ or

$y_j = \sum_{i=1}^{M} w_j^i v_i, j = \overline{1,M}$. If $d_i = \mu_i, i = \overline{1,M}$ then we have the above-mentioned solution of FDS, but for $d_i = \lambda_i$ we obtain the solution of FDSES.

Example 1.4. The solution of the problem (1.10) for
$f(x) = \exp(-ax)(\frac{a\pi}{L}\cos(x\pi/L) + (\frac{\pi}{L})^2\sin(x\pi/L))$
is $u(x) = \exp(-ax)\sin(x\pi/L)$.
The calculations with MATLAB for different methods by $L = 1, a = 5$ give following results for maximal error:
1) FDS upwind and FDSES:
0.0424; 0.0225(N=10), 0.0329; 0.0156(N=20), 0.0245;0.0110(N=40),
2) Iljyn FDS and FDSES:
0.00166; 0.01040(N=10), 0.0006; 0.0041(N=20), 0.0002; 0.0015 (N=40),
3) Samarsky FDS and FDSES:
0.0046;0.0115(N=10), 0.002;0.005(N=20), 0.0008;0.0017(N=40),
4) Central FDS and FDSES:
0.0062;0.0099(N=10), 0.0022;0.0040(N=20), 0.0008;0.0009(N=40).
The MATLAB m.file **"PDS2veid" :**

```
%ODE -U''-a U'=f   U(0)=0,  U(1)=0
%Analyt.sol., Exact Iljin FDS,spectral probl.,FDSES
function PDS2veid(N)
N1=N+1;N2=N-1; L=1;a=10;x=linspace(0,L,N1)';
x=x(2:N);h=L/N;a1=a*h/2;
%g=a1*coth(a1);% Iljina FDS
g=1; % centr. dif.
%g=1+abs(a1);% upwind
%g=1+a1^2/(1+abs(a1));% Samarskij FDS
B1=zeros(N2,N2);
B1=B1+2*g*diag(ones(N2,1))-(g-a1)*diag(ones(N2-1,1),-1)-. . .
g+a1)*diag(ones(N2-1,1),1);
B1=B1/(h^2);% 3-diag. matr.$
F=exp(-a*x).*(a*pi/L*cos(pi*x/L)+ (pi/L)^2*sin(pi*x/L));
prec=exp(-a*x).*sin(pi*x/L);
u=B1\F;
lk0=(pi/L*[1:N2]').^2+a^2/4;
lk=2/(h^2)*(g-sqrt(g^2-a1^2)*cos([1:N2]'*pi*h/L));
CK1=sqrt(2*h/L);g1=(g-a1)/(g+a1);
W=(sin(pi/L*[1:N2]'*x').*(ones(N2,1)*(g1.^(0.5*x/h))'))';
W1=(sin(pi/L*[1:N2]'*x').*(ones(N2,1)*(g1.^(-0.5*x/h))'))';
for j=1:N2
    W1(:,j)=W1(:,j).*CK1;
W(:,j)=W(:,j).*CK1;end;
%W'*W1,A2=W1*diag(lk)*W' % control
%U=W*diag(lk.^(-1))*W1'*F;%FDS
```

```
U=W*diag(lk0.^(-1))*W1'*F; %FDSES
%[V1,D1]=eig(B1);lkk=diag(D1);[lk,lkk,lk0] % control
figure,plot(x,prec,'k*',x,u,'ko',x,U,'rO')
legend('exact sol.', 'Matlab sol.', 'FDS or FDSES')
title(sprintf('Solution  on x,N= %3.0f',N))
kMat=abs(u-prec);kU=abs(U-prec);norm(kMat),norm(kU)
figure,plot(x,kMat,'k*',x,kU)
legend('Matlab err.', 'FDS or FDSES err.')
title(sprintf('Error Sol.on x,N=\%2.0f,kM=\%6.5f, kU=%6.5f',. . .
N,norm(kMat),norm(kU)))
```

The results can be represented operator **"PDS2veid(20)"** by $a = 5, N = 40$.

1.5 The finite difference operator with periodical BCs

We consider following boundary value problem with periodical conditions

$$-u''(x) = f(x), x \in (0,L), u(0) = u(L), u'(0) = u'(L). \tag{1.15}$$

This problem has unique solutions by $\int_0^L f(x)dx = 0, u(x_0) = u_0$, where $x_0 \in [0,L], u_0$ are fixed constant.
The analytical solution of this problem is

$$u(x) = \int\limits_0^x (t-x)f(t)dt - \frac{x}{L}\int\limits_0^L tf(t)dt + u_0,$$

where $u_0 = 0$ if $x_0 = 0$.
Using the uniform grid and finite differences with the second order of approximation we obtain the FDS $Ay = f$ with 3-diagonal matrix A of $M = N$ order in the following form

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ .. & .. & .. & \dots & .. & .. & \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

where y, f are the column-vectors with following elements $y_j, f_j = f(x_j), j = \overline{1,N}, y_0 = y_N, y_{N+1} = y_1$.
The matrix A is the circulant matrix and this can determined wth the first row $A = [2\ -1\ 0 \ldots 0\ 0\ -1]$.

The calculations of circulant matrixes
$A = [a_1, a_2, \ldots, a_M], B = [b_1, b_2, \ldots, b_M], C = [c_1, c_2, \ldots, c_M]$
and column-vectors
$b = (b_1, b_2, \ldots, b_M)^T$, $c = (c_1, c_2, \ldots, c_M)^T$ of the M order
can be carried out using following formulae:
1)Matrix A invertion

$$B = A^{-1}, b_k = \frac{1}{M}\sum_{j=0}^{M-1}(\alpha_j \cos\frac{2\pi kj}{M} - \beta_j \sin\frac{2\pi kj}{M})/(\alpha_j^2 + \beta_j^2), k = \overline{1,M},$$

where $\alpha_j = \sum_{i=0}^{M-1} a_{i+1}\cos\frac{2\pi ij}{M}, \beta_j = \sum_{i=0}^{M-1} a_{i+1}\sin\frac{2\pi ij}{M}$.
Here α_j, β_j are the real and imaginar part of the matrix A eigenvalues. The complex eigenvalues of matrix B are in the form $\lambda_j = \sum_{i=0}^{M-1} b_{i+1}(\cos\frac{2\pi ij}{M} + i_* \sin\frac{2\pi ij}{M}) =$
$\frac{1}{\alpha_j + i_*\beta_j} = \frac{\alpha_j - i_*\beta_j}{\alpha_j^2 + \beta_j^2}$, where i_* is the imaginary units. By multiplay and summed this expression with $\sum_{j=0}^{M-1}\cos\frac{2\pi kj}{M}, \sum_{j=0}^{M-1}\sin\frac{2\pi kj}{M}$, and used the orthonormed conditions $\sum_{j=0}^{M-1}\cos\frac{2\pi kj}{M}\cos\frac{2\pi ij}{M} = \frac{M}{2}\delta_{k,i}$,
$\sum_{j=0}^{M-1}\sin\frac{2\pi kj}{M}\sin\frac{2\pi ij}{M} = \frac{M}{2}\delta_{k,i}$,
we obtain $\frac{M}{2}b_k = \sum_{j=0}^{M-1}\alpha_j\cos\frac{2\pi kj}{M}/(\alpha_j^2 + \beta_j^2)$,
$\frac{M}{2}b_k = -\sum_{j=0}^{M-1}\beta_j\sin\frac{2\pi kj}{M}/(\alpha_j^2 + \beta_j^2)$.
2)Matrices A and B multiplication

$$C = A.B, c_s = \sum_{k=1}^{s} a_k b_{s-k+1} + \sum_{k=s+1}^{M} a_k b_{M+s-k+1}, s = \overline{1,M}.$$

3) Matrix A multiplication with vector b

$$c = A.b, c_s = \sum_{k=1}^{s-1} a_{M-s+k+1} b_k + \sum_{k=s}^{M} a_{k-s+1} b_k, s = \overline{1,M}.$$

4) Matrix A eigenvalues

$$\mu_k = \sum_{j=0}^{M-1} a_{j+1}\phi_k(x_j), \phi_k(x) = \exp(2\pi ikx/L), i = \sqrt{-1}, x_j = jh, j,k = \overline{1,M}.$$

5) Matrix A orthonormed eigenvectors

$$w^k = (w_1^k, w_2^k, \ldots, w_M^k)^T, w_j^k = \sqrt{\frac{h}{L}}\phi_k(x_j) = \sqrt{\frac{1}{M}}\exp(2\pi ikj/M), j = \overline{1,M}.$$

6) Matrix A spectral representation
$A = WDW^*$, where w_j^k, w_{*j}^k, μ_k, are the elements of matrices W, W^* and diagonal matrix D of the M-order,
$w_{*j}^k = \sqrt{\frac{1}{M}}\exp(-2\pi ikj/M), j = \overline{1,M}$.
7) Matrix A norm: $||A|| = \max|\mu_k| \leq \sum_{j=1}^{M}|a_j|$.
This algorithm can be easily realized by MATLAB.
We have following MATLAB m.files
1)**B=Cikl(A)** for matrix A invertion:

```
function B=Cikl(A)
% A,B=vectors-rows, circular matrix A invertion,B=inv(A)
M=length(A);M1=M-1;h=2*pi/M;P1=h*([0:M1]'*[0:M1]);
CS=cos(P1);A1=CS*A';SN=sin(P1);A2=SN*A';
A3=A1.^2+A2.^2;AJ=A1./A3; BJ=A2./A3;
B=(CS*AJ-SN*BJ)'/M;
```

2) **C=CMR(A,B)** for two matrices multiplication:

```
function C=CMR(A,B)
% A,B,C -vectors- rows, two matrices A un B multiplication
M=length(B);C(M)=A*B(M:(-1):1)';
for s=1:M-1
    C(s)=A(1:s)*B(s:(-1):1)'+A(s+1:M)*B(M:(-1):s+1)';
end
```

3) **c=CMRV(A,b)** for matrix A multiplication with vector b:

```
function c=CMRV(A,b)
% A,b -vectors- rows and column, circular matrix A
% multiplication with  vector b, c=A*b- column-vector
M=length(A);c1(1)=A*b;
for s=2:M
   c1(s)=A(M-s+2:M)*b(1:s-1)+A(1:M-s+1)*b(s:M);
end
c=c1';
```

The corresponding spectral problem is $\Lambda w = \mu w$ or in the index form

$$w_{j+1} - 2(1-0.5\mu h^2)w_j + w_{j-1} = 0, j = \overline{1,N}, w_0 = w_N, w_{N+1} = w_1.$$

From the properties of the circulant matrices follows that
$\mu_k = \frac{4}{h^2}(\sin(k\pi/N)^2$, $w_j^k = C_k \exp(2\pi ikj/N), k, j = \overline{1,N}$,
where the constants C_k can be determined from the scalar product $(w^k, w_*^m) = \sum_{j=1}^N w_j^k w_{*,j}^m = \delta_{k,m}$,
$w_{*,j}^m = C_m \exp(-2\pi imj/N), m, j = \overline{1,N}$.
We obtain that $C_k = \sqrt{\frac{1}{N}}$.
The eigenvalues μ_k are symmetrical as regards $k = N/2$ or $\mu_{N/2+m} = \mu_{N/2-m}, m = \overline{1,N/2}$, where N is even number.
Using the matrices W, W_* with the eigenvectors w^k, w_*^k in the matrices columns we get

$$AW = WD, WW_* = E, W^{-1} = W_*, W_*^{-1} = W, A = WDW_*, A^{-1} = W_* D^{-1} W,$$

where the elements of the diagonal matrix D is $d_k = \mu_k, k = \overline{1,N}$.
The solution of the FDS $Ay = f$ in the form $y = A^{-1}f = W_* D^{-1} W f$ it is not possible because $\mu_N = 0, det(A) = 0$.
For the differential spectral problem
$-w''(x) = \lambda w(x), x \in (0,L), w(0) = w(L), w'(0) = w'(L)$
we obtain $\lambda_k = (2\pi k/L)^2, w^k(x) = \sqrt{\frac{1}{L}} \exp(2\pi ikx/L)$,
$w_*^k(x) = \sqrt{\frac{1}{L}} \exp(-2\pi ikx/L) = w^{-k}(x)$,
$(w^k, w_*^m)_* = \int_0^L w^k(x) w_*^m(x) dx = \delta_{k,m}$, $k, m = \overline{-\infty, +\infty}$.
The solution of (1.15) with the Fourier method can be obtained in following form:
$f(x) = \sum_{k=-\infty}^{\infty} b_k w^k(x)$, $b_k = (w_*^k, f)$, $u(x) = \sum_{k=-\infty}^{\infty} a_k w^k(x), a_k = b_k/\lambda_k$.
For the solution the discrete problem $Ay = f$ we use the transformation
$W_* y = v$ or $y = Wv$. Then $Dv = W_* f$ or $d_j v_j = (W_* f)_j, j = \overline{1,N}$.
For $j = N$ we have the expression $0 = \sum_{k=1}^N f_(x_k)$ where consist with the integral condition, $x_0 \in [0,L]$.
The value v_N is indeterminable and we can take $v_N = 0$. For $j = \overline{1,N-1}$ we have $v_j = \frac{1}{d_j}(W_* f)_j$ and the solution is in the form $y = Wv$.

If $d_k = \lambda_k$ then we can obtain the solution of FDSES in following way:
1)$d_k = \lambda_k$ for $k = \overline{1, N_2}$, where $N_2 = N/2$.
2)$d_k = \lambda_{N-k}$ for $k = \overline{N_2, N-1}, d_N = 0$.
For periodical function $f(x)$ follows the complex expression
$f(x) = \sum_{k=-\infty}^{\infty} b_k w^k(x) = \sum_{k=1}^{\infty}(b_k w^k(x) + b_{-k} w^{-k}(x)) + \frac{b_0}{\sqrt{L}} =$
$\frac{1}{2}\sum_{k=1}^{\infty}((b_k + b_k)(w^k(x) + w_*^k(x)) + (b_k - b_{-k})(w^k(x) - w_*^k(x))) + \frac{b_0}{\sqrt{L}} =$
$\sum_{k=1}^{\infty}(b_{kc}\cos\frac{2\pi kx}{L} + b_{ks}\sin\frac{2\pi kx}{L}) + \frac{b_{0c}}{2}$, $b_k = (f, w_*^k)$,
where
$b_{kc} = \frac{1}{\sqrt{L}}(b_k + b_{-k}) = \frac{1}{\sqrt{L}}\int_0^L f(x)(w_*^k(x) + w^k(x))dx) =$
$\frac{2}{L}\int_0^L f(t)\cos\frac{2\pi kt}{L}dt$,
$b_{ks} = \frac{i}{\sqrt{L}}(b_k - b_{-k}) = \frac{i}{\sqrt{L}}\int_0^L f(x)(w_*^k(x) - w^k(x))dx) =$
$\frac{2}{L}\int_0^L f(t)\sin\frac{2\pi kt}{L}dt$.

Therefore the solution of (1.15) we can also in real form:
$u(x) = \sum_{k=1}^{\infty}(a_{kc}\cos\frac{2\pi kx}{L} + a_{ks}\sin\frac{2\pi kx}{L}) + \frac{a_{0c}}{2}$,
where $a_{kc} = \frac{b_{kc}}{\lambda_k}, a_{ks} = \frac{b_{ks}}{\lambda_k}$.
For vector f of N order with component $f_j, j = \overline{1, N}$ using $w_*^k = w^{N-k} = w^{-k}, w_j^N = w_j^0 = 1, j = \overline{1, N}$ we have similarly following expressions: $f = \sum_{k=1}^{N} b_k w^k = \sum_{k=1}^{N_2-1}(b_k w^k + b_{N-k} w^{N-k}) + b_{N_2} w^{N_2} + b_N w^N = \frac{1}{2}\sum_{k=1}^{N_2-1}((b_k + b_{N-k})(w^k + w^{N-k})$
$+ (b_k - b_{N-k})(w^k - w^{N-k})) + b_{N_2} w^{N_2} + b_N w^0, b_k = (f, w_*^k)$,
or $f_j = \sum_{k=1}^{*N_2}(b_{kc}\cos\frac{2\pi kj}{N} + b_{ks}\sin\frac{2\pi kj}{N}) + \frac{b_{0c}}{2}$,
where
$b_{kc} = \frac{1}{\sqrt{N}}(b_k + b_{N-k}) = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} f_j(w_{j*}^k + w_j^k) =$
$\frac{2}{N}\sum_{j=1}^{N} f_j\cos\frac{2\pi kj}{N}$,
$b_{ks} = \frac{i}{\sqrt{N}}(b_k - b_{N-k}) = \frac{i}{\sqrt{N}}\sum_{j=1}^{N} f_j(w_{j*}^k - w_j^k) =$
$\frac{2}{N}\sum_{j=1}^{N} f_j\sin\frac{2\pi kj}{N}, k = \overline{1, N_2 - 1}$,
$b_0 = b_N = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} f_j, b_{0c} = b_{Nc} = \frac{2}{\sqrt{N}} b_0$,
$b_{N_2c} = \frac{2}{\sqrt{N}} b_{N_2} = \frac{2}{N}\sum_{j=1}^{N}\cos(j\pi)$,
$b_{N_2} w_k^{N_2} = \frac{1}{N}\sum_{j=1}^{N}\cos(j\pi)\cos(k\pi), N_2 = \frac{N}{2}, b_{N_2s} = 0$,
$\sum_{k=1}^{*N_2}\beta_k = \sum_{k=1}^{N_2-1}\beta_k + \frac{\beta_{N/2}}{2}$, for periodic functions $b_0 = b_N = 0$.
This discrete Fourier expression we can representation in the following form [19]:
$f_j = \sum_{k=1}^{N_2-1}(b_{kc}\cos\frac{2\pi kj}{N} + b_{ks}\sin\frac{2\pi kj}{N}) + \frac{b_{N_2c}\cos(\pi j)}{2} + \frac{b_{0c}}{2}$.

Similarly the solution of the discrete problem can be represented in the
following form: $y_j = \sum_{k=1}^{*N_2}(a_{kc}\cos\frac{2\pi kj}{N} + a_{ks}\sin\frac{2\pi kj}{N}) + \frac{a_{0c}}{2}$,
where $a_{kc} = \frac{b_{kc}}{\mu_k}, a_{ks} = \frac{b_{ks}}{\mu_k}, a_{0c} = 0$.
This solution we can obtained also from the orthonormed trigonometrical functions. Using the relations [19]
$\sum_{j=1}^{N}\sin(ax_j) = \frac{\sin(0.5a(L+h))\sin(0.5aL)}{\sin(0.5ah)}$,
$\sum_{j=1}^{N}\cos(ax_j) = \frac{\sin(0.5a(L+h))\cos(0.5aL)}{\sin(0.5ah)} - 1$
we obtain $\sum_{j=1}^{N}\sin_k\cos_s = 0, \sum_{j=1}^{N}\sin_k\sin_s = \sum_{j=1}^{N}\cos_k\cos_s = N_2\delta_{k,s}$.
$\sum_{j=1}^{N}\sin_k = \sum_{j=1}^{N}\cos_k = 0$, where $\sin_k, \cos_k$ are N-order column-vectors with the elements $\sin\frac{2\pi kj}{N}, \cos\frac{2\pi kj}{N}$, $k,s = \overline{1,N_2}, L = Nh$.

We have the relation
$\sum_{j=1}^{N} f_j^2 = N_2(\frac{b_{0c}^2}{2} + \frac{b_{N_2c}^2}{2} + \sum_{k=1}^{N_2-1}(b_{kc}^2 + b_{ks}^2))$.
From $Ay = \sum_{k=1}^{*N_2}(a_{kc}A\cos_k + a_{ks}A\sin_k) +$
$\frac{a_{0c}}{2}A\cos_0$ and $A\cos_k = \mu_k\cos_k, A\sin_k = \mu_k\sin_k$
follows $a_{kc}\mu_k = b_{kc}, a_{ks}\mu_k = b_{ks}$.
If in the discrete Fourier expression
$f_j^M = \sum_{k=1}^{*M}(b_{kc}\cos\frac{2\pi kj}{N} + b_{ks}\sin\frac{2\pi kj}{N}) + \frac{b_{0c}}{2}, M < N_2$
then using the least -squares method we can proved, that the Fourier coefficients $b_{kc}, b_{ks}, k = \overline{1,M}$ maintain own form and we can estimate the error
$\sum_{j=1}^{N}(f_j - f_j^M)^2 = \sum_{j=1}^{N} f_j^2 - N_2(\frac{b_{0c}^2}{2} + \sum_{k=1}^{M}(b_{kc}^2 + b_{ks}^2))$.
Compared the discrete Fourier coefficients B_{kc}, B_{ks} with to Fourier series coefficients b_{kc}, b_{ks} we have [19]
$B_{0c} = b_{0c} + \sum_{m=1}^{\infty} b_{(Nm)c}, B_{kc} = b_{kc} + \sum_{m=1}^{\infty}(b_{(N(m-k))c} + b_{(N(m+k))c})$,
$B_{ks} = b_{ks} + \sum_{m=1}^{\infty}(-b_{(N(m-k))s} + b_{(N(m+k))s})$.
For $f(x) \in C^\nu(0,L)$ the coefficients decrease as $k^{-(\nu+1)}$. We can obtain the solution of FDSES by replasing the discrete eigenvalues μ_k with the first N eigenvalues $\lambda_k, k = \overline{1,N}$.

Example 1.5. The solution of the problem (1.15) for $f(x) = x - 0.5, L = 1$ is $u(x) = -(2x^3 - 3x^2 + x)/12, u_0 = 0$.
The calculations with MATLAB by FDS give following results for maximal error: 0.0086 (N=6), 0.0062 (N=10), 0.0042 (N=20).
The FDSES give 0.0077 (N=6), 0.0058 (N=10), 0.0041 (N=20).

The MATLAB m.file **"PDSper" :**

```
%ODE  -U''=f with periodicalU(0)=U(L), U'(0)=U'(L)
%Analyt.sol., spectral probl. -v''=lam^2 v
%Au=lam1^2 *u,A={-1/h^2(u_j+1-2u_j +u_j-1);discrete probl.
% example:f(x)=x-0.5,L=1,U(x)=-(2x^3-3x^2 +x)/12;
function PDSper(N)% N-even
N1=N+1;N2=N-1; NH=N/2;NH1=NH+1;
L=1;x=linspace(0,L,N1)';x=x(2:N1);h=L/N;
B1=zeros(N,N);
B1=B1+2*diag(ones(N,1))-diag(ones(N2,1),-1). . .
-diag(ones(N2,1),1);
B1(1,N)=-1; B1(N,1)=-1;
B1=B1/(h^2);%3-diag. matrix
F=x-0.5;V=zeros(N,1);d=zeros(N,1);
prec=-(2*x.^3 -3*x.^2 +x)/12;
u=B1\F;
lk=4/(h^2)*(sin((1:N)'*h*pi/L)).^2;
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
%d=lk; %FDS
d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
V(1:N2)=W1(1:N2,:)*F(:)./d(1:N2);
U=W*V;ui=imag(max(abs(U)));
%W1*W,A2=W*diag(lk)*W1% control
figure,plot(x,prec,'k*',x,U,'b*')
legend('prec. atr.', 'FDS or FDSES')
title(sprintf('Solution  on x,N=%3.0f,imag=%e',N,ui))
kMat=abs(u-prec); kU=abs(U-prec);norm(kU)
figure,plot(x,kU,'b*')
legend('FDS or FDSES')
title(sprintf('Error Sol.on x,N=%2.0f,kM=%6.5f,. . .
kFDS=%6.5f',N,norm(kMat),norm(kU)))
```

The results can be represented weith the operator **"PDSper(40)"**.

The solution of the problem (1.15) for $f(x) = -f_0\cos(2\pi x), L = 1, u_= 0, f_0 = const$ is $u(x) = \frac{f_0}{4\pi^2}(2-\cos(2\pi x)$.
This solution we can also obtained with Fourier method:
$b_{ks} = 0, b_{1c} = f_0, b_{kc} = 0 (k \neq 1), a_{1c} = \frac{f_0}{\lambda_1}, \lambda_1 = 4\pi^2, u(0) = 0$.
For FDS using discrete Fourier method in similar way:
$a_{1c} = \frac{f_0}{\mu_1}, \mu_1 = \frac{4}{h^2}(\sin(\pi/N)^2 y_0 = 0$.
This solution we can obtain directly solved the difference equation:
$y_{j+1} - 2y_j + y_{j-1} = h^2 f_0\cos(2\pi x_j), j = \overline{1,N}, y_0 = y_N, y_{N+1} = y_1$.

The solution is in the form $y_j = C_1 + C_2 x_j - \frac{f_0 h^2}{4\sin^2(\pi h)}\cos(2\pi x_j)$, where C_1, C_2 are unknown constants. From $y_0 = y_N = 0$ follows $C_1 = \frac{f_0 h^2}{4\sin^2(\pi h)}, C_2 = 0$.
The maximal error is $2|f_0(\mu_1^{-1} - \lambda_1^{-1}|$ and for FDSES by replacing in the matrix equation $WDW_* y = f$ the elements μ_k of the diagonal matrix D with $\lambda_k, k = \overline{1,N}$ we obtain the exact solution and the error is equal to zero (similar Fourier method).
Easy we can solved also the following nonlinear problem $-(v^\beta)''(x) = f(x), x \in (0,L), v(0) = v(L), v'(0) = v'(L)$, using the transformation $u = v^\beta, v = u^{1/\beta}, \beta \geq 1$.

1.6 The higher order FDS by periodical BCs

In this section we consider the finite difference approximation for derivatives $u'(x_j), -u''(x_j), -u'''(x_j), u''''(x_j)$ using the uniform grid $x_j = jh, j = \overline{0,N}$ with multi points stencil.

1.6.1 Derivative of second order

We consider the finite difference approximation for second order derivative $-u''(x_j)$ using the uniform grid $x_j = jh$ with $2n+1$ points stencil $(x_{j-n}, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_{j-n})$. Used the method of unknown coefficients C_k, E_p we consider the approximation of the $O(h^{2n})$ order in following form:

$$u''(x_j) = \frac{1}{h^2}\sum_{k=-n}^{n} C_k u(x_{j-k}) + E_{2n}\frac{h^{2n}u^{(2n+2)}(\xi)}{(2n+2)!}, x_{j-n} < \xi < x_{j+n}.$$

With the transformation $t = \frac{x-x_j}{h}, u(x) = u(x_j + th) = \bar{u}(t)$,
$u''(x) = \frac{1}{h^2}\bar{u}''(t), u^{(2n+2)} = \frac{1}{h^{2n+2}}\bar{u}^{(2n+2)}$
we obtain

$$\bar{u}''(0) = \sum_{k=-n}^{n} C_k \bar{u}(-k) + E_{2n}\frac{\bar{u}^{(2n+2)}(\bar{\xi})}{(2n+2)!}, -n < \bar{\xi} < n.$$

Using properties of symmetry $C_m = C_{-m}$ and the power functions $\bar{u}(t) = t^m, m = \overline{0, 2n+1}$ we get the systems of linear algebraic equations for determined the coefficients C_m. For $m = 0$ follows the equation $C_0 = -2\sum_{m=1}^{n} C_m$. For the others coefficients $(m > 0)$ we get the system of linear algebraic system $Vc = e$ or $\sum_{k=1}^{n} C_k k^{2m} = \delta_{m,1}, m = \overline{1,n}$,
where $e = (1, 0, \cdots, 0)^T$ is the unit column-vector, $c = (C_1, C_2, \cdots, C_n)^T$ and V is the matrix Vandermonde of the n order tn following form:

$$V = \begin{pmatrix} 1^2 & 2^2 & 3^2 & \dots & (n-1)^2 & n^2 \\ 1^4 & 2^4 & 3^4 & \dots & (n-1)^4 & (n)^4 \\ .. & .. & .. & \dots & \dots & \\ 1^{2n-2} & 2^{2n-2} & 3^{2n-2} & \dots & (n-1)^{2n-2} & (n)^{2n-1} \\ 1^{2n} & 2^{2n} & 3^{2n} & \dots & (n-1)^{2n} & (n)^{2n} \end{pmatrix}$$

If $m = 2n+2$ then we obtain
$E_{2n} = -2(C_1 + 2^{2n+2}C_2 + \cdots + (n)^{2n+2}C_n) = -2\sum_{m=1}^{n} C_m m^{2n+2}$.
The matrix B is the inverse of the Vandermonde matrix
with the elements $b_{j,k}$ $(V^{-1} = B, c = B^{-1}e)$. Then $C_j = b_{j,1}, j = \overline{1,n}$.
We consider the polynomial of the $(n-1)$-order $P_j^1(x) = \sum_{k=1}^{n} b_{j,k} x^{k-1}$ and polynomial of n order $P_j^2(x) = P_j^1(x)x = \sum_{k=1}^{n} b_{j,k} x^k$ and $R_j(x) = j^2 P_j^1(x)$ [24]. From $BV = E$ follows that $P_j^2(k^2) = \sum_{l=1}^{n} b_{j,l} k^{2l} = \delta_{k.j}$ and $P_j^1(k^2) = 1/j^2 \delta_{k,j}, R_j(k^2) = \delta_{k,j}.k, j = \overline{1,n}$. Then the polynomial $R_j(x)$ is the Lagrange characteristic polynomials in following form $R_j(x) = \prod_{k=1 \neq j}^{n} \frac{x-k^2}{j^2-k^2}$ and $C_j = b_{j,1} = 1/j^2 R_j(0) = \frac{1}{j^2} \prod_{k=1 \neq j}^{n} \frac{-k^2}{(j^2-k^2)}$ [21]. Therefore

$$C_m = \frac{2(n!)^2(-1)^{m-1}}{m^2(n-m)!(n+m)!}, m = \overline{1,n}.$$

Therefore we have following coefficients:
1)$n = 1 : C_1 = 1, C_0 = -2, E_2 = -2$,
2)$n = 2 : C_1 = \frac{4}{3}, C_2 = -\frac{1}{12}, C_0 = -\frac{5}{2}, E_4 == 8$,
3)$n = 3 : C_1 = \frac{3}{2}, C_2 = -\frac{3}{20}, C_3 = \frac{1}{90}, C_0 = -\frac{49}{18}, E_4 = -72$,
4)$n = 6 : C_1 = \frac{8}{5}, C_2 = -\frac{1}{5}, C_3 = \frac{8}{315}, C_4 = -\frac{1}{560}, C_0 = -\frac{205}{72}$,
$E_8 = 1152$.
In this case the circulant finite difference matrix A approximed the second order derivative $-u''(x_j)$ is in the form

$$A = -\frac{1}{h^2}[C_0, C_1, \cdots, C_n, 0, \cdots, 0, C_n, C_{n-1}, \cdots, C_2, C_1].$$

The eigenvalues of matrix A are $\mu_k = -\frac{1}{h^2}\sum_{m=0}^{N-1} C_m \exp(2\pi i m/N)$= $-\frac{1}{h^2}(C_0 + 2\sum_{m=1}^{n} C_m \cos\frac{2\pi km}{N})$ or

$$\mu_k = \frac{4}{h^2}\sum_{m=1}^{n} C_m \sin^2(\pi km/N), k = \overline{1,N}.$$

For obtaining the unknown coefficients Q_m in the relation
$\mu_k = \frac{4}{h^2}\sum_{m=1}^{n} Q_m \sin^{2m}(\pi k/N)$ undepending on n we use
the following expressions for sinus function $(a = \frac{\pi k}{N})$:
$\sin^2(2a) = 4\sin^2(a) - 4\sin^4(a); \sin^2(3a) = 9\sin^2(a) - 24\sin^4(a) + 16\sin^6(a)$;
$\sin^2(4a) = 16\sin^2(a) - 80\sin^4(a) + 128\sin^6(a) - 64\sin^8(a)$;
$\sin^2(5a) = 25\sin^2(a) - 200\sin^4(a) + 560\sin^6(a) - 640\sin^8(a) + 256\sin^{10}(a)$;
$\sin^2(ma) = m^2\sin^2(a) + ... + D_m \sin^{2m}(a)$,
where $D_m = (-4)^{m-1}, m \geq 0$. The for $m = n$ by $Q_n = D_n C_n$ we obtain $Q_n = \frac{2(n!)^2 4^{n-1}}{n^2(2n)!}$. We can assume that $Q_m = \frac{2(m!)^2 4^{m-1}}{m^2(2m)!}$ for all $m = \overline{1,n}$.
The strong prove can be jused the m-th degree of Spread polynomial $S_m(s) = \sin^2(ma), s = \sin^2(a)$, in [20] S. Goh is proved that

$$S_m(s) = s\sum_{j=0}^{m-1} \frac{m}{m-j} C_{2m-1-j}^{j} (-4s)^{m-1-j} = s\sum_{l=1}^{m} \frac{m}{l} C_{m+l-1}^{m-l} (-4s)^{l-1},$$

where $l = m - j$. This formula is proved also in [24], using the Chebyshev polynomials of the first kind $\cos(m \arccos(x)) = T_m(x) = m\sum_{k=0}^{m}(-2)^k \frac{(m+k-1)!}{(m-k)!(2k)!}(1-x)^k$.
Then $\mu_k = \frac{4}{h^2}\sum_{m=1}^{n} C_m S_m(s) = \frac{4}{h^2}\sum_{m=1}^{n} C_m \sum_{l=1}^{m} \alpha_{m,l} s^l$,
where $\alpha_{m,l} = \frac{m}{l}\frac{(m+l-1)!}{(m-l)!(2l-1)!}(-4)^{l-1}$.
Using $\sum_{m=1}^{n}\sum_{l=1}^{m} \alpha_{m,l} = \sum_{l=1}^{n}\sum_{m=l}^{n} \alpha_{m,u}$, and denoting
$Q_l = \sum_{m=l}^{n} C_m \alpha_{m,l}$ we get $\mu_k = \frac{4}{h^2}\sum_{l=1}^{n} Q_l s^l$.
We need proved that $Q_l = \frac{2((l-1!))^2 4^{l-1}}{(2l)!}$. From $C_m = \frac{1}{m^2}\prod_{k=1 \neq m}^{n} \frac{-k^2}{(m^2-k^2)}$ follows that we need proved the expression
$\sum_{m=l}^{n}\prod_{k=1\neq m}^{n} \frac{k^2}{(k^2-m^2)}\frac{1}{m}\frac{(m+l-1)!}{(m-l)!}(-1)^{l-1} = ((l-1)!)^2$

or $\sum_{m=v+1}^{n}(\prod_{k=1\neq m}^{n}\frac{k^2}{(k^2-m^2)})\frac{1}{m}\frac{(m+v)!}{(m-v-1)!}(-1)^v=(v!)^2$, where $v=l-1$.
This expresion we can write in following form [20]:
$\sum_{m=v+1}^{n}(\prod_{k=1\neq m}^{n}\frac{k^2}{(k^2-m^2)})(\prod_{k=1}^{v}(k^2-m^2))=(v!)^2$ or
$\sum_{m=v+1}^{n}(\prod_{k=1\neq m}^{n}\frac{k^2}{(k^2-m^2)})(\prod_{k=1}^{v}\frac{(k^2-m^2)}{k^2})=1[24]$.
Then $\mu_k=\frac{4}{h^2}\sum_{m=1}^{n}Q_m\sin^{2m}(\pi k/N)$.
We have $Q_1=1,Q_2=\frac{1}{3},Q_3=\frac{8}{45},Q_4=\frac{4}{35},\cdots$.
For FDSES we can replace the discrete eigenvalues $d_k=\mu_k,k=\overline{1,N}$ with the continues values $\lambda_k=(\frac{2\pi k}{L})^2$ in following way:
$d_k=\lambda_k,k=\overline{1,N/2},d_{k+N/2}=\lambda_{N/2-k-1},k=\overline{1,N/2},d_N=0$ (see Figs. 1.1, 1.2).

Example 1.6. We added following operators by the MATLAB m.file **”PDSper”**

```
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; % O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+ . . .
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
```

we can by $N=6$ with the operator **”PDSper(6)”** for FDS obtain the following maximal errors:
$0.00857(O(h^2)),0.00786(O(h^4)),0.00775(O(h^6)),0.00772(O(h^8))$, $0.00767(for\ FDSES)$.

Example 1.7. The solution of the problem (1.15) for $f(x)=x-0.5L$ is $u(x)=-(2x^3-3Lx^2+xL^2-12u_0)/12,u_0=1$.
The calculations with MATLAB by $L=10,N=100$ for FDS give following results for maximal error:
$0.6250(O(h^2)),0.6214(O(h^4)),0.6207(O(h^6)),0.6205(O(h^8))$.
The FDSES give 0.6199.
The exact solution of the problem with $f(x)=\cos(2\pi x/L)\exp(\sin(2\pi x/L))$ can be obtained use the Matlab operator ”quad”
(see the **Matlab listing**). The calculations with MATLAB by $L=10,N=10$ for FDS give following maximal error:
$0.2097(O(h^2)),0.0131(O(h^4)),0.0049(O(h^6)),0.0023(O(h^8))$, but for FDSES, 3.210^{-5}.

If $f(x) = \cos(2P\pi x/L)\exp(\sin(2P\pi x/L))$ then we obtain following results:
1) $P = 2, N = 10 : 0.22(O(h^2)), 0.052(O(h^4)), 0.049(O(h^6)), 0.045(O(h^8)), 0.034(FDSES)$,
2) $P = 3, N = 20 : 0.054(O(h^2)), 0.0087(O(h^4)), 0.0047(O(h^6)), 0.0034(O(h^8)), 0.0011(FDSES)$,
3)1) $P = 3, N - 40 : 0.013(O(h^2)), 6.10^{-4}(O(h^4)), 10^{-4}(O(h^6)), 4.10^{-5}(O(h^8)), 6.410^{-8}(FDSES)$,
4)1) $P = 4, N - 40 : 0.013(O(h^2)), 8.10^{-4}(O(h^4)), 3.10^{-4}(O(h^6)), 1.510^{-4}(O(h^8)), 2.10^{-6}(FDSES)$,
5)1) $P = 4, N - 100 : 0.0021(O(h^2)), 3.10^{-5}(O(h^4)), 2.10^{-6}(O(h^6)), 2.10^{-7}(O(h^8)), 1.510^{-8}(FDSES)$.

Matlab listing:

```
gg=@(t) cos(2*pi*t/L).*exp(sin(2*pi*t/L));
gg1=@(t)t.*cos(2*pi*t/L).*exp(sin(2*pi*t/L));
pre=quad(gg1,0,L,1.e-10);
for j=1:N
prec(j)= -x(j)*quad(gg,0,x(j),1.e-10)+. . .
quad(gg1,0,x(j),1.e-10) -x(j)/L*pre;
end
F=cos(2*pi*x/L).*exp(sin(2*pi*x/L));
```

The following MATLAB m.file **"PDSper1" is used:**

```
%ODE  -U''=f(x) with periodicalU(0)=U(L), U'(0)=U'(L),U(L)=U0
% example:f(x)=x-0.5L,U(x)=-(2x^3-3L x^2 +L^2 x-12 U0)/12;
function PDSper1(N)% N-even
N1=N+1;N2=N-1; NH=N/2;NH1=NH+1;U0=1; L=10;
x=linspace(0,L,N1)';x=x(2:N1);h=L/N;
B1=zeros(N,N);V=zeros(N,1);d=zeros(N,1);
VV=zeros(N,1);dd=zeros(N,1);
U=zeros(N,1); UU=zeros(N,1);
%B1=B1+...
%2*diag(ones(N,1))-diag(ones(N2,1),-1)-diag(ones(N2,1),1);
%B1(1,N)=-1; B1(N,1)=-1;
%B1=B1/(h^2);%3-diag. matrix for O(h^2)
F=x-0.5*L;
prec=-(2*x.^3 -3*L*x.^2 +x*L^2-12*U0)/12;
%u=B1\F;
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %2.order FDS
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%4.ord.
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
%8/45*(sin(pi*h*NT)).^6);%6.order FDS
```

```
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
%8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%8.ord.
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
dd=lk; %FDS
d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
V(1:N2)=W1(1:N2,:)*F(:)./d(1:N2);
VV(1:N2)=W1(1:N2,:)*F(:)./dd(1:N2);
U=W*V;ui=max(abs(imag(U)));U=U-U(N,1)+U0;%FDSES sol.
UU=W*VV;uui=max(abs(imag(UU)));UU=UU-UU(N,1)+U0;%FDS sol.
%W1*W,A2=W*diag(lk)*W1% control, A2=B1 for O(h^2)
figure
plot(x,prec,'k*',x,U,'b*',x,UU,'-','LineWidth',2,'MarkerSize',3)
legend('prec. atr.', 'FDSES','FDS')
title(sprintf('Solution  on x,N=...
%3.0f,imPDS=%6.5d,imDS=%6.5d',N,ui,uui))
kUU=abs(UU-prec); kU=abs(U-prec);norm(kUU),norm(kU)
figure,plot(x,kU,'b*',x,kUU,'-','LineWidth',2,'MarkerSize',3)
legend('FDSES','FDS')
title(sprintf('Error N=...
%2.0f,nFDSES=%6.5d,nFDS=%6.5d',N,norm(kU),norm(kUU)))
```

Example 1.8. The solution of the problem (1.15) for $f(x) = \sin(2\pi Px/L), P = 2;4$ is $u(x) = (\frac{L}{2\pi p})^2 f(x), u(0) = 0$.
We have the exact solution for FDSES, if $N \geq 2*P$. The results for FDS and FDSES of the maximal error we can see in the Tab. 1.1.

Table 1.1 The values of FDS $O(h^k), k = 2;4;6;8, \delta(FDS), \delta(FDSES)$ for diferent values of p, N, L

FDS	$pNL = 2,6,1$	$pNL = 2,6,10$	$pNL = 2,6,100$	$pNL = 2,12,10$	$pNL = 2,12,100$	$pNL = 4,16,10$
$O(h^2)$	0.0025	0.2535	25.39	0.0530	5.299	0.0371
$O(h^4)$	$9.3e-4$	0.0931	9.310	0.0067	0.6727	0.0091
$O(h^6)$	$4.5e-4$	0.0456	4.552	0.0011	0.1092	0.0030
$O(h^8)$	$2.5e-4$	0.0251	2.513	$2.0e-4$	0.0196	0.0010
$FDSES$	$1.1e-17$	$1.7e-15$	$1.7e-13$	$5.1e-15$	$3.6e-13$	$6.4e-15$

We use following MATLAB m.file **"PDSexat":**

```
%ODE  -U''=f(x), U(0)=U(L), U'(0)=U'(L),U(L)=U0
% example:f(x)=sin(2 pi px/L),U(x)=(L/(2 pi p))^2 f(x);p=2;4
function PDSexat(N)% N-even
```

```
N1=N+1;N2=N-1; NH=N/2;U0=0; L=100;
x=linspace(0,L,N1)';x=x(2:N1);h=L/N;
V=zeros(N,1);d=zeros(N,1);VV=zeros(N,1);dd=zeros(N,1);p=2;
F=sin(2*pi*p*x/L);
prec=(L/(2*p*pi))^2*F;
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %2.order FDS
%lk=4/h^2*((sin(pi*h*NT)).^2+ . . .
%1/3*(sin(pi*h*NT)).^4);%4.order FDS
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
%8/45*(sin(pi*h*NT)).^6);%6.order FDS
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
%8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%8.order FDS
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
dd=lk; %FDS
d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
V(1:N2)=W1(1:N2,:)*F(:)./d(1:N2);
VV(1:N2)=W1(1:N2,:)*F(:)./dd(1:N2);
U=W*V;ui=max(abs(imag(U)));U=U-U(N,1)+U0;%FDSES sol.
UU=W*VV;uui=max(abs(imag(UU)));UU=UU-UU(N,1)+U0;%FDS sol.
%real(W1*W),A2=real(W*diag(lk)*W1)% control, A2=B1 for O(h^2)
figure,plot(x,prec,'k*',x,U,'r*',x,UU,'-',. . .
'LineWidth',2,'MarkerSize',3)
legend('prec. atr.', 'FDSES','FDS')
title(sprintf('Solution  on x,N=%3.0f,. . .
imPDS=%6.5d,imDS=%6.5d',N,ui,uui))
kUU=abs(UU-prec); kU=abs(U-prec);max(kUU),max(kU)
figure,plot(x,kU,'r*',x,kUU,'-',. . .
'LineWidth',2,'MarkerSize',3)
legend('FDSES','FDS')
title(sprintf('Error N=%2.0f,nFDSES=%6.5d,. . .
nFDS=%6.5d',N,max(kU),max(kUU)))
```

We can also obtain with FDSES the exact solution in the case for linear combination of the functions $\sin(2\pi p_1 x/L), \cos(2\pi p_2 x/L)$ for $f(x)$ if $N \geq 2 * \max(p_1, p_2)$.

1.6.2 Derivative of first order

The finite difference approximation $O(h^{2n})$ for first order derivative $u'(x_j)$ using the uniform grid $x_j = jh$ with $2n+1$ points stencil we can obtained in following form:

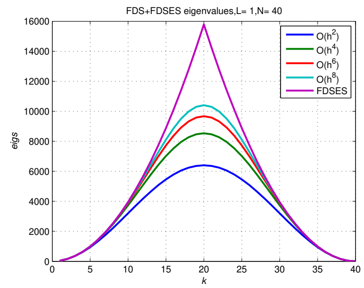


Fig. 1.1 Eigenvalues for $-u''$ by $N = 40, L = 1, n = 1;2;3;4$

$$u'(x_j) = \frac{1}{h}\sum_{k=-n}^{n} c_k u(x_{j-k}) + e_{2n}\frac{h^{2n}u^{(2n+2)}(\xi)}{(2n+2)!}, x_{j-n} < \xi < x_{j+n}.$$

Using propertys of anti-symmetry $c_m = -c_{-m}, c_0 = 0$ we get the systems of linear algebraic equations for determined the coefficients c_m in the form $\sum_{k=1}^{n} 2\frac{c_k}{k}k^{2m} = \delta_{m,1}, m = \overline{1,n}$, with the the matrix Vandermonde.

We have following matrix representation:
$A^0 = \frac{1}{h}[0, c_1, c_2, ..., c_n, 0, 0, ... - c_n, -c_{n-1}, ..., -c_2, -c_1]$,
where $c_m = 0.5mC_m = \frac{(n!)^2(-1)^{m-1}}{m(n-m)!(n+m)!}, m = \overline{1,n}$.
Then the eigenvalues are: $\mu_k^0 = \frac{2i}{h}\sum_{m=1}^{n} c_m \sin\frac{2\pi mk}{N}$.
We have following expressions:
1) $m = 2$; $\sin\frac{4\pi k}{N} = \sin\frac{2\pi k}{N}(2 - 4\sin^2\frac{\pi k}{N})$,
2)$m = 3$; $\sin\frac{6\pi k}{N} = \sin\frac{2\pi k}{N}(3 - 16\sin^2\frac{\pi k}{N} + 16\sin^4\frac{\pi k}{N})$,
3) $m = 4$; $\sin\frac{8\pi k}{N} = \sin\frac{2\pi k}{N}(4 - 40\sin^2\frac{\pi k}{N} + 96\sin^4\frac{\pi k}{N} - 64\sin^6\frac{\pi k}{N})$,
4) $m = n$; $\sin\frac{2n\pi k}{N} = \sin\frac{2\pi k}{N}(n - ... + (-4)^{n-1}\sin^{2n-2}\frac{\pi k}{N})$.
Therefore $\mu_k^0 = \frac{2i}{h}\sin\frac{2\pi k}{N}\sum_{m=1}^{n} q_m \sin^{2m-2}\frac{\pi k}{N}$,
where the coefficients $q_m = c_m(-4)^{m-1} = 0.5mQ_m = \frac{(m!)^2 4^{m-1}}{m(2m)!}$ are not depending on n. We have $q_1 = \frac{1}{2}, q_2 = \frac{1}{3}, q_3 = \frac{4}{15}, Q_4 = \frac{8}{35}, \cdots$.
For FDSES we can replace the imaginary part of the discrete eigenvalues $d_k = Im(\mu_k^0), k = \overline{1,N}$ with the continues values $d1_k = Im(\lambda_k^0) = \frac{2\pi k}{L}$ in following way:
$d_k = d1_k, k = \overline{1, N/2-1}, d_{k+N/2} = -d1_{N/2-k}, k = \overline{1, N/2}, d_{N/2} = 0$ (see Fig. 1.3).

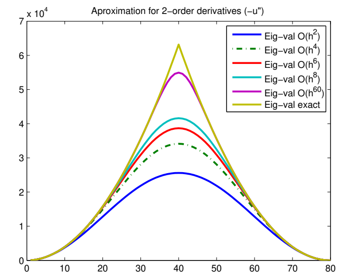


Fig. 1.2 Eigenvalues for $-u''$ by $N = 80, L = 1, n = 1;2;3;4;30$

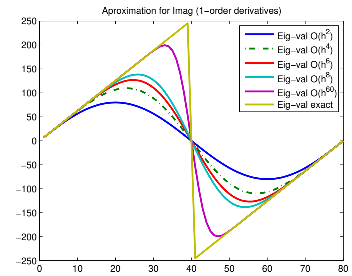


Fig. 1.3 Imaginary part of eigenvalues for u' by $N = 80, L = 1, n = 1;2;3;4;30$

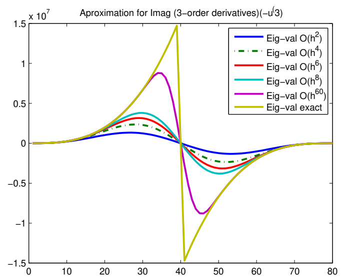


Fig. 1.4 Imaginary part of eigenvalues for $-u'''$ by $N = 80, L = 1, n = 1;2;3;4;30$

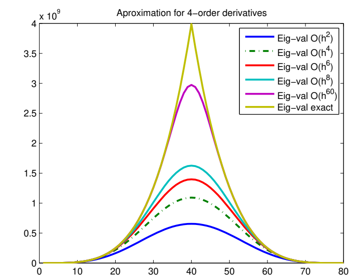


Fig. 1.5 Eigenvalues for u'''' by $N = 80, L = 1, n = 1;2;3;4;30$

1.6.3 Derivative of fourth order

The finite difference approximation $O(h^{2n})$ for fourth order derivative $u''''(x_j)$ using the uniform grid $x_j = jh$ with $2n+3$ points stencil we can obtain in following form:

$$u''''(x_j) = \frac{1}{h^4}\sum_{k=-n}^{n} C_k u(x_{j-k}) + E_{2n}\frac{h^{2n}u^{(2n+4)}(\xi)}{(2n+4)!}, x_{j-n} < \xi < x_{j+n}.$$

Using properties of symmetry $C_m = C_{-m}, C_0 = -2\sum_{k=1}^{n+1}$ we get the systems of linear algebraic equations for determine the coefficients $C_m, m = 1\overline{1,n+1}$ in the form $\sum_{k=1}^{n+1} C_k k^{2m} = 12\delta_{m,2}, m = \overline{1,n+1}$ or $Vc = e$ with the the matrix Vandermonde V of the $n+1$ order and $e = (0,1,0,\cdots,0)^T$ is the unit column-vector of the $n+1$ order.
If $m = 2n+4$ then we obtain $E_{2n} = -2\sum_{m=1}^{n+1} C_m m^{2n+4}$.
For the the inverse matrix B of the Vandermonde matrix with the

elements $b_{j,k}$ follows that $C_j = 12b_{j,2}, j = \overline{1,n+1}$. Similarly we consider the polynomials $P_j^1(x) = \sum_{k=1}^{n+1} b_{j,k}x^{k-1}$, $P_j^2(x) = P_j^1(x)x = \sum_{k=1}^{n+1} b_{j,k}x^k$ and $R_j(x) = j^2 P_j^1(x)$. From $BV = E$ follows that $P_j^2(k^2) = \sum_{l=1}^{n+1} b_{j,l}k^{2l} = \delta_{k.j}$ and $P_j^1(k^2) = 1/j^2\delta_{k,j}, R_j(k^2) = \delta_{k,j}.k, j = \overline{1,n+1}$. Then the polynomial $R_j(x)$ is the Lagrange polynomial in following form $R_j(x) = \prod_{k=1\neq j}^{n+1} \frac{x-k^2}{j^2-k^2}$ and $C_j = 12b_{j,2} = 12/j^2 R_j'(0)$. Using derivative of logarithm follows $\ln'(R_j(x)) = \frac{R_j'(x)}{R_j(x)}$ and $R_j'(x) = R_j(x)\sum_{k=1\neq j}^{n+1}(x-k^2)^{-1}$. Then $R_j'(0) = -R_j(0)S_j$, where $S_j = \sum_{k=1\neq j}^{n+1} \frac{1}{k^2}, R_j(0) = \prod_{k=1\neq j}^{n+1} \frac{-k^2}{j^2-k^2}$. Therefore

$$C_m = \frac{24((n+1)!)^2(-1)^m}{m^2(n+1-m)!(n+1+m)!}S_m, m = \overline{1,n+1}.$$

For $n = 1$ we have $C_0 = 6, C_1 = -4, C_2 = 1, E_2 = -120$. In this case the circulant finite difference matrix A approximed the derivative $u''''(x_j)$ is in the form

$$A = \frac{1}{h^4}[C_0, C_1, \cdots, C_{n+1}, 0, \cdots, 0, C_{n+1}, C_n, \cdots, C_2, C_1].$$

The eigenvalues of matrix A are $\mu_k = \frac{1}{h^4}\sum_{m=0}^{N-1} C_m \exp(2\pi i m/N)$= $\frac{1}{h^4}(C_0 + 2\sum_{m=1}^{n+1} C_m \cos\frac{2\pi km}{N})$ or

$$\mu_k = -\frac{4}{h^4}\sum_{m=1}^{n+1} C_m \sin^2(\pi km/N), k = \overline{1,N}.$$

For obtaining the unknown coefficients Q_m in the relation
$\mu_k = \frac{4}{h^4}\sum_{m=1}^{n+1} Q_m \sin^{2m}(\pi k/N)$ undepending on n
we use the following expressions for sinus function $(a = \frac{\pi k}{N})$:
$\sin^2(ma) = m^2\sin^2(a) + ... + D_m \sin^{2m}(a)$,
where $D_m = (-4)^{m-1}, m \geq 0$.
Then for $m = n$ by $Q_{n+1} = -D_{n+1}C_{n+1}$ we obtain $Q_{n+1} = \frac{24((n+1)!)^2 4^n}{(n+1)^2(2n+2)!}S_{n+1}$.
We can assume that
$Q_m = \frac{24(m!)^2 4^{m-1}}{m^2(2m)!}\sum_{s=1}^{m-1}\frac{1}{s^2}$ for all $m = \overline{2,n+1}, Q_1 = 0$.
For the strong prove can be used the m-th degree of
Spread polynomials $S_m(s) = \sin^2(ma), s = \sin^2(a), a = \frac{\pi k}{N}$. Then we

need proved that $Q_l = -\sum_{m=l}^{n+1} C_m \alpha_{m,l}$ or
$0.5((l-1)!)^2 \sum_{s=1}^{l-1} s^{-2} =$
$-((n+1)!)^2 \sum_{m=l}^{n+1} \frac{(-1)^{m+l-1}(m+l-1)!}{m(n+1-m)!(n+1+m)!(m-l)!} S_m,\ l = \overline{1,n+1},$
where $\alpha_{m,l} = \frac{m}{l} \frac{(m+l-1)!}{(m-l)!(2l-1)!} (-4)^{l-1}$.
For $l = n+1 (m = n+1), l = n (m = n, m = n+1), l = n-1 (m = n-1, m = n, m = n+1)$ we can easily obtain identity. We have $Q_2 = 4, Q_3 = \frac{8}{3}, Q_4 = \frac{196}{105}, \cdots$.
For FDSES we can replace the discrete eigenvalues $d_k = \mu_k, k = \overline{1,N}$ with the continues values $\lambda_k = (\frac{2\pi k}{L})^4$ in following way:
$d_k = \lambda_k, k = \overline{1,N/2}, d_{k+N/2} = \lambda_{N/2-k-1}, k = \overline{1,N/2}, d_N = 0$ (see Fig. 1.5).

1.6.4 Derivative of third order

The finite difference approximation $O(h^{2n})$ for third order derivative $-u'''(x_j)$ using the uniform grid $x_j = jh$ with $2n+3$ points stencil we can obtain in following form :

$$u'''(x_j) = \frac{1}{h^3} \sum_{k=-n}^{n} c_k u(x_{j-k}) + e_{2n} \frac{h^{2n} u^{(2n+3)}(\xi)}{(2n+3)!}, x_{j-n} < \xi < x_{j+n}.$$

Using properties of anti-symmetry $c_m = -c_{-m}, c_0 = 0$ we get the systems of linear algebraic equations for determine the coefficients $c_m, m = 1\overline{1,n+1}$ in the form
$\sum_{k=1}^{n+1} c_k k^{2m-1} = \sum_{k=1}^{n+1} \frac{c_k}{k} k^{2m} = 3\delta_{m,2}, m = \overline{1,n+1}$, with the the matrix Vandermonde of the $n+1$ order.
Using the coefficients for derivative of fourth order we obtain

$$c_m = \frac{6((n+1)!)^2(-1)^m}{m(n+1-m)!(n+1+m)!} S_m, m = \overline{1,n+1}.$$

For the derivatives $-u'''(x_j), j = \overline{1,N}$ we have following matrix representation:
$A^0 = \frac{1}{h^3}[0, c_1, c_2, ..., c_{n+1}, 0, ..., 0, -c_{n+1}, -c_n, ..., -c_2, -c_1]$.
Then the eigenvalues are: $\mu_k^0 = -\frac{2i}{h^3} \sum_{m=1}^{n} c_m \sin \frac{2\pi mk}{N}$ or
$\mu_k^0 = -\frac{2i}{h^3} \sin \frac{2\pi k}{N} \sum_{m=2}^{n+1} q_m \sin^{2m-2} \frac{\pi k}{N}$,

where the coefficients $q_m = 6\frac{(m!)^2 4^{m-1}}{m(2m)!}\sum_{s=1}^{m-1}\frac{1}{s^2}, q_1 = 0$
are not depending on n. We have $q_2 = 2, q_3 = 2, q_4 = \frac{196}{105}, \cdots$.
For FDSES we can replace the imaginary part of the discrete eigenvalues $d_k = Im(\mu_k^0), k = \overline{1,N}$ with the continues values $d1_k = Im(\lambda_k^0) = \frac{2\pi k}{L}^3$ in following way:
$d_k = d1_k, k = \overline{1, N/2-1}, d_{k+N/2} = -d1_{N/2-k}, k = \overline{1, N/2}, d_{N/2} = 0$
(see Fig. 1.4).

1.6.5 Derivative of k-th order

The finite difference approximation $O(h^{2n+1-k})$ for k-th order derivative $u^k(x_j)$ using the uniform grid $x_j = jh$ with $2n+1$ points stencil we can obtained in following form ($j = \overline{0,N}, 2n+1 \le N, k \le 2n$):

$$u^{(k)}(x_j) = \frac{1}{h^k}\sum_{m=-n}^{n} D_m^{k,n} u(x_{j+m}) + e_{2n}^k \frac{h^{2n+1-k}u^{(2n+1)}(\xi)}{(2n+1)!}, x_{j-n} < \xi < x_{j+n}.$$

For every polynomial $p(x) \in P_{2n}$ can be proved, that (M. Kokainis)

$$\begin{cases} p^{(k)}(x_j) = \frac{1}{h^k}\sum_{m=-n}^{n} D_m^{k,n} p(x_{j+m}), \\ u^{(k)}(x_j) = \frac{1}{h^k}\sum_{m=-n}^{n} D_m^{k,n} u(x_{j+m}) + O(h^{2n+1-k}), k \equiv 1(mod 2), \\ u^{(k)}(x_j) = \frac{1}{h^k}\sum_{m=-n}^{n} D_m^{k,n} u(x_{j+m}) + O(h^{2n+2-k}), k \equiv 0(mod 2), \end{cases} \tag{1.16}$$

where $D_{-m}^{k,n} = (-1)^k D_m^{k,n}, m = \overline{1,n}, D_0^{k,n} = -\sum_{m=1}^{n}(1+(-1)^k)D_m^{k,n}$.

$\diamondsuit$ **The idea of prove**. Denote $q(t) = p(x_j + th)$, then $p(x_{j+m}) = q(m), p^{(k)}(x_j) = \frac{d^k q(0)}{dt^k}\frac{1}{h^k}$. If $R_m(t) \in P_{2n}$ is the Lagrange characteristic polynomials with $R_m(r) = \delta_{r,m}, (r,m) = \overline{-n,n}$, then $q(t) = \sum_{s=-n}^{n} q(s)R_s(t)$. Define $D_m^{k,0} = R_m^{(k)}(0)$, then the first equation of (1.16) is valid by $p = R_m$. From $R_m^{(r)}(0) = \sum_{r=-n}^{n} R_m(r)D_r^{k,n} = \sum_{r=-n}^{n}\delta_{r,m}D_r^{k,n} = D_m^{k,n}$ follows $q^{(k)}(0) = \sum_{m=-n}^{n} q(m)R_m^{(r)}(0) = \sum_{m=-n}^{n} q(m)D_m^{k,n}$ and $p^{(k)}(x_j) = \frac{1}{h^k}\sum_{m=-n}^{n} q(m)D_m^{k,n}$.$\diamondsuit$

For the derivatives $u^{(k)}(x_j), j = \overline{1,N}$ we have following matrix representation:
$A_{k,n} = \frac{1}{h^k}[D_0^{k,n}, D_1^{k,n}, ..., D_n^{k,n}, 0, ..., 0, D_{-n+1}^{k,n}, D_{-n}^{k,n}, ..., D_{-2}^{k,n}, D_{-1}^{k,n}]$.

Then the eigenvalues are: $\mu_s = \frac{2i}{h^k}\sum_{m=1}^n D_m^{k,n}\sin\frac{2\pi sm}{N}, k\equiv 1(mod2)$,
$\mu_s = -\frac{4}{h^k}\sum_{m=1}^n D_m^{k,n}\sin^2\frac{\pi sm}{N}, k\equiv 0(mod2)$
or $\mu_s = -\frac{i}{h^k}\cos(\pi s/N)\sum_{m=1}^n P^{m,k}\sin^{2m-1}\frac{\pi s}{N}, k\equiv 1(mod2)$,
$\mu_s = \frac{1}{h^k}\sin(\pi s/N)\sum_{m=1}^n mP^{m,k}\sin^{2m-1}\frac{\pi s}{N}, k\equiv 0(mod2), s=\overline{1,N}$,
where the coefficients $P^{r,k} = \frac{(-4)^r}{r(2r-1)!}\sum_{m=r}^n D_m^{k,n}\frac{(m+r-1)!}{(m-r)!}, k\equiv 1(mod2)$,
$P^{r,k} = \frac{(-4)^r}{r(2r-1)!}\sum_{m=r}^n D_m^{k,n}\frac{m(m+r-1)!}{(m-r)!}, k\equiv 0(mod2)$,
are not depending on n,
$D_m^{k,n} = R_m^{(k)}(0) = (-1)^{n+m-k}e_{2n-k}^m\frac{k!}{(n-m)!(n+m)!}, e_{2n-k}^m =$
$e_{2n-k}(-n,...,-1,1,...m-1,m+1,...n)$,
$e_j(t_1,t_2,,....,t_K) = \sum_{1\le t_1\le t_2\le ...t_j\le t_K} t_{i_1}t_{i_2}...t_{i_j}$ are the elementary symmetric polynomials.

Example 1.9. For $k=1: e_{2n-1}^m = \prod_{j=-n}^{-1} j\prod_{j=1}^{m-1} j\prod_{j=m+1}^n j = n!(-1)^n n!/m = (-1)^n(n!)^2/m$ and $D_m^{1,n} = (-1)^{m+1}\frac{(n!)^2}{m(n-m)!(n+m)!}$,
$\mu_s = \frac{i}{h}\cos(\pi s/N)\sum_{m=1}^n\frac{4^m m((m-1)!)^2}{(2m)!}\sin^{2m-1}\frac{\pi s}{N}$.
For $k=2: e_{2n-2}^m = \sum_{j=-n,j\neq(m,0)}^n\frac{e_{2n-1}^m}{j} = e_{2n-1}^m\sum_{j=-n,j\neq(m;0)}^n j^{-1} = e_{2n-1}^m(\sum_{j=-n}^{-1}\frac{1}{j}+\sum_{j=1}^n\frac{1}{j}-\frac{1}{m}) = -\frac{1}{m}e_{2n-1}^m$
and $D_m^{2,n} = (-1)^{m+1}\frac{2(n!)^2}{m^2(n-m)!(n+m)!} = \frac{2}{m}D_m^{1,n}$,
$\mu_s = -\frac{2}{h^2}\sum_{m=1}^n\frac{4^m((m-1)!)^2}{(2m)!}\sin^{2m}\frac{\pi s}{N}$.
Similarly
$e_{2n-3}^m = e_{2n-1}^m\sum_{i\neq(0,n}\sum_{j>i,j\neq m}\frac{1}{ij} = e_{2n-1}^m(\frac{1}{m}-\sigma_{m-1,2})$,
$D_m^{3,n} = 6(\frac{1}{m^2}-\sigma_{m-1,2})D_m^{1,n}$,
$\mu_s = -\frac{6i}{h^3}\cos(\pi s/N)\sum_{m=2}^n\frac{4^m m((m-1)!)^2}{(2m)!}\sigma_{m-1,2}\sin^{2m-1}\frac{\pi s}{N}$,
$e_{2n-4}^m = -e_{2n-1}^m\frac{1}{m}(\frac{1}{m^2}-\sigma_{n,2}), D_m^{4,n} = \frac{24}{m}(\frac{1}{m^2}-\sigma_{m-1,2})D_m^{1,n}$,
$\mu_s = \frac{24}{h^4}\sum_{m=2}^n\frac{4^m((m-1)!)^2}{(2m)!}\sin^{2m}\frac{\pi s}{N}$, where $\sigma_{r,l} = \sum_{j=1}^r j^{-l}$.

1.7 The self-conjugate finite difference operator with third kind BC

For finite value $\sigma_1 > 0, \sigma_2 > 0$ we consider following bondary value problem for ODEs:

$$\begin{cases} -u''(x) = f(x), x \in (0,L), \\ u'(0) - \sigma_1 u(0) = 0, \\ u'(L) + \sigma_2 u(L) = 0. \end{cases} \tag{1.17}$$

The corresponding FDS for second order of approximation is in following form:

$$\begin{cases} -2(y_1 - y_0)/h^2 + 2\sigma_1 y_0/h = f(0), \\ -(y_{j+1} - 2y_j + y_{j-1})/h^2 = f(x_j), \quad j = \overline{1,N-1}, \\ -2(y_{N-1} - y_N)/h^2 + 2\sigma_2 y_N/h = f(L), \end{cases} \tag{1.18}$$

where $h = \frac{L}{N}, x_j = jh$.
We obtain the FDS $Ay = f$ with 3-diagonal matrix A of $M = N+1$ order in the following form

$$A = \frac{1}{h^2} \begin{pmatrix} 2+2h\sigma_1 & -2 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ .. & .. & .. & \dots & .. & .. & \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -2 & 2+2\sigma_2 \end{pmatrix}$$

where y, f is column-vector of $N+1$ order with elements $y_j, f(x_j), j = \overline{0,N}$.
Using two vectors y^1, y^2 scalar product
$[y^1, y^2] = h(\sum_{j=1}^{N-1} y_j^1 y_j^2 + 0.5(y_0^1 y_0^2 + y_N^1 y_N^2))$ can prove, that the operator A is symmetrical and $[Ay, y] \geq 0$ [3].

1.7.1 The discrete spectral problem for difference operator

The corresponding discrete spectral problem $Aw^n = \mu_n w^n, n = \overline{1,N+1}$ have following solution [3]

$$\begin{cases} w^n = C_n^{-1}(\frac{1}{\sqrt{2}}w_0^n, w_1^n, ..., w_{N-1}^n, \frac{1}{\sqrt{2}}w_N^n)^T, \\ \mu_n = \frac{4}{h^2}\sin^2(p_n h/2), \end{cases} \tag{1.19}$$

where
$w_j^n = \frac{\sin(p_n h)}{h}\cos(p_n x_j) + \sigma_1 \sin(p_n x_j), j = \overline{0,N}$ are the components of column-vector w^n, p_n are the positive roots of the following transcendental equation

$$\cot(p_n L) = \frac{\sin^2(p_n h) - h^2\sigma_1\sigma_2}{h(\sigma_1+\sigma_2)\sin(p_n h)}, n = \overline{1,N+1} \tag{1.20}$$

The solutions of the finite difference equations

$$-(w_{j+1} - 2w_j + w_{j-1})/h^2 = \mu w_j, \quad j = \overline{1,N-1}$$

can obtain in the classical form
$1 - \frac{\mu h^2}{2} = \cos(ph)$ or $\mu = \frac{4}{h^2}\sin^2\frac{ph}{2}$.
Then $w_j = C_1\sin(px_j) + C_2\cos(px_j)$.
The constants C_1, C_2 are determined from the difference equations by $j = 0, j = N$ and for the values p_n we obtain of the previous transcendental equation.

The constants $C_n^2 = [w^n, w^n]$ can be obtained in following form

$$\begin{cases} C_n^2 = h(A_1^2 S_1 + 2A_1A_2S_3 + A_2^2 S_2 + 0.5(A_1^2(1+\cos^2(p_n L)) + \\ A_2^2\sin^2(p_n L) + A_1A_2\sin(2p_n L)), \end{cases} \tag{1.21}$$

where $A_1 = \frac{\sin(p_n h)}{h}, A_2 = \sigma_1$
$S_1 = \sum_{j=1}^{N-1}\cos^2(p_n x_j) = 0.5(N - 1 + \frac{\cos(p_n L)\sin(p_n(L-h))}{\sin(p_n h)})$,
$S_2 = \sum_{j=1}^{N-1}\sin^2(p_n x_j) = 0.5(N - 1 - \frac{\cos(p_n L)\sin(p_n(L-h))}{\sin(p_n h)})$,
$S_3 = 0.5\sum_{j=1}^{N-1}\sin(2p_n x_j) = 0.5\frac{\sin(p_n L)\sin(p_n(L-h))}{\sin(p_n h)})$.
Then we have the orthonormed eigenvectors w^n, w^m with the scalar product $[w^n, w^m] = \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker symbol.

1.7.2 The special solution for the discrete spectral problem

The experimental calculations with MATLAB shows, that the first roots $p_n, n = \overline{N-1}$ of equations (1.20) are closed in the interval

$((n-1)\pi, n\pi)$, but the two roots p_N, p_{N+1} for finite values σ_1, σ_2 can not obtained from (1.20) (the spectral problem also is solved with MATLAB using the operator **"eig "**).
Depending on the parameter

$$Q = \frac{L\sigma_1\sigma_2}{\sigma_1+\sigma_2}$$

we can be one $(Q < 1)$ or two $(Q \geq 1)$ roots from following new transcendental equations obtained

$$\coth(p_nL) = \frac{\sinh^2(p_nh)+h^2\sigma_1\sigma_2}{h(\sigma_1+\sigma_2)\sinh(p_nh)}, n = \overline{N, N+1} \tag{1.22}$$

and the eigenvalues and eigenvectors are in the following form

$$\begin{cases} w^n = C_n^{-1}(\frac{1}{\sqrt{2}}w_0^n, w_1^n, ..., w_{N-1}^n, \frac{1}{\sqrt{2}}w_N^n)^T, \\ \mu_n = \frac{4}{h^2}\cosh^2(p_nh/2), \end{cases} \tag{1.23}$$

where $w_j^n = (-1)^j(\frac{\sinh(p_nh)}{h}\cosh(p_nx_j) - \sigma_1\sinh(p_nx_j)), j = \overline{0,N}$, $n = N$ or $n = N+1$.
This special solutions of the finite difference equations

$$-(w_{j+1} - 2w_j + w_{j-1})/h^2 = \mu w_j, \quad j = \overline{1, N-1}$$

from $\mu \geq \frac{4}{h^2}$ can obtain in the following form
$-1 + \frac{\mu h^2}{2} = \cosh(ph)$ or $\mu = \frac{4}{h^2}\cosh^2\frac{ph}{2}$.
Then $w_j = (-1)^j[C_1\sinh(px_j) + C_2\cosh(px_j)]$.
The constants C_1, C_2 are determined from the difference equations by $j = 0, j = N$ and the values p_N, p_{N+1} are obtained from the previous transcendental equation.

The constants C_n^2 in this case are

$$\begin{cases} C_n^2 = h(A_1^2S_1 + 2A_1A_2S_3 + A_2^2S_2 + 0.5(A_1^2(1+\cosh^2(p_nL)) + \\ A_2^2\sinh^2(p_nL) + A_1A_2\sinh(2p_nL)), \end{cases} \tag{1.24}$$

where $A_1 = \frac{\sinh(p_nh)}{h}, A_2 = \sigma_1$
$S_1 = \sum_{j=1}^{N-1}\cosh^2(p_nx_j) = 0.5(N - 1 + \frac{\cosh(p_nL)\sinh(p_n(L-h))}{\sinh(p_nh)})$,
$S_2 = \sum_{j=1}^{N-1}\sinh^2(p_nx_j) = 0.5(-N + 1 + \frac{\cosh(p_nL)\sinh(p_n(L-h))}{\sinh(p_nh)})$,

$S_3 = 0.5\sum_{j=1}^{N-1}\sinh(2p_nx_j) = 0.5\frac{\sinh(p_nL)\sinh(p_n(L-h))}{\sinh(p_nh)})$.
Then we have the orthonormed eigenvectors w^n, w^m with the scalar product $[w^n, w^m] = \delta_{n,m}$ for all $n, m = \overline{1, N+1}$.
The expression $Q = 1$ follows from (1.22) by the limit case when $p_n \to 0$.
If $Q > 1$ then we have two positive eigenvalues of (1.22) $(n = N; N+1)$,
but for $Q = 1$ the first eigenvalue $(n = N)$ is $\mu_N = 4/h^2, (p_N = 0)$ and the corresponding components w_j^N of the orthonormed eigenvector w^N are obtained from the limit case $(p_N \to 0)$ for expression $\frac{w_j^N}{C_N}$ (1.23) in the following form

$$w_j^N = (-1)^j(1-\sigma_1x_j)\sqrt{6h/(6L+2\sigma_1^2L^3+L\sigma_1^2h^2-6\sigma_1L^2)}, j = \overline{0,N}. \tag{1.25}$$

1.7.3 The discrete spectral problem for mixed BC

If $\sigma_1 = \infty$ then we have the parameter $Q = L\sigma_2$ and from (1.20, 1.22) follows the transcendental equations

$$\cot(p_nL) = -\frac{\sigma_2 h}{\sin(p_nh)}, n \geq 1 (Q < 1), \coth(p_NL) = \frac{\sigma_2 h}{\sinh(p_Nh)} (Q \geq 1).$$

Then depending of the parameter Q we have the following solutions of the spectral problem:
1) for $Q < 1$

$$\begin{cases} w_j^n = C_n^{-1}\sin(p_nx_j),\ j = \overline{1,N}, \\ \mu_n = \frac{4}{h^2}\sin^2(p_nh/2), \\ C_n^2 = 0.5(L - 0.5h\sin(2p_nL)\cot(p_nh)), n = \overline{1,N} (Q < 1); \\ n = \overline{1, N-1} (Q \geq 1), \end{cases} \tag{1.26}$$

2) for $Q > 1$

$$\begin{cases} w_j^N = C_N^{-1}(-1)^j\sinh(p_nx_j),\ j = \overline{1,N}, \\ \mu_N = \frac{4}{h^2}\cosh^2(p_Nh/2), \\ C_N^2 = 0.5(L - 0.5h\sinh(2p_NL)\coth(p_Nh)), \end{cases} \tag{1.27}$$

3) for $Q=1$ and when $p_N \to 0$ from $\frac{w_j^N}{C_N}$ we have the following components of the orthonormed eigenvector w^N

$$w_j^N = 2(-1)^j x_j \sqrt{3h/(2h^2L+4L^3)},\ j=\overline{1,N}. \tag{1.28}$$

This expression follows also from(1.25) when $\sigma_1 \to \infty$.

If $\sigma_2 = \infty$ then from value of the parameter $Q = L\sigma_1$ we have the following transcendental equations:

$$\cot(p_n L) = -\frac{\sigma_1 h}{\sin(p_n h)}, n \geq 1 (Q<1), \coth(p_N L) = \frac{\sigma_1 h}{\sinh(p_N h)} (Q \geq 1).$$

Then depending of the parameter Q we have the following eigenvectors (the corresponding norms C_n and eigenvalues μ_n are remained):

$$\begin{cases} w_j^n = C_n^{-1} \sin(p_n(L-x_j)),\ j=\overline{0,N-1}, (Q<1), \\ w_j^N = C_N^{-1}(-1)^j \sinh(p_N(L-x_j)),\ j=\overline{0,N-1},\ (Q>1), \\ w_j^N = 2(-1)^j (l-x_j)\sqrt{3h/(2h^2L+4L^3)},\ j=\overline{0,N-1}, (Q=1). \end{cases} \tag{1.29}$$

Therefore the matrix A can be represented in form $A = WDW^T$, where the column of the matrix W and the diagonal matrix D contains M orthonormed eigenvectors w^n and eigenvalues $\mu_n, n=\overline{1,M}$ correspondly, where $M = N+1$ for finite value of σ_1 and σ_2, $M=N$ for infinite σ_1 or σ_2. From $W^TW = E$ follows that $W^{-1} = W^T$. From the solution of (1.18) follows $Ay = WDW^Ty = f, DW^Ty = W^Tf, W^Ty = D^{-1}W^Tf, y = WD^{-1}W^Tf$,
where $D^{-1} = diag(\mu_k^{-1})$ is the diagonal matrix with the elements $\mu_k^{-1}, k=\overline{1,M}$.

Note 1.1. To follow the scalar product of vectors in this solution
1) for $M = N+1$ the first f_1 and last f_M components of the vector f are **divided** with $\sqrt{2}$, but the components y_1, y_M of the solution vector y need to **multiply** with $\sqrt{2}$;
2) for $M=N$ the above-mentioned can be apply to first ($\sigma_2 = \infty$) or last ($\sigma_1 = \infty$) component for vectors y, f.

If $\sigma_1 = \infty$ **and** $\sigma_2 = \infty$ then the matrix A represented in the form $A = WDW$, $(W = W^T = W^{-1})$ is the symmetrical orthogonal matrix of $N-1$ order with elements $p_{i,j} = \sqrt{\frac{2}{N}} \sin\frac{\pi i j}{N}$, $i, j = \overline{1,N-1}$

and $p_n = n\pi/L, \mu_n = \frac{4}{h^2}\sin^2\frac{n\pi}{2N}, n = \overline{1, N-1}$.
In this case $M = N-1$ and the orthonormed eigenvectors $w_n(x_j) = \sqrt{2/L}\sin(n\pi x_j/L)$ are equal for discrete and continuous problems, but eigenvalues are different ($\lambda_n^2 = (\frac{n\pi}{L})^2$ for the differential operator).

1.7.4 The spectral problem for differential equation and FDSES

The solution of the spectral problem for differential equations

$$w''(x) + \lambda^2 w(x) = 0, x \in (0,l), w'(0) - \sigma_1 w(0) = 0, w'(L) + \sigma_2 w(L) = 0,$$

is in following form $w_n(x) = C_n^{-1}(\lambda_n\cos(\lambda_n x) + \sigma_1\sin(\lambda_n x))$, $C_n^2 = 0.5(L(\lambda_n^2+\sigma_1^2) + \frac{\sigma_2(\lambda_n^2+\sigma_1^2)}{\lambda_n^2+\sigma_2^2} + \sigma_1)$,
where $(w_n, w_m) = \int_0^L w_n(x)w_m(x)dx = \delta_{n,m}$ and λ_n are positive roots of the following transcendental equation:

$$\cot(\lambda_n L) = \frac{\lambda_n}{\sigma_1+\sigma_2} - \frac{\sigma_1\sigma_2}{\lambda_n(\sigma_1+\sigma_2)}, n = \overline{1, N+1}. \tag{1.30}$$

For the scalar product $[w^n, w^m]$ the integral (w_n, w_m) is approximated with trapezoidal formula and in the limit case if $h \to 0$ then from (1.22,1.30) follows that $\mu_n \to \lambda_n^2$.
If $\sigma_1 = \infty$ then $w_n(x) = C_n^{-1}\sin(\lambda_n x)$, $C_n^2 = 0.5L(L + \frac{\sigma_2}{L(\sigma_2^2+\lambda_n^2)})$,
where λ_n are positive roots of the following equation

$$\cot(\lambda_n L) = -\frac{\sigma_2}{\lambda_n}.$$

If $\sigma_2 = \infty$ then $w_n(x) = C_n^{-1}\sin(\lambda_n(L-x))$, $C_n^2 = 0.5L(L + \frac{\sigma_1\sin^2(\lambda_n L)}{\lambda_n^2})$,
where λ_n are positive roots of the following equation

$$\cot(\lambda_n L) = -\frac{\sigma_1}{\lambda_n}.$$

We can used also the **Fourier method** for solving (1.17) in the form

$$u(x) = \sum_{k=1}^{\infty}\alpha_k w_k(x), f(x) = \sum_{k=1}^{\infty}\beta_k w_k(x)$$

where $w_k(x)$ are the orthonormed eigenvectors, $\alpha_k = \frac{\beta_k}{d_k}, \beta_k = (f, w_k)$, $d_k = \lambda_k^2$.
For the discrete solution (1.18) we get $y = \sum_{k=1}^{M} \alpha_k w^k, f = \sum_{k=1}^{M} \beta_k w^k$ and $\beta_k = [f, w^k], \alpha_k = \frac{\beta_k}{d_k}$, where $d_k = \mu_k$. For this solution the note 1.1. is valid.
For the **difference scheme with exact spectrum** (FDSES) the matrix A is represented in the form form $A = WDW^T$ and the diagonal matrix D contain the first $N+1$ eigenvalues $d_k = \lambda_k^2$, $k = \overline{1, N+1}$ from the differential operator $-(\frac{\partial^2}{\partial x^2})$ correspondly.
If $d_k = \mu_k$, then we have the method of finite difference approximation with tridiagonal matrix A.
If $\sigma_1 = \sigma_2 = \infty$ then we have the boundary conditions of first kind and the matrix A is represented in the form $A = WDW$, where $W = W^{-1}$ is the symmetrical orthogonal matrix with elements $p_{i,j} = \sqrt{\frac{2}{N}} \sin \frac{\pi i j}{N}$, $i, j = \overline{1, N-1}$, $d_k = (k\pi/L)^2, \mu_k = \frac{4}{h^2} \sin^2 \frac{k\pi}{2N}$, $k = \overline{1, N-1}$, $(AW = WD)$.
The FDSES method is more stable as the method of finite difference by approximation with central difference, because the eigenvalues are larger $d_k > \mu_k$.

1.7.5 The examples of boundary value problems for ODEs

Example 1. The boundary value problem (1.17) is solved by $f(x) = 12x^2 C_0 + \sigma_1 \sin(x), C_0 = \frac{\sigma_1 \cos(L) + \sigma_1 \sigma_2 \sin L + \sigma_2}{4L^3 + \sigma_2 L^4}$.
The exact solution of the differential problem is
$u(x) = -x^4 C_0 + 1 + \sigma_1 \sin(x)$.
The exact solution of the corresponding FDS (1.18) is
$y_j = C_1(1 + \sigma_1 x_j) + (x_j^2 h^2 - x_j^4) C_0 + \sigma_1 h (0.25 h \sin(x_j)/\sin^2(0.5h) - 0.5 x_j \cot(0.5h))$, where
$C_1 = \frac{C_0((1+\sigma_2 h)(L^4 - L^2 h^2) + (L-h)^2 h^2 - (L-h)^4 + 6h^2 L^2) + C_2}{(1+\sigma_2 h)(1+\sigma_1 L) - (1+\sigma_1 (L-h))}$,
$C_2 = 0.5 h^2 \sigma_1 \sin(L) + 0.25 \frac{\sigma_1 h^2}{\sin^2(0.5h)} (\sin(L-h) - (1+\sigma_2 h)\sin(L))$
$+ 0.5 \sigma_1 h^2 \cot(0.5h)(1 + L\sigma_2)$.
Using the Fourier method we have following coefficients:
$\alpha_k = 12 C_0 (\lambda_k^{-1} (\frac{\sin(\lambda_k L)}{\lambda_k} (L^2 - \frac{2}{\lambda_k^2}) + \frac{2L}{\lambda_k^2} \cos(\lambda_k L)) +$

$\frac{\sigma_1}{\lambda_k^2}(\frac{\cos(\lambda_k L)}{\lambda_k}(\frac{2}{\lambda_k^2}-L^2)+$
$\frac{2L}{\lambda_k^2}\sin(\lambda_k L)-\frac{2}{\lambda_k^3}))+0.5\sigma_1(\frac{\sigma_1}{\lambda_k^2}(\frac{\sin((\lambda_k-1)L)}{\lambda_k-1}-\frac{\sin((\lambda_k+1)L)}{\lambda_k+1})+$
$\lambda_k^{-1}(\frac{\cos((\lambda_k-1)L)-1}{\lambda_k-1}-\frac{\cos((\lambda_k+1)L)-1}{\lambda_k+1}))/$
$(0.5(L(\lambda_k^2+\sigma_1^2)+\sigma_2\frac{\lambda_k^2+\sigma_1^2}{\lambda_k^2+\sigma_2^2}+\sigma_1))$.

The solution of the problem (1.18) are obtained with MATLAB, using the spectral problem (1.20)-(1.28) in the form $y=A^{-1}F$ or $y=PD^{-1}P^TF$, where D^{-1} is the diagonal matrix with elements $\mu_k^{-1}, k=\overline{1,N+1}$. In Tab. 1.3 we can see the last eigenvalues $p_N, p_{N+1}, \mu_N, \mu_{N+1}$ obtained from (1.20), (1.22) and the maximum norm of the difference from the solution of finite difference scheme $(\delta(FDS))$ and FDSES $(\delta(FDSES))$ between the exact solution. The calculated eigenvalues are coincidence with the eigenvalues, obtained with the MATLAB operator "eig" for matrix A.
In the Figs. 1.6, 1.7 we can see the solution of the test problem by $\sigma_1=2, \sigma_2=0.1, N=15$ and $L=11(Q>1$, the Fig. 1.9), $L=10(Q<1)$, the Fig. 1.10).

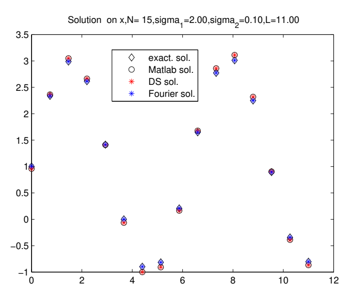


Fig. 1.6 Solution for $\sigma_1=2, \sigma_2=0.1, l=11, Q>1$

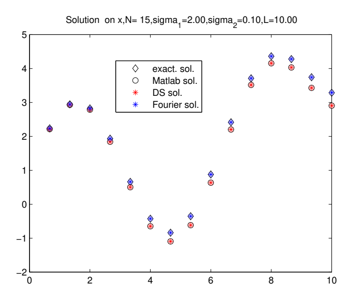


Fig. 1.7 Solution for $\sigma_1=2, \sigma_2=0.1, l=10, Q<1$

In following Figs. 1.8, 1.9 we can see the eigenvalues by $N=10, \sigma_1=1, \sigma_2=1;2, L=1;5$, obtained with MATLAB m.file "ipas3v" and following m.file **EXPP:**

Table 1.2 The values of $\sigma_1, \sigma_2, N, L, \delta(FDS), \delta(FDSES), p_N, \mu_N, p_{N+1}, \mu_{N+1}$.

σ_1	σ_2	N	L	$\delta(FDS)$	$\delta(FDSES)$	p_N	μ_N	p_{N+1}	μ_{N+1}
2	0.1	10	11.0	0.4795	0.1788	0.0370	3.3072	1.3906	5.6473
2	0.1	20	11.0	0.1582	0.0525	0.0370	13.2245	1.7297	16.4404
2	0.1	15	11.0	0.2484	0.0856	0.0370	7.4394	1.6038	10.3208
1	2.0	10	2.0	0.0099	0.00540	0.840	100.70	1.950	103.87
1	2.0	15	2.0	0.00516	0.00270	0.840	225.71	1.950	228.95
1	2.0	30	2.0	0.00175	0.00089	0.840	900.70	1.950	904.00
1	2.0	60	2.0	0.00062	0.00031	0.840	3600.7	1.950	3604.0
0	1.0	10	2.0	0.0053	0.0036	14.480	98.500	1.030	101.06
0	1.0	20	2.0	0.0018	0.0013	30.190	398.49	1.030	401.06

```
function EXPP(N)
N1=(N+1);sig1=1;sig2=2;L=1.5;Q=L*sig1*sig2/(sig1+sig2);
[lk0,lk,W]=ipas3v(N,sig1,sig2,L);lk(N),lk(N1)
figure,plot((1:N1)',lk,'*',(1:N1)',lk0.^2,'O')
title(sprintf('Eig-val.sigma_1=%3.2f,sigma_2=. . .
%3.2f,Q=%6.4f',sig1,sig2,Q))
xlabel('\itn'), ylabel('\it eig-val.')
legend('discrete', 'differential')
```

The last eigenvalues μ_N, μ_{N+1} are $(N = 10, \sigma_1 = 1)$: 394.64; 402.38 $(\sigma_2 = 1, L = 1, Q = 0.5)$, 16.94; 19.31 $(\sigma_2 = 2, L = 5, Q = 3.333)$, 177.78; 181.81 $(\sigma_2 = 1, L = 1.5, Q = 1)$,

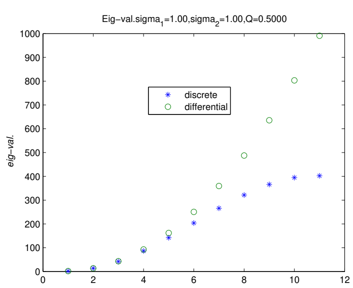


Fig. 1.8 Eigenvalues for $\sigma_1 = \sigma_2 = L = 1, Q = 0.5$

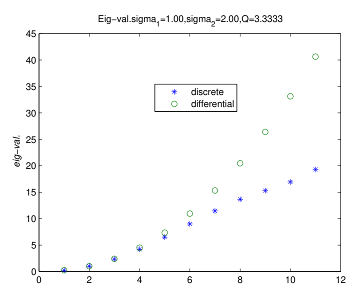


Fig. 1.9 Eigenvalues for $\sigma_1 = 1, \sigma_2 = 2, L = 5, Q = 3.333$

Example 2. For $\sigma_1 = \infty$ we consider following problem:

$$\begin{cases} -u''(x) = f(x), x \in (0, L), \\ u(0) = 0, u'(L) + \sigma_2 u(L) = 0, \end{cases} \tag{1.31}$$

where $f(x) = 12x^2C_0, C_0 = \frac{\sigma_2 L+1}{4L^3+\sigma_2 L^4}$.
The exact solution of the differential problem is $u(x) = -x^4C_0 + x$.
The corresponding FDS for second order of approximation is in following form:

$$\begin{cases} y_1 = 0, -(y_{j+1} - 2y_j + y_{j-1})/h^2 = f(x_j), \quad j = \overline{1, N-1}, \\ -2(y_{N-1} - y_N)/h^2 + 2\sigma_2 y_N/h = f(l), \end{cases} \tag{1.32}$$

The exact solutions of the discrete problem is
$y_j = C_0(C_1 x_j + x_j^2 h^2 - x_j^4)$, where
$C_1 = \frac{(L^4-L^2h^2)(1+\sigma_2 h)+(L-h)^2h^2-(L-h)^4+6h^2L^2}{h(1+\sigma_2 L)}$. We have following Fourier coefficients:
$\alpha_k = 12C_0(\frac{\sin(\lambda_k L)}{\lambda_k^4}(L^2\sigma_2 - \frac{2\sigma_2}{\lambda_k^2} + 2L) - \frac{2}{\lambda_k^5})/(0.5(L + \frac{\sigma_2}{\sigma_2^2+\lambda_k^2}))$.
The solution of this problem is obtained with MATLAB (see the Tab. 1.3), using the spectral problem (1.26)-(1.28) in the form $y = PD^{-1}P^TF$.

Table 1.3 The values of $\sigma_2, N, L, \delta(FDS), \delta(FDSES), p_N, \mu_N$ for $\sigma_1 = \infty$

σ_2	N	L	$\delta(FDS)$	$\delta(FDSES)$	p_N	μ_N
5	10	0.2	0.00204	0.00150	157.08	10000.
5	20	0.20	0.00069	0.00051	314.16	40000.
5	10	0.1	0.00118	0.00087	303.00	39865.
5	10	0.3	0.00269	0.00180	43.00	4462.7
1	10	1.5	0.01346	0.00890	0.850	178.51
1	10	1.0	0.01021	000730	31.420	400.0
1	10	0.9	0.00945	0.00690	34.305	493.47

1.8 MATLAB for solving spectral problems with BCs of 3. kind

We consider special m.file **"ipas3v"** with
"[lk0,lk,W]=ipas3v(N,sig1,sig2,L)" for solving the discrete spectral problem and diferential spectral problem for BC of the third kind, where $W = P$ is the matrix with orthonomed eigenvetors,
$lk0, lk$ are the column-vectors with eigenvalues corresponding

of differential and discrete operators
$\mu_j = lk(j), \lambda_j = lk0(j), j = \overline{1, N+1}$ $(\sigma_1 = sig1, \sigma_2 = sig2)$.

```
%lk0^2--differential, lk--discr. eig-val., W- matrix
% u'(0)-sig1 u(0)=0,u'(L)+sig2 u(L)=0,-u''(x)=$\lambda^2 u(x)
function [lk0,lk,W]=ipas3v(N,sig1,sig2,L)
N1=N+1;N2=N-1;x=linspace(0,L,N1)';e1=1.e-4;
lk0=zeros(N1,1);lk1=zeros(N1,1);h=L/N;
%figure,fplot(@msak1,[0,40,0,5],[],[],'-',sig1,sig2,h,L)
% plot for discrete eig-val. estimation
%figure,fplot(@msak,[0,40,0,10],[],[],'-',sig1,sig2,L)
% plot for diferential eig-val. estimation
for j=1:N1
lk0(j)=fzero(@msak,[pi/L*(j-1)+e1,pi/L*j-e1],[],sig1,sig2,L);
end
for j=1:N2
lk1(j)=fzero(@msak1,[pi/L*(j-1)+e1,pi/L*j-e1],[],sig1,sig2,h,L)
end \% the val. p_j
lk=4/(h^2)*(sin(0.5*lk1(1:N2)*h)).^2;
CK1=sqrt(2*h./(L*((sin(lk1(1:N2)*h)/h).^2+sig1^2). . .
+0.25*sin(2*lk1(1:N2)*L).*sin(2*lk1(1:N2)*h)/h-...
0.5*sig1^2*h*sin(2*lk1(1:N2)*L).*cot(lk1(1:N2)*h)+. . .
2*sig1*(sin(lk1(1:N2)*L)).^2.*cos(lk1(1:N2)*h)));
W=(0.5*(sin(lk1(1:N2)*(h+x'))-sin(lk1(1:N2)*(-h+x')))/h. . .
 +sig1*sin(lk1(1:N2)*x'))';
sgk=L*(sig1*sig2)/(sig1+sig2 ) % criterium
if sgk <  1,
lk1(N)=fzero(@msak1,[pi/L*(N-1)+e1,pi/L*N-e1],[],. . .
sig1,sig2,h,L);
lk(N)=4/(h^2)*(sin(0.5*lk1(N)*h)).^2; k1=lk1(N);
CK1(N)=sqrt(2*h./(L*((sin(k1*h)/h)$.^2+sig1^2). . .
+0.25*sin(2*k1*L).*sin(2*k1*h)/h-0.5...
*$sig1^2$*h*sin(2*k1*L).*cot(k1*h)+. . .
2*sig1*(sin(k1*L)).^2.*cos(k1*h)));
W(:,N)=(0.5*(sin(k1*(h+x))-sin(k1*(-h+x)))/h +sig1*sin(k1*x));
% figure,fplot(@f1,[0,30,-3,10],[],[],'-',sig1,sig2,h,L)
% plot for  special eig-val. by Q<1
%title(sprintf('Spec. eig-val.Q<1$,Q=\%4.1f',sgk))
lk1(N1)=fzero(@f1,[0.011,7],[],sig1,sig2,h,L);
lk(N1)=2/h^2*(cosh(lk1(N1)*h)+1);
k1=lk1(N1);
CK1(N1)=sqrt(2*h/(L*((sinh(k1*h)/h)^2-sig1^2). . .
+0.25*sinh(2*k1*L)*sinh(2*k1*h)/h+0.5...
*sig1^2*h*sinh(2*k1*L)*cosh(k1*h)/sinh(k1*h). . .
-2*sig1*(sinh(k1*L)).^2.*cosh(k1*h)));
W(:,N1)=(-1).^{[1:N1]'}.*(0.5*(sinh(k1*(h+x))-. . .
sinh(k1*(-h+x)))/h-sig1*sinh(k1*x));
end
if sgk == 1
lk1(N)=N*pi/L;lk(N)=4/(h^2)*(sin(0.5*lk1(N)*h)).^2;
CK1(N)=sqrt(h/(-sig1*L^2 +L +sig1^2*L^3/3 +L*sig1^2*h^2/6));
```

```
W(:,N)=$(-1).^{[1:N1]'}$.*(1-sig1*x);
%figure,fplot(@f1,[0,30,-1,10],[],[],'-',sig1,sig2,h,L)
% plot for special eig-val. by Q=1
%title(sprintf('Spec eig-val. Q=1,Q=%4.1f',sgk))
lk1(N1)=fzero(@f1,[0.001,30],[],sig1,sig2,h,L);
lk(N1)= 2/h^2*(cosh(lk1(N1)*h)+1);
k1=lk1(N1);
CK1(N1)=sqrt(2*h/(L*((sinh(k1*h)/h)^2-sig1^2). . .
+0.25*sinh(2*k1*L)*sinh(2*k1*h)/h+...
0.5*$sig1^2*h*sinh(2*k1*L)*cosh(k1*h)/sinh(k1*h)-. . .
2*sig1*(sinh(k1*L)).^2.*cosh(k1*h)));
W(:,N1)=(-1).^{[1:N1]'}$.*(0.5*(sinh(k1*(h+x)). . .
-sinh(k1*(-h+x)))/h-sig1*sinh(k1*x));
end
if sgk > 1
%figure,fplot(@f1,[0,30,-1,10],[],[],'-',sig1,sig2,h,L)
% plot for special eig-val. by Q>1
%title(sprintf('Spec pavrt.Q>1,Q=%4.1f',sgk))
lk1(N)=fzero(@f1,[0.0001,0.1],[],sig1,sig2,h,L);
lk(N)=2/h^2*(cosh(lk1(N)*h)+1);
k1=lk1(N);
CK1(N)=sqrt(2*h/(L*((sinh(k1*h)/h)^2-sig1^2). . .
+0.25*sinh(2*k1*L)*sinh(2*k1*h)/h+...
0.5*sig1^2*h*sinh(2*k1*L)*cosh(k1*h)/sinh(k1*h). . .
-2*sig1*(sinh(k1*L)).^2.*cosh(k1*h)));
W(:,N)=(-1).^{[1:N1]'}.*(0.5*(sinh(k1*(h+x)). . .
-sinh(k1*(-h+x)))/h-sig1*sinh(k1*x));
lk1(N1)=fzero(@f1,[0.2,3],[],sig1,sig2,h,L);
lk(N1)=2/h^2*(cosh(lk1(N1)*h)+1);
k1=lk1(N1);
CK1(N1)=sqrt(2*h/(L*((sinh(k1*h)/h)^2-sig1^2). . .
+0.25*sinh(2*k1*L)*sinh(2*k1*h)/h+...
0.5*sig1^2*h*sinh(2*k1*L)*cosh(k1*h)/sinh(k1*h). . .
-2*sig1*(sinh(k1*L)).^2.*cosh(k1*h)));
W(:,N1)=(-1).^{[1:N1]'}.*(0.5*(sinh(k1*(h+x)). . .
-sinh(k1*(-h+x)))/h-sig1*sinh(k1*x));
end
for j=1:N1
W(:,j)=W(:,j).*CK1(j,1);end;
W(1,:)= W(1,:)/sqrt(2);
W(N1,:)=W(N1,:)/sqrt(2); % orthnorm. eig-vect.
%A2=W*diag(lk)*W',by solving  A2*u=f,
%vect. f,u first and last elem. need coresp.. . .
 divide and multiply with sqrt(2)
function y=f1(x,sig1,sig2,h,L)
y=tanh(x*L)*(sig1*sig2*h^2+(sinh(x*h))^2)/h/sinh(x*h). . .
-(sig2+sig1);
function y=msak(x,sig1,sig2,L)
y= cot(x*L)-(x- sig1*sig2/x)/(sig1+sig2);
function y=msak1(x,sig1,sig2,h,L)
y= cot(x*L)-(sin(x*h)/h -h*sig1*sig2/sin(x*h))/(sig2+sig1);
```

1.9 The non-conjugate finite difference operator with the periodical BC

We consider following boundary value problem

$$\tilde{L}(u) = u''(x) + au'(x) = -f(x), u(0) = u(L), u'(0) = u'(L) \quad (1.33)$$

where a is constant parameter. This problem has unique solutions by $\int_0^L f(x)dx = 0, u(x_0) = u_0$, where $x_0 \in [0,L], u_0$ are fixed constant.
The analytical solution of this problem is
$u(x) = -\frac{1}{a}\ \big(\ \frac{\exp(-ax)-1}{\exp(aL)-1}\int_0^L \exp(at)f(t)dt +$
$\int_0^x (\exp(-a(x-t)) - 1)f(t)dt\ \big),$
where $u(0) = 0, u'(0) = \int_0^L \exp(at)f(t)dt/(\exp(aL) - 1)$.

In this case we have corresponding A. Iljin FDS $(M = N)$

$$\Lambda(y) = -(\gamma y_{x\bar{x}} + a y_{\dot{x}}) = f(x), y(0) = y(L), y(h) = y(L+h), x = x_j = jh, \quad (1.34)$$

where $j = \overline{1,N}, \gamma = \alpha\coth(\alpha)$, $\alpha = 0.5ah$.
The FDS 1.34 we can obtain also from the exact FDS
using the integro-interpolation method for solving the equation $(pu')' = -p(x)f(x), (p(x) = \exp(ax))$, in the form:

$$-(a_{j+1}(u_{j+1} - u_j) - a_j(u_j - u_{j-1})) = F_j^- + F_j^-, j = \overline{1,N}, \quad (1.35)$$

where $u_0 = u_N, u_1 = u_{N+1}$,
$a_j = \big(\ \int_{x_{j-1}}^{x_j} \frac{dx}{p(x)}\ \big)^{-1} = \frac{\exp(ax_j)}{\exp(ah)-1}$, $a_{j+1} = \frac{\exp(ax_j)}{1-\exp(-ah)}$,
$F_j^- = \int_{x_{j-1}}^{x_j}\ \big(\ 1 - a_j\int_x^{x_j}\frac{dt}{p(t)}\ \big)\ p(x)f(x)dx =$
$\frac{1}{\exp(ah)-1}\int_{x_{j-1}}^{x_j}(\exp(a(x+h)) - \exp(ax_j))f(x)dx,$
$F_j^+ = \int_{x_j}^{x_{j+1}}\ \big(\ 1 - a_{j+1}\int_{x_j}^{x}\frac{dt}{p(t)}\ \big)\ p(x)f(x)dx =$
$\frac{1}{1-\exp(-ah)}\int_{x_j}^{x_{j+1}}(\exp(ax_j) - \exp(a(x-h)))f(x)dx.$
Multiply the equation 1.35 with $\exp(-ax_j)/h$ we obtain the exact FDS

$$\Lambda(u) = \frac{1}{h^2}\ \big(\ -(\gamma+\alpha)u_{j+1} + 2\gamma u_j - (\gamma-\alpha)u_{j-1}\ \big) = F_j, \quad (1.36)$$

where
$j = \overline{1,N}, u_0 = u_N, u_1 = u_{N+1},$

$F_j = \frac{1}{h}\left(\int_{x_{j-1}}^{x_j} \frac{\exp(a(x-x_{j-1}))-1}{\exp(ah)-1} f(x)dx + \int_{x_j}^{x_{j+1}} \frac{1-\exp(a(x-x_{j+1}))}{1-\exp(-ah)} f(x)dx \right)$.
If $f = const$ then this FDS is A. ILjin FDS.

The corresponding spectral problem is $\Lambda w = \mu w$ or in the index form $(w_0 = w_N, w_1 = w_{N-1},)$

$$\frac{1}{h^2}(-(\gamma+\alpha)w_{j+1} + 2\gamma w_j - (\gamma-\alpha)w_{j-1}) = \mu w_j, j = \overline{1,N},$$

or

$$Aw^k = \mu_k w^k, k = \overline{1,N}$$

where $A = \frac{1}{h^2}[2\gamma, -(\gamma+\alpha), 0, 0, ...0, -(\gamma-\alpha)]$, w^k are the circulant matrix and column-vector of the N order with the elements w_j^k.
From the propertys of the circulant matrixes follows that
$\mu_k = \frac{4}{h^2}(\sin(k\pi/N)^2(\gamma - i\alpha\cot\frac{k\pi}{N})$,
$w_j^k = \sqrt{\frac{1}{N}}\exp(2\pi ikj/N), w_{*j}^k = \sqrt{\frac{1}{N}}\exp(-2\pi ikj/N)k, j = \overline{1,N}$, and the scalar product of two eigenvectors $(w^k, w_*^m) = \sum_{j=1}^N w_j^k w_{*,j}^m = \delta_{k,m}$.

The complex eigenvalues μ_k are complex conjugate as regards $k = N_2 = N/2$ or $\mu_{N/2+m} = \mu_m^*, m = \overline{1,N/2}$, where N is even number.
For the solution the discrete problem $Ay = f$ we use the transformation $W_* y = V$ or $y = WV$. Then $DV = W_* f$ or in components $d_j v_j = (W_* f)_j, j = \overline{1,N}$.
For $j = N$ we have the expression $0 = \sum_{k=1}^N f_(x_k)$ where consist with the integral condition.
The value v_N is indeterminable and we can take $v_N = 0$. For $j = \overline{1,N-1}$ we have $v_j = \frac{1}{d_j}(W_* f)_j$ and the solution is in the form $y = WV$. If $d_k = \lambda_k$ then we can obtain the solution of FDSES in following way:
1)$d_k = \lambda_k$ for $k = \overline{1,N_2}$, where $N_2 = N/2$.
2)$d_k = \lambda_{N-k}$ for $k = \overline{N_2, N-1}, d_N = 0$.

The solution of the FDS $Ay = f$ we can obtain by spectral method also in the real form. For vector f of N order with component $f_j, j = \overline{1,N}$ using $w_*^k = w^{N-k} = w^{-k}, w_j^N = w_j^0 = 1, j = \overline{1,N}$ we have following expressions:
$f = \sum_{k=1}^N b_k w^k = \sum_{k=1}^{N_2-1}(b_k w^k + b_{N-k} w^{N-k}) + b_{N_2} w^{N_2} + b_N w^N =$
$\frac{1}{2}\sum_{k=1}^{N_2-1}((b_k + b_{N-k})(w^k + w^{N-k}) + (b_k + b_{N-k})(w^k - w^{N-k}))+$

$b_{N_2}w^{N_2}+b_Nw^0$, $b_k=(f,w^k_*)$,
or $f_j=\sum_{k=1}^{*N_2}(b_{kc}\cos\frac{2\pi kj}{N}+b_{ks}\sin\frac{2\pi kj}{N})+\frac{b_{0c}}{2}$,
where $b_{kc}=\frac{1}{\sqrt{N}}(b_k+b_{N-k})=$
$\frac{1}{\sqrt{N}}\sum_{j=1}^{N}f_j(w^k_{j*}+w^k_j)=\frac{2}{N}\sum_{j=1}^{N}f_j\cos\frac{2\pi kj}{N}$, $b_{ks}=\frac{i}{\sqrt{N}}(b_k-b_{N-k})=$
$\frac{i}{\sqrt{N}}\sum_{j=1}^{N}f_j(w^k_{j*}-w^k_j)=\frac{2}{N}\sum_{j=1}^{N}f_j\sin\frac{2\pi kj}{N}, k=\overline{1,N_2}$,
$b_0=b_N=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}f_j, b_{0c}=b_{Nc}=\frac{2}{\sqrt{N}}b_0, b_{N_2,c}=\frac{2}{\sqrt{N}}$,
$b_{N_2}=\frac{2}{N}\sum_{j=1}^{N}\cos(j\pi)$,
$N_2=\frac{N}{2}, b_{N_2s}=b_{Ns}=0$, $\sum_{k=1}^{*N_2}\beta_k=\sum_{k=1}^{N_2-1}\beta_k+\frac{\beta_{N/2}}{2}$.

Similarly the solution of the discrete problem can be represented in the following form $y_j=\sum_{k=1}^{N}a_kw^k_j$.
Then $f_j=Ay_j=\sum_{k=1}^{N}a_k\mu_kw^k_j=$
$\frac{1}{2}\sum_{k=1}^{N_2-1}((a_k\mu_k+a_{N-k}\mu_{N-k})(w^k_j+w^{N-k}_j)+$
$(a_k\mu_k+a_{N-k}\mu_{N-k})(w^k_j-w^{N-k}_j))+a_{N_2}\mu_{N/2}w^{N_2}_j=$
$\sum_{k=1}^{*N_2}((a_{kc}Re(\mu_k)+a_{ks}Im(\mu_k))\cos\frac{2\pi kj}{N}+(a_{ks}Re(\mu_k)-a_{kc}Im(\mu_k))\sin\frac{2\pi kj}{N})$,
because $(a_k\mu_k+a_{N-k}\mu_{N-k})/\sqrt{N}=a_{kc}Re(\mu_k)+a_{ks}Im(\mu_k)$,
$i(a_k\mu_k-a_{N-k}\mu_{N-k})/\sqrt{N}=a_{ks}Re(\mu_k)-a_{kc}Im(\mu_k)$,
where $a_{kc}=\frac{a_k+a_{N-k}}{\sqrt{N}}, a_{ks}=\frac{i(a_k-a_{N-k}}{\sqrt{N}}$ are the coefficients in
the expression $y_j=\sum_{k=1}^{N}a_kw^k_j=\sum_{k=1}^{*N_2}(a_{kc}\cos\frac{2\pi kj}{N}+a_{ks}\sin\frac{2\pi kj}{N})$.

Therefore we have following system of two algebraic equations:

$$\begin{cases}a_{kc}Re(\mu_k)+a_{ks}Im(\mu_k)=b_{kc},\\ a_{ks}Re(\mu_k)-a_{kc}Im(\mu_k)=b_{ks}, k=\overline{1,N_2}\end{cases}\tag{1.37}$$

or

$$a_{kc}=\frac{1}{|\mu_k|^2}\left(b_{kc}Re(\mu_k)-b_{ks}Im(\mu_k)\right), a_{ks}=\frac{1}{|\mu_k|^2}\left(b_{ks}Re(\mu_k)+b_{kc}Im(\mu_k)\right),\tag{1.38}$$

where
$Re(\mu_k)=\frac{4}{h^2}(\sin(k\pi/N)^2\gamma, Im(\mu_k)=-\frac{4}{h^2}(\sin(k\pi/N)^2\alpha\cot\frac{k\pi}{N}), k=\overline{1,N_2}$.
We can obtain the equations (1.37) from the real N-order matrix B (matrix-representation for the approximations of the first and second derivatives) spectral problems:
$Bw^k=\mu_kw^k, Bw^k_*=\mu^*_kw^k_*$,
$B(w^k+w^k_*)=0.5((\mu_k+\mu^*_k)(w^k+w^k_*)+(\mu_k-\mu^*_k)(w^k-w^k_*))$,

$B(w^k - w_*^k) = 0.5((\mu_k + \mu_k^*)(w^k - w_*^k) + (\mu_k - \mu_k^*)(w^k + -w_*^k))$,
or $B\cos_k = Re(\mu_k)\cos_k - Im(\mu_k)\sin_k$, $B\sin_k = Re(\mu_k)\sin_k + Im(\mu_k)\cos_k$,
where $\sin_k, \cos_k$ are the column-vectors with the elements $\sin\frac{2\pi k}{N}, \cos\frac{2\pi k}{N}$.

For the differential spectral problem
$-\tilde{L} = -(w''(x) + aw'(x) = \lambda w(x), x \in (0,L), w(0) = w(L), w'(0) = w'(L)$
we obtain $\lambda_k = (2\pi k/L)^2 - 2\pi kai/L, w^k(x) = \sqrt{\frac{1}{L}}\exp(2\pi ikx/L)$,
$w_*^k(x) = \sqrt{\frac{1}{L}}\exp(-2\pi ikx/L) = w^{-k}(x)$,
$(w^k, w_*^m)_* = \int_0^L w^k(x)w_*^m(x)dx = \delta_{k,m},\ k,m = \overline{-\infty,+\infty}$.
The solution of (1.33) with the Fourier method can be obtained in following form:
$f(x) = \sum_{k=-\infty}^{\infty} b_k w^k(x),\ b_k = (w_*^k, f),\ u(x) = \sum_{k=-\infty}^{\infty} a_k w^k(x), a_k = b_k/\lambda_k$.
For periodical function $f(x)$ follows the complex expansion
$f(x) = \sum_{k=-\infty}^{\infty} b_k w^k(x) = \sum_{k=1}^{\infty}(b_k w^k(x) + b_{-k}w^{-k}(x)) + \frac{b_0}{\sqrt{L}} =$
$\frac{1}{2}\sum_{k=1}^{\infty}((b_k + b_k)(w^k(x) + w_*^k(x)) + (b_k - b_{-k})(w^k(x) - w_*^k(x))) + \frac{b_0}{\sqrt{L}} =$
$\sum_{k=1}^{\infty}(b_{kc}\cos\frac{2\pi kx}{L} + b_{ks}\sin\frac{2\pi kx}{L}) + \frac{b_{0c}}{2}$, $b_k = (f, w_*^k)$,
where $b_{kc} = \frac{1}{\sqrt{L}}(b_k + b_{-k}) = \frac{1}{\sqrt{L}}\int_0^L f(x)(w_*^k(x) + w^k(x))dx) =$
$\frac{2}{L}\int_0^L f(t)\cos\frac{2\pi kt}{L}dt$,
$b_{ks} = \frac{i}{\sqrt{L}}(b_k - b_{-k}) = \frac{i}{\sqrt{L}}\int_0^L f(x)(w_*^k(x) - w^k(x))dx) =$
$\frac{2}{L}\int_0^L f(t)\sin\frac{2\pi kt}{L}dt$.
Therefore the solution of (1.33) we can obtain also in real form:
$u(x) = \sum_{k=1}^{\infty}(a_{kc}\cos\frac{2\pi kx}{L} + a_{ks}\sin\frac{2\pi kx}{L}) + \frac{a_{0c}}{2}$,
where a_{kc}, a_{ks} are unknown coefficients.
From $f(x) = -(u''(x) + au'(x)) = -\sum_{k=-\infty}^{\infty} a_k(w''(x)^k + aw'(x)^k) = \sum_{k=\infty}^{\infty} a_k\lambda_k w(x)^k$ follows $f(x) = \frac{1}{2}\sum_{k=1}^{\infty}((a_k\lambda_k + a_{-k}\lambda_{-k})(w(x)^k + w(x)^{-k}) + (a_k\lambda_k + a_{-k}\lambda_k)(w(x)^k - w(x)^{-k})) =$
$\sum_{k=1}^{\infty}((a_{kc}Re(\lambda_k) + a_{ks}Im(\lambda_k))\cos\frac{2\pi kx}{L} + (a_{ks}Re(\lambda_k) - a_{kc}Im(\lambda_k))\sin\frac{2\pi kx}{L})$,
because $(a_k\lambda_k + a_{-k}\lambda_{-k})/\sqrt{L} = a_{kc}Re(\lambda_k) + a_{ks}Im(\lambda_k)$,
$i(a_k\lambda_k - a_{-k}\lambda_{-k})/\sqrt{L} = a_{ks}Re(\lambda_k) - a_{kc}Im(\lambda_k)$,
where $a_{kc} = \frac{a_k + a_{-k}}{\sqrt{L}}, a_{ks} = \frac{i(a_k - a_{-k}}{\sqrt{L}}$ are the coefficients in the expression from the solution $u(x)$, where can obtained from (1.37,1.38) replacing μ_k with $\lambda_k(Re(\lambda_k) = \frac{4\pi^2k^2}{L^2}, Im(\lambda_k) = -\frac{2\pi ka}{L})$.
If $h \to 0$ then $\mu_k \to \lambda$.

We have following expressions:
$-\tilde{L}(\cos\frac{2\pi kx}{L}) = Re(\lambda_k)\cos\frac{2\pi kx}{L} - Im(\lambda_k)\sin\frac{2\pi kx}{L}$,
$-\tilde{L}(\sin\frac{2\pi kx}{L}) = Re(\lambda_k)\sin\frac{2\pi kx}{L} + Im(\lambda_k)\cos\frac{2\pi kx}{L}$,
$\Lambda(\cos\frac{2\pi kj}{N}) = Re(\mu_k)\cos\frac{2\pi kj}{N} - Im(\mu_k)\sin\frac{2\pi kj}{N}$,
$\Lambda(\sin\frac{2\pi kj}{N}) = Re(\mu_k)\sin\frac{2\pi kj}{N} + Im(\mu_k)\cos\frac{2\pi kj}{N}$.
Using the ortonormed conditions of
$g = \sin, p = \cos$ or $\frac{2}{L}\int_0^L g(\frac{2\pi kx}{L})p(\frac{2\pi nx}{L})dx = \delta_{k,n}$,
$\frac{2}{N}\sum_{j=1}^N g(\frac{2\pi kx_j}{L})p(\frac{2\pi kx_j}{L}) = \delta_{k,n}$, we can obtain the solution of the problem with simple date, when the function $f(x)$ is proportional to the eigenfunctions.

Example 1.10. If $f(x) = \sin(2\pi x) = \frac{1}{2i}(w^1(x) - w^{-1}(x)), L = 1$,
then $b_1 = \frac{1}{2i}, b_{-1} = -\frac{1}{2i}, b_{1c} = 0, b_{1s} = 1$,
$u(x) = a_1 w^1(x) + a_{-1}w^{-1}, a_1 = \frac{1}{2i\lambda_1}, a_{-1} = -\frac{1}{2i\lambda_{-1}}$,
$u(x) = \frac{4\pi^2\sin(2\pi x)+2\pi a\cos(2\pi x)}{(4\pi^2)^2+4\pi^2a^2}, u'(0) = \frac{8\pi^3}{(4\pi^2)^2+4\pi^2a^2},.$
$a_{1c} = \frac{4\pi^2 b_{1c}+2\pi ab_{1s}}{|\lambda_1|^2} = \frac{2\pi a}{|\lambda_1|^2}, a_{1s} = \frac{4\pi^2 b_{1s}-2\pi ab_{1c}}{|\lambda_1|^2} = \frac{4\pi^2}{|\lambda_1|^2}$,
$u(x) = a_{1c}\cos(2\pi x) + a_{1s}\sin(2\pi x), |\lambda_1|^2 = 4\pi^2)^2 + 4\pi^2a^2$.
This solution we have obtain also in following way:
$u(x) = d_1\cos(2\pi x) + d_2\sin(2\pi x)(d_1, d_2-$ unknown coeficients,
$-\tilde{L}(u) = d_1(-\tilde{L}(\cos(2\pi x))) + d_2(-\tilde{L}(\sin(2\pi x))) =$
$d_1(Re(\lambda_1)\cos(2\pi x) - Im(\lambda_1)\sin(2\pi x)) + d_2(Re(\lambda_1)\sin(2\pi x) +$
$Im(\lambda_1)\cos(2\pi x)) = f(x) = \sin(2\pi x)$
or $d_1 Re(\lambda_1) + d_2 Im(\lambda_1) = 0, -d_1 Im(\lambda_1) + d_2 Re(\lambda_1) = 1$,
$d_1 = -\frac{Im(\lambda_1}{||\lambda_1|^2}, d_2 = \frac{Re(\lambda_1}{||\lambda_1|^2}$.
Similaly $y_j = d_1\cos(2\pi j/N) + d_2\sin(2\pi j/N), d_1 = -\frac{Im(\mu_1}{||\mu_1|^2}, d_2 = \frac{Re(\mu_1}{||\mu_1|^2}$.

We can obtain the solution of FDSES by replasing the discrete eigenvalues μ_k with the first N eigenvalues $\lambda_k, k = \overline{1, N/2}$. For FDS with the central differences $\gamma = 1$.
The exact solution of the problem with $f(x) = \cos(2P\pi x/L)\exp(\sin(2P\pi x/L))$ can be obtained use the Matlab operator "quad" (see the **listing**). The calculations with MATLAB by $L = 2, a = 3$ give following maximal errors:
1 $P = 2, N = 20 : 0.044(FDS), 0.0049(FDSES)$,
2) $P = 2, N = 40 : 0.0082(FDS), 1.510^{-6}(FDSES)$, (see Figs. 1.10, 1.11),

3) $P=4, N=40: 0.0111(FDS), 0.0114(FDS, \gamma=1), 0.0012(FDSES)$,(see Fig. 1.12),
4) $P=4, N=80: 0.0020(FDS), 0.0021(FDS, \gamma=1), 3.10^{-7}(FDSES)$ (see Fig. 1.13).
If $a=10, L=P=2$, then
1) $N=40: 0.010(FDS), 0.014(FDS, \gamma=1), 0.003(FDSES)$,
2) $N=20: 0.043(FDS), 0.060(FDS, \gamma=1), 0.006(FDSES)'$.
The results obtained with real and complex expressions are equally.
The FDS with central difference is conditionally stable for $|\alpha| \leq 2$.

```
%ODE  U''+a U'=f(x) with periodical BC
% example cos(2piPx/L)exp(sin(2piPx/L),Iljin FDS
function PDSper1(N)% N-even
N1=N+1;N2=N-1; NH=N/2; L=2;x=linspace(0,L,N1)';
    x=x(2:N1);h=L/N;B1=zeros(N,N);V=zeros(N,1);d=zeros(N,1);
VV=zeros(N,1);d1=zeros(N,1);prec=zeros(N,1);
P=2;U0=0;a=3;alfa=a*h/2;gamma=alfa*coth(alfa);
B1=B1+2*gamma*diag(ones(N,1))-(gamma-alfa)*diag(ones(N2,1),-1). . .
-(gamma+alfa)*diag(ones(N2,1),1);
B1(1,N)=-(gamma -alfa); B1(N,1)=-(gamma+alfa);
B1=B1/(h^2);%3-diag. matrix for O(h^2)
gg=@(t) cos(2*pi*P*t/L).*exp(5*sin(2*pi*P*t/L));
gg1=@(t)exp(a*t).*cos(2*pi*P*t/L).*exp(5*sin(2*pi*P*t/L));
pre=quad(gg1,0,L,1.e-10);
for j=1:N
prec(j)= (exp(-a*L)-exp(-a*(x(j)+L)))/(1-exp(-a*L))/a*pre +. . .
(quad(gg,0,x(j),1.e-10)-exp(-a*x(j))*quad(gg1,0,x(j),1.e-10))/a ;
end   % quadrature formula
F=-cos(2*pi*P*x/L).*exp(5*sin(2*pi*P*x/L));
NT=(1:N)'/L;
lk=4*alfa/h^2*((sin(pi*h*NT)).^2 *coth(alfa) -...
0.5*i*sin(2*pi*h*NT));
%FDS
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2 - 2*(1:N)'*pi/L*a*i; %exact eigenvalues
dd=lk; %FDS
d1(1:NH)=conj(lk0(1:NH));
d1(NH:N2)=(lk0(NH:-1:1));
d(1:NH)=(lk0(1:NH));
d(NH:N2)=(lk0(NH:-1:1));%FDSES
r1=real(d);i1=imag(d);r2=real(dd);i2=imag(dd);
W=Ck*exp(2*pi*i*(1:N)'*x'/L)'; % Eigen vectors
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';% Eigen vectors-conjugate
V(1:N2)=W1(1:N2,:)*F(:)./(d1(1:N2)); %transfor. sol.FDSES
VV(1:N2)=W1(1:N2,:)*F(:)./conj(dd(1:N2));%transf. sol.FDS
%Real eig-func.
WC=cos(2*pi*(1:NH)'*x'/L)';WS=sin(2*pi*(1:NH)'*x'/L)';
Ak=W1*F;FF=W*Ak; MMM=max(abs(FF-F))% control Fourier expr.
Bkc=2/N*WC'*F;Bks=2/N*WS'*F;B0= 0; %coef-Fourier
```

```
Akc1=(r1(1:NH).*Bkc -i1(1:NH).*Bks)./(abs(d(1:NH))).^2;%FDSEs
Aks1=(r1(1:NH).*Bks +i1(1:NH).*Bkc)./(abs(d(1:NH))).^2;%FDSEs
Akc=(r2(1:NH).*Bkc -i2(1:NH).*Bks)./(abs(dd(1:NH))).^2;%FDS
Aks=(r2(1:NH).*Bks +i2(1:NH).*Bkc)./(abs(dd(1:NH))).^2;%FDS
Akc(end)=Akc(end)/2; Akc1(end)=Akc1(end)/2;
Aks(end)=Aks(end)/2; Aks1(end)=Aks1(end)/2;
UU=WC*Akc+WS*Aks;% FDS real solutions;
U=WC*Akc1+WS*Aks1;% FDSES real solutions;
U1=W*V;%FDSES sol.
UU1=W*VV;% FDS sol.
ui1=max(abs(imag(U1)));U1=U1-U1(N,1)+U0;%FDSES sol.error
uui1=max(abs(imag(UU1)));UU1=UU1-UU1(N,1)+U0;%FDS sol.err
ui=max(abs(imag(U)));U=U-U(N,1)+U0;%FDSES sol.error-real
uui=max(abs(imag(UU)));UU=UU-UU(N,1)+U0;%FDS sol.error-real
%W1*W;A2=W*diag(lk)*W1% control, A2=B1 for O(h^2)
figure
plot(x,prec,'k*',x,U1,'-.',x,UU1,'-','LineWidth',2,...
'MarkerSize',5)
legend('prec. atr.', 'FDSES','FDS')
title(sprintf('Solution  on x,N=...
%3.0f,imPDS=%6.5d,imDS=%6.5d',N,ui1,uui1))
figure
plot(x,prec,'k*',x,U,'-.',x,UU,'-','LineWidth',2,...
'MarkerSize',5)
legend('prec. atr.', 'FDSES-real','FDS-real')
title(sprintf('Sol.-real  on x,N=...
%3.0f,imPDS=%6.5d,imDS=%6.5d',N,ui,uui))
kUU=abs(UU-prec); kU=abs(U-prec);max(kUU),max(kU)
kUU1=abs(UU1-prec); kU1=abs(U1-prec);max(kUU1),max(kU1)
figure
plot(x,kU1,'r*',x,kUU1,'-','LineWidth',2,'MarkerSize',3)
legend('FDSES','FDS')
title(sprintf('Error N=...
%2.0f,nFDSES=%6.5d,nFDS=%6.5d',N,max(kU1),max(kUU1)))
figure
plot(x,kU,'r*',x,kUU,'-','LineWidth',2,'MarkerSize',3)
legend('FDSES-real','FDSreal')
title(sprintf('Error-real N=...
%2.0f,nFDSES=%6.5d,nFDS=%6.5d',N,max(kU),max(kUU)))
```

Using for first derivative $u'(x_j)$ the higher order approksimation $O(h^{2n})$ we have following matrix representation (see sect. 1.6):
$B = \frac{1}{h}[0, C_1, C_2, ..., C_n, 0, 0, ... - C_n, -C_{n-1}, ..., -C_2, -C_1]$,
where $C_m = \frac{(n!)^2(-1)^{m-1}}{m(n-m)!(n+m)!}, m = \overline{1,n}$. Then the eigenvalues are:
$\mu_k = \frac{2i}{h}\sum_{m=1}^{n} C_m \sin\frac{2\pi mk}{N}$.
We have following eigenvalues $(k = \overline{1,N})$:
1) $n = 1 : \mu_k^0 = \frac{i}{h}\sin(2\pi k/N)$,
2)$n = 2 : \mu_k^0 = \frac{2i}{h}(\frac{2}{3}\sin(2\pi k/N) - \frac{1}{12}\sin(4\pi k/N))$,
3)$n = 3 : \mu_k^0 = \frac{2i}{h}(\frac{3}{4}\sin(2\pi k/N) - \frac{3}{20}\sin(4\pi k/N) + \frac{1}{60}\sin(6\pi k/N))$,

4) $n = 4 : \mu_k^0 = \frac{2i}{h}(\frac{24}{30}\sin(2\pi k/N) - \frac{1}{5}\sin(4\pi k/N) + \frac{4}{105}\sin(6\pi k/N) - \frac{1}{280}\sin(8\pi k/N))$.
We added following operators by the MATLAB m.file **"PDSper1"**

```
%lk=4/h^2*((sin(pi*h*NT)).^2 -0.5*alfa*i*sin(2*pi*h*NT));
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4 . . .
-0.5*alfa*i*(4/3*sin(2*pi*h*NT)-1/6*sin(4*pi*h*NT))); %4.ord
%lk=4*alfa/h^2*((sin(pi*h*NT)).^2 *coth(alfa) -. . .
0.5*i*sin(2*pi*h*NT));%2.order FDS Iljin
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4 +. . .
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8-. . .
0.5*alfa*i*(24/15*sin(2*pi*h*NT). . .
-2/5*sin(4*pi*h*NT)+8/105*sin(6*pi*h*NT)-1/140*sin(8*pi*h*NT)));
%8.order,lk=4/h^2*((sin(pi*h*NT)).^2+ . . .
0.5*alfa*i*(3/2*sin(2*pi*h*NT)-3/10*sin(4*pi*h*NT)+. . .
1/30*sin(4*pi*h*NT))); %6.order
```

The calculations with MATLAB by $L = P = a = 2, N = 40$ give following maximal errors:
$8.6\ 10^{-3}(O(h^2)), 8.2\ 10^{-3}(IljinFDS), 1.24\ 10^{-3}(O(h^4)), 2.60\ 10^{-3}(O(h^6)), 1.2\ 10^{-4}(O(h^8)), 1.27\ 10^{-6}(FDSES)$.
If $a = 10$, then
$1.4\ 10^{-2}(O(h^2)), 1.0\ 10^{-2}(IljinFDS), 3.7\ 10^{-3}(O(h^4)), 7.9\ 10^{-3}(O(h^6)), 3.1\ 10^{-3}(O(h^8)), 3.0\ 10^{-3}(FDSES)$.
Therefore, FDS with $O(h^4)$ is precised than FDS with $O(h^6)$ (see also [22], [23]).

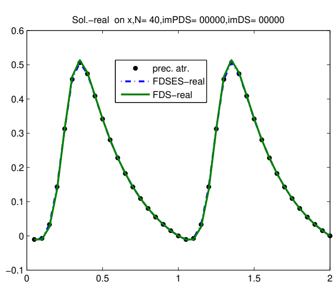


Fig. 1.10 Solution for $a = 3, N = 40, L = 2, P = 2$

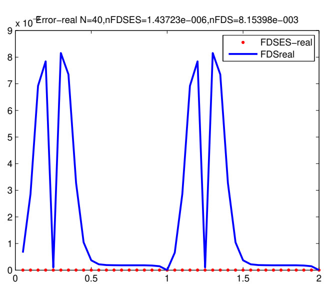


Fig. 1.11 Errors for $a = 3, N = 40, L = 2, P = 2$

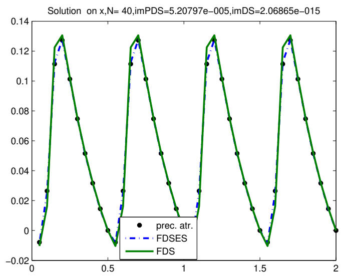


Fig. 1.12 Solution for $a = 3, N = 40, L = 2, P = 4$

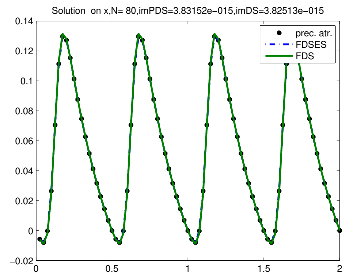


Fig. 1.13 Solution for $a = 3, N = 80, L = 2, P = 4$

1.10 The conjugate operators with modified equations

We consider following boundary value problems

$$\begin{cases} u''(x) + a^2 u(x) = -f(x), u(0) = u(L) = 0, \\ u''(x) + a^2 u(x) = -f(x), u(0) = u(L), u'(0) = u'(L), \end{cases} \tag{1.39}$$

$$\begin{cases} u''(x) - a^2 u(x) = -f(x), u(0) = u(L) = 0, \\ u''(x) - a^2 u(x) = -f(x), u(0) = u(L), u'(0) = u'(L), \end{cases} \tag{1.40}$$

where a is constant parameter.

1.10.1 The BCs with the first kind

The analytical solution of the problem (1.39) is

$$u(x) = u(L)\frac{\sin(ax)}{\sin(aL)} + u(0)\frac{\sin(a(L-x))}{\sin(aL)} - \frac{1}{a\sin(aL)}\int_0^L K(x,t)f(t)dt,$$

where $K(t,x)$ is the Green function in following way:

$$K(t,x) = \begin{cases} \sin(ax)\sin(a(L-t)),\ 0 \le t \le x, \\ \sin(at)\sin(a(L-x)),\ x \le t \le L. \end{cases}$$

For (1.40) the form of the solution remained by replacing the trigonometrical functions with hyperbolical.

Using discretization of (1.39) we have corresponding V. Bahvalov FDS (M=N-1)

$$\Lambda y = Ay = -(\gamma y_{x\bar{x}} + a^2 y) = f(x), y(0) = y(L) = 0, x_j = jh, \quad j = \overline{1,M}, \tag{1.41}$$

where $\gamma = \left(\frac{\alpha}{\sin(\alpha)} \right)$, $\alpha = 0.5ah$.
This FDS is exact for linear function of $f(x) = bx + c$, where b, c are constants. In this case we have
$u(x) = C_1 \sin(ax) + \frac{bx+c}{a^2}, y_j = C_1 \sin(px_j) + \frac{bx_j+c}{a^2}$,
where $C_1 = -\frac{bL+c}{\sin(aL)a^2}$, $\cos(ph) = 1 - \frac{a^2h^2}{2\gamma} = \cos(ah)$ or $p = a$.
For the FDS with central differences $\gamma = 1$.
The corresponding spectral problem is
$\Lambda w^k = \mu_k w^k$ $\mu_k = \frac{4}{h^2} \sin(k\pi/2N)^2 \gamma - a^2$,
$w_j^k = \sqrt{\frac{2}{N}} \sin(\pi kj/N)$, $k, j = \overline{1,M}$.
Every vector f of M order with the elements $f_j, j = \overline{1,M}$ we can be expanded in the basis of eigenvectors
$f_j = \sum_{k=1}^M b_k \sin \frac{\pi kj}{N}, b_k = \frac{2}{N} \sum_{j=1}^M f_j \sin \frac{\pi kj}{N}$.
The solution of the boundary value problem we can write in following form:
$y_j = \sum_{k=1}^M a_k \sin \frac{k\pi j}{N}, a_k = \frac{b_k}{\mu_k}$.

The solution of the spectral problem for the corresponding differential problem $-w''(x) = \lambda w(x), w(0) = w(L) = 0$ is in following form:
$w^k(x) = \sqrt{\frac{2}{L}} \sin \frac{k\pi x}{L}, \lambda_k = (\frac{k\pi}{L})^2 - a^2$.
Using the expression
$f(x) = \sum_{k=1}^\infty b_k \sin \frac{k\pi x}{L}, b_k = \frac{2}{L} \int_0^L \sin \frac{k\pi x}{L} dx$,
we can the solution write in the form
$u(x) = \sum_{k=1}^\infty a_k \sin \frac{k\pi x}{L}, a_k = \frac{b_k}{\lambda_k}$.
If $f(x) = \sum_{k=1}^K c_k \sin \frac{\pi kx}{L}, K \leq M$, ($c_k$ are constant coefficients) then FDSES method at least with M summands are exact methods, but FDS is the method of the second order approximation.
In this case the exact solution is $u(x) = \sum_{k=1}^K \frac{c_k}{\lambda_k} \sin \frac{\pi kx}{L}$.
We can constructed the FDSES when in the representation $A = WDW$ the diagonal elements μ_k of matrix D are replaced with the eigenvalues λ_k from the differential problem.
For (1.40) we have similar expressions with $\mu_k = \frac{4}{h^2} \sin(k\pi/2N)^2 \gamma + a^2$,

$\gamma = (\frac{\alpha}{\sinh(\alpha)}), \lambda_k = (\frac{k\pi}{L})^2 + a^2.$
Similarly we can consider following problem

$$Lu(x) = -(u''(x) + au'(x) - bu(x)) = f(x), u(0) = u(L) = 0. \quad (1.42)$$

For the differential spectral problem
$Lw(x) = -(w''(x) + aw'(x) - bw(x)) = \lambda w(x), x \in (0,L), w(0) = w(L) = 0$
we obtain $\lambda_k = (\pi k/L)^2 + a^2/4 + b, w^k(x) = \sqrt{\frac{2}{L}}\exp(-ax/2)\sin(k\pi x/L)$,
$w_*^k(x) = \sqrt{\frac{2}{L}}\exp(ax/2)\sin(k\pi x/L)$,
$(w^k, w_*^m) = \int_0^L w^k(x)w_*^m(x)dx = \delta_{k,m}$, $k,m = \overline{1,\infty}$.
The solution of (1.42) with the Fourier method can be obtained
in following form:
$f(x) = \sum_{k=1}^{\infty} b_k w^k(x)$, $b_k = (w_*^k, f)$, $u(x) = \sum_{k=1}^{\infty} a_k w^k(x), a_k = b_k/\lambda_k$.
In the discrete case we use corresponding A. Iljin FDS

$$\Lambda y = -(\gamma y_{x\bar{x}} + a y_{\dot{x}} - bx) = f(x), y(0) = y(L) = 0, x = x_j = jh, \quad (1.43)$$

where $j = \overline{1, N-1}, \gamma = \alpha\coth(\alpha)$, $\alpha = 0.5ah$.

1.10.2 The periodical BCs

The analytical solution of the problem (1.39) is

$$u(x) = \frac{1}{2a\sin(aL/2)}\int_0^L K(t,x)f(t)dt,$$

where $K(t,x)$ is the Green function in following way:

$$K(t,x) = \begin{cases} \cos(a(L/2 + t - x)), \; 0 \le t \le x, \\ \cos(a(L/2 + x - t)), \; x \le t \le L. \end{cases}$$

For (1.40) the form of the solution remained by replacing the trigonometrical functions with hyperbolical.

We use the discretization of (1.39) with the V. Bahvalov FDS by $M=N-1$. The corresponding spectral problem $Aw^k=\mu_k w^k$ with the circulant matrix A have following solution $\mu_k=\frac{4}{h^2}\sin(k\pi/N)^2\gamma-a^2$, $w_j^k=\sqrt{\frac{1}{N}}\exp(\pi ikj/N), w_{*j}^k=\sqrt{\frac{1}{N}}\exp(-2\pi ikj/N)$, $k,j=\overline{1,M}$.

The solution of the FDS $Ay=f$ we can obtain by spectral method in the real form. For vector f of N order with component $f_j, j=\overline{1,N}$ using we have following expressions:
$f_j=\sum_{k=1}^{*N_2}(b_{kc}\cos\frac{2\pi kj}{N}+b_{ks}\sin\frac{2\pi kj}{N})+\frac{b_{0c}}{2}$,
where $b_{kc}=\frac{2}{N}\sum_{j=1}^{N}f_j\cos\frac{2\pi kj}{N}$, $b_{ks}=\frac{2}{N}\sum_{j=1}^{N}f_j\sin\frac{2\pi kj}{N}, k=\overline{1,N_2}$,
$b_{N_2}=\frac{2}{N}\sum_{j=1}^{N}\cos(j\pi)$, $N_2=\frac{N}{2}, \sum_{k=1}^{*N_2}\beta_k=\sum_{k=1}^{N_2-1}\beta_k+\frac{\beta_{N/2}}{2}$.
Similarly the solution of the discrete problem can be represented in the following form:
$y_j=\sum_{k=1}^{*N_2}(a_{kc}\cos\frac{2\pi kj}{N}+a_{ks}\sin\frac{2\pi kj}{N})+\frac{a_{0c}}{2}$, where $a_{kc}=\frac{b_{kc}}{\mu_k}, a_{ks}=\frac{b_{ks}}{\mu_k}$.

For the differential spectral problem
$\lambda_k=(2\pi k/L)^2, w^k(x)=\sqrt{\frac{1}{L}}\exp(2\pi ikx/L)$,
$w_*^k(x)=\sqrt{\frac{1}{L}}\exp(-2\pi ikx/L)=w^{-k}(x)$.
The solution with the Fourier method can be obtained in following form:
$f(x)=\sum_{k=1}^{\infty}(b_{kc}\cos\frac{2\pi kx}{L}+b_{ks}\sin\frac{2\pi kx}{L})+\frac{b_{0c}}{2}$,
where $b_{kc}=\frac{2}{L}\int_0^L f(t)\cos\frac{2\pi kt}{L}dt$, $b_{ks}=\frac{2}{L}\int_0^L f(t)\sin\frac{2\pi kt}{L}dt$.
Therefore $u(x)=\sum_{k=1}^{\infty}(a_{kc}\cos\frac{2\pi kx}{L}+a_{ks}\sin\frac{2\pi kx}{L})+\frac{a_{0c}}{2}$,
where $a_{kc}=\frac{b_{kc}}{\lambda_k}, a_{ks}=\frac{b_{ks}}{\lambda_k}$.

For (1.40) we have similar expressions with $\mu_k=\frac{4}{h^2}\sin(k\pi/N)^2\gamma+a^2$,
$\gamma=\left(\frac{\alpha}{\sinh(\alpha)}\right), \lambda_k=(\frac{k\pi}{L})^2+a^2$.

1.11 Conclusions

The algebraical spectral problems for spectral represention of the quadratic matrix A with the real different eigenvalues and eigenvectors are considered.
The finite difference scheme with second order approximation (FDS) and the boundary value problem for differential equation

(1-D Poisson equation) $-u''(x) = f(x), u(0) = u(L) = 0$, are used.
For FDSES in the representation $A = WDW_*$ the diagonal elements of matrix D are replaced with the eigenvalues λ_k from the differential problem, where W, W_* are the corresponding matrices of eigenvectors (W_* is the conjugate-transposed matrix of W). Then the matrix A is not in the 3-diagonal form but this is full matrix.
We can construct the FDSES when in the representation $A = WDW_*$ the diagonal elements μ_k of matrix D are replaced with the eigenvalues λ_k from the differential problem.
For the symmetric matrix A the matrix W is also symmetric and $WW = E, A = WDW$. The higher order FDS by periodical BCs also are constructed.

Chapter 2
Mathematical models for heat transfer equation

We consider a simple model for description the process of diffusion with PDE: we have an infinitely long tube divided into the following fourth parts - two average parts S_1 ans S_2 are finite, but other two S_0, S_3 are infinite [4] (see the Fig. 2.1).

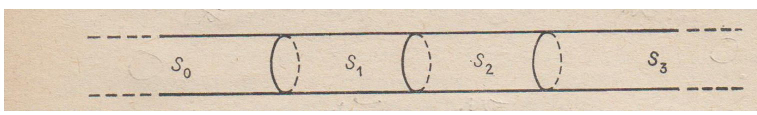


Fig. 2.1 Model of diffusion

In the initial time t= 0 the average parts S_1 ans S_2 contain some chemical solvent with the concentrations v_0, w_0, but concentrations of two infinite parts S_0, S_3 are equal zero for every time moment. Diffusion process begins for $t > 0$ and the velocity of the diffusion between two parts is equal to difference of concentration. This process is continuous in the time and discontinuous in the space. The unknown functions are $v(t), w(t)$ in the two parts S_1, S_2. The concentration $v(t)$ in part S_1 depends on diffusion in the boundary parts S_0 and S_2. The velocity of this concentration is equal

$$\frac{dv}{dt} = (w - v) + (0 - v) = w - 2v.$$

Similarly the velocity of this concentration $w(t)$ in S_2 is

$$\frac{dw}{dt} = (0 - w) + (v - w) = v - 2w.$$

Therefore $\frac{du}{dt} = Au$, where $u(t)$ is the column-vectors of the second order with elements $v(t), w(t)$, A is the matrix of the second order

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

We have solution in the form $u(t) = exp(At)u_0$, where u_0 is the column-vectors of the second order with elements v_0, w_0.
The eigenvalues (-1, -3) characterize the solution depending on the time
$(exp(-t), exp(-3t))$ and for $t \to \infty$ we have $u(t) = v(t) \to 0.5(v_0 + w_0)exp(-t)$.
For obtaining the finite difference approximation for the heat transfer equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$
the average domains S_1, S_2 must be divided into small parts with the length $h = 1/N$.
Then we have the equation $\frac{du}{dt} = Au$, where $u(t)$ is the column-vectors of N order with elements $u_1, ..., u_N$ and A is the 3-diagonal matrix of N order

$$A = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 \ldots 0 & 0 & 0 \\ 1 & -2 & 1 \ldots 0 & 0 & 0 \\ .. & .. & .. \ldots .. & .. & \\ 0 & 0 & 0 \ldots 1 & -2 & 1 \\ 0 & 0 & 0 \ldots 0 & 1 & -2 \end{pmatrix}.$$

We consider the linear initial-boundary heat transfer problem in the following form:

$$\begin{cases} \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}(\bar{k}\frac{\partial T(x,t)}{\partial x}) + f(x,t), x \in (0,L), t \in (0,t_f), \\ \frac{\partial T(0,t)}{\partial x} - \sigma_1 T(0,t) = 0, \frac{\partial T(L,t)}{\partial x} + \sigma_2 T(L,t) = 0, t \in (0,t_f), \\ T(x,0) = T_0(x), x \in (0,L), \end{cases} \quad (2.1)$$

where $\bar{k} > 0, \sigma_1 \geq 0, \sigma_2 \geq 0(\sigma_1^2 + \sigma_2^2 \neq 0)$, are the constant parameters, t_f is the final time, T_0, f are given functions (for boundary conditions of first kind $\sigma_1 = \sigma_2 = \infty$).
We consider a uniform grid in the space $x_j = jh, j = \overline{0,N}, Nh = L$.
Using the finite differences of second order approximation for partial derivatives with second order of x we obtain from (2.1) the initial value problem for system of ordinary differential equations (ODEs) in

the following matrix form

$$\begin{cases} \dot{U}(t) + \bar{k}AU(t) = F(t), \\ U(0) = U_0, \end{cases} \tag{2.2}$$

where A is the 3-diagonal matrix of $N+1$ order, $U(t), \dot{U}(t), U_0, F(t)$ are the column-vectors of $N+1$ order with elements
$u_j(t) \approx T(x_j,t)), \dot{u}_j(t) \approx \frac{\partial T(x_j,t)}{\partial t}, u_j(0) = U_0(x_j), f_j(t) = f(x_j,t), j = \overline{0,N}$.
The solution of the spectral problem for difference operator A is described in chapter 1.

2.1 The discrete problem: H. Kalis, S. Rogovs, 2011 [74]

We can consider the **analytical solutions** of (2.2) using the spectral representation of matrix $A = WDW^T$ (see chapter 1) . From transformation $V = W^T U (U = WV)$ follows the seperate system of ODEs

$$\begin{cases} \dot{V}(t) + \bar{k}DV(t) = G(t), \\ V(0) = W^T U_0, \end{cases} \tag{2.3}$$

where $V(t), \dot{V}(t), V(0), G(t) = W^T F(t)$ are the column-vectors of M order with elements $v_k(t), \dot{v}_k(t), v_k(0), g_k(t) k = \overline{1,M}$.
The solution of this system is the function

$$v_k(t) = v_k(0)\exp(-\kappa_k t) + \int_0^t exp(-\kappa_k(t-\tau)) g_k(\tau) d\tau, \tag{2.4}$$

where $\kappa_k = \bar{k}\mu_k$.
The matrix A from (2.2) $(M = N+1)$ have the first and last rows in the form:
$h^{-2}(2 + 2\sigma_1 h \;\; -\sqrt{2} \dots 0 \;\; 0)$; $h^{-2}(0 \;\; 0 \dots -\sqrt{2} \;\; 2 + 2\sigma_2 h)$.
Therefore the first and last equations of (2.2) are valid when the first and last components of vector AU are divided with $\sqrt{2}$.

Note 2.1. In (2.2) for $M = N+1$ the first $u_1(0), f_1(t)$ and last $u_M(0)$, $f_M(t)$ components of vectors $U_0, F(t)$ are **divided** with $\sqrt{2}$, but the components $u_1(t)$, $u_M(t)$ of the solution vector $U(t)$ in (2.3) need to

multiply with $\sqrt{2}$; for $M = N$ the above-mentioned can be apply to first $(\sigma_2 = \infty)$ or last $(\sigma_1 = \infty)$ component for vectors U, F.

2.2 The linear equation with the BC of the first kind

The analytical solution of heat transfer problem (2.1) by $\bar{k} = L = 1, f = 0, T_0 = 1, \sigma_1 = \sigma_2 = \infty$ (the first kind BC) with discontinuous initial and boundary data can obtain from following Fourier series:

$$T(x,t) = \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{1}{2i+1}\exp(-(2i+1)^2\pi^2 t)\sin((2i+1)\pi x).$$

The corresponding solution with FDS (2.2) is in the following form:

$$U(t) = W\exp(Dt)WU_0,$$

where U_0 is the column vector with ones,
the diagonal matrix D contain the discrete eigenvalues
$\mu_k = \frac{4}{h^2}\sin^2(\frac{k\pi h}{2L}), k = \overline{1, N-1}$.
For the FDSES the elements of matrix D are replaced with the first $N-1$ continuous eigenvalues $\lambda_k = \frac{k^2\pi^2}{L^2}$. The maximal error by $t = 0.02, N = 10$ is 0.089 for FDS and 0.0102 for FDSES. The results obtained with Fourier series contain on $x = 0, x = L$ oscillations (Gibbs phenomenon). For FDSES method these oscillations disappear. The maximal error by $t = 0.9, N = 10$ is 0.0000118 for FDS and 0.0000015 for FDSES. We have following MATLAB m.file **Siltm1:**

```
%system ODE U_t+ AU=0 with first kind hom. BC
%t=Tb,u(x,0)=1;u(0,t)=u(L,t)=0
function Siltm1(N)
N1=N+1;MK=2;Tb=0.9;L=1;x=linspace(0,L,N1)';
t=linspace(0,Tb,MK);
h=L/N;N2=N-1;x=x(2:N);MF=100;
NT=0.5*(1:N2)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2}
Ck=sqrt(2*h/L);
lk0=((1:N2)'*pi/L).^2;
d=lk0; %FDS or FDSES
W=Ck*sin(pi*(1:N2)'*x'/L);
A2=W*diag(d)*W;
```

```
y1=ones(N2,1); % init-cond
P=zeros(N2,1);P=W*y1;P1=zeros(MK,N2);
for k=1:N2
    b=d(k); %FDS or FDSES
    P1(:,k)=exp(-b*t')*P(k);
    end
P2=(W*P1')';
P21=W*diag(exp(-d*Tb))*W*y1;%okei !
nM=2*[1:MF]-1;
     eM=exp(-(pi/L)^2*Tb*nM.^2)./nM;
     up=4/pi*sin(pi*x*nM)*eM';%Fourier
     Ma1=max(abs(up-P21));M=max(abs(P2(end,1:N2)'-up));
figure,plot(x,P21,'ko',x,up,'*',x,P2(end,1:N2),'-')
grid on
legend('MATLAB','Fourier','Analytic')
title(sprintf('Sol.an.on x by err1=%9.7f,err=%9.7f',Ma1,M))
xlabel('\itx'), ylabel('\itu')
X1=ones(MK,1)*x';Y1=t'*ones(1,N2);
figure, surfc(X1,Y1,P2)% error anl.
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('an.sol.,max err1.=%9.7f,err=%8.7f',Ma1,M))
```

If the functions $f(x,t), T_0(x)$ are proportional to the eigenvector $w_p(x) = \sqrt{2/L}\sin(\pi px/L), f(x,t) = g(t)w_p(x), T_0(x) = a_0 w_p(x)$, then the solution we can obtained in the form
$T(x,t) = y(t)w_p(x)$, where for function $y(t)$ follows the ODEs
$\dot{y}(t) = -\bar{k}\lambda_p y(t) + g(t)$ with $y(0) = a_0, \lambda_p = (\frac{\pi p}{L})^2$.
We have the exact solution
$y(t) = \exp(-\bar{k}\lambda_p t)a_0 + \int_0^t \exp(-\bar{k}\lambda_p(t-\xi)g(\xi)d\xi$.

From Fourier series $T(x,t) = \sum_{k=1}^{\infty} a_k(t)w_k(x), f(x,t) = \sum_{k=1}^{\infty} b_k(t)w_k(x)$, $b_k(t) = (f, w_k)_*$, follows ODEs: $\dot{a}_k(t) = -\bar{k}\lambda_k a_k(t) + b_k(t)$
with $a_k(0) = (T_0, w_k)_*, \lambda_k = (\frac{\pi k}{L})^2$.
We have following solutions
$a_k(t) = \exp(-\bar{k}\lambda_k t)a_k(0) + \int_0^t \exp(-\bar{k}\lambda_k(t-\xi)b_k(\xi)d\xi$.
From orthonormal eigenvectors follows, that
$b_p(t) = g(t), a_p(0) = a_0, b_k(t) = a_k(0) = 0, k \neq p$ and $a_p(t) = y(t)$.

From the discrete case (FDS) the solution of the matrix equation (2.2) is
$U(t) = \exp(-\bar{k}tA)U(0) + \int_0^t \exp(-\bar{k}A(t-\xi)F(\xi)d\xi$.
Using the matrix A representation $A = WDW$ and transformation $V = WU$ follows that for every matrix function $f(A) = Wf(D)W$ and $V = \exp(-\bar{k}tD)V(0) + \int_0^t \exp(-\bar{k}D(t-\xi)G(\xi)d\xi$.
Therefore we have the solution in the form (2.4).
If $p \leq N-1$, then the components $v_k(0) = (WU_(0))_k = \sum_{j=1}^{N-1} w_j^k T_0(x_j) =$

$a_0 \sum_{j=1}^{N-1} w_j^k w_p(x_j) = \frac{a_0}{\sqrt{h}}(w^k, w^p)$,
$g_k(t) = (WF)_k, = \frac{g(t)}{\sqrt{h}}(w^k, w^p), k = \overline{1, N-1}$.
We get $v_p(0) = \frac{a_0}{\sqrt{h}}, g_p(t) = \frac{g(t)}{\sqrt{h}}, v_k(0) = g_k(t) = 0, k \neq p$ and from (2.4) follows $v_p(t) =$
$\frac{1}{\sqrt{h}}(\exp(-\bar{k}\mu_p t)a_0 + \int_0^t \exp(-\bar{k}\mu_p(t-\xi)g(\xi)d\xi), v_k(t) = 0, k \neq p$.
For FDSES from $U = WV, w_j^k = \sqrt{h}w_k(x_j)$ and replacing the discrete eigenvalue $\mu_p = \frac{4}{h^2}(\sin\frac{\pi ph}{2L})^2$
with λ_p we obtain the exact solutions $T(x_j,t) = y(t)w_p(x_j), j = \overline{0,N}$.

2.3 The nonlinear equations: H. Kalis, I. Kangro et al., 2009 [1]

We shall consider the initial-boundary value problem with $\sigma_1 = \sigma_2 = \infty$ for solving the following nonlinear heat transfer equation:

$$\frac{\partial T}{\partial t} = \frac{\partial^2(g(T))}{\partial x^2} + f(T), T(0,t) = 0, T(L,t) = 0, T(x,0) = T_0(x), \tag{2.5}$$

where $g(T)$ is nonlinear continuously differentiable function with $\frac{\partial g}{\partial T} = g'(T) > 0, T = T(x,t)$ $f(T)$ is nonlinear continuous source function. For the power functions
$g(T) = T^{\sigma+1}, g'(T) = (\sigma+1)T^{\sigma}, f(T) = aT^{\beta}$,
$a > 0, \beta \geq 1, \sigma \geq 0$, follows $T(x,t) \geq 0$ for all $t \geq 0$, if $T_0(x) \geq 0$.
From $T(0,t) = T(L,t) = 0$ follows that $T'(u) = 0$ by $x = 0; x = L$ and the solution of the problem (2.5) is not classical.
In paper [5] is proved that
1) by $\beta < \sigma + 1$ exists global bounded solution for all t,
2) by $\beta \geq \sigma + 1$ exists global bounded solution for sufficient small $||T_0||$,
but for larger $||T_0||$, exists finite value T_* when $u(x,t) \to \infty$ if $t \to T_*$ ("blow up" solutions)

The initial value problem (2.2) is in the vector form

$$\dot{U} + AG = F, U(0) = U_0, \tag{2.6}$$

where $A = PDP$ is the standard 3-diagonal matrix of $N-1$ order with elements $\frac{1}{h^2}\{-1;2;-1\}$,
G, F are the vectors-column of $N-1$ order with elements $g_k = g(u(x_k,t)), f_k = af(u(x_k,t))$, $k = \overline{1, N-1}$.
The numerical experiment with $L = 1$ and $T_0(x) = x(1-x) \geq 0$,
is produced by MATLAB 7.4 solvers ”ode23s” for different value of σ and β [1].
For example by $a = 5, \sigma = \beta = 3, (\beta < \sigma + 1), t = 10, N = 6, 10, 20$ are obtained following maximal error with FDS and FDSES methods:
1) $N = 5 - -0,0125(FDS), 0.0011(FDSES)$;
2) $N = 10 - -0.0046(FDS), 0.0003(FDSES)$;
3) $N = 20 - -0.0013(FDS), 0.0001(FDSES)$.
Figs. 2.2, 2.3 we can see two type solutions for three time moments $(t = 0, t = T1, t = T2 > T1)$, by $\sigma = 3$ depending on
the parameters β, a, obtained with the FDSES method $(N = 80)$:
1) $\beta = 4(\beta = \sigma + 1), a = 12$, the solution $T(x,t) \to \infty$ globally for all $x \in (0,1)$, when $t \to T_* < \infty$ (T_* is finite value of time, this is global ”blow up” solution),
2) $\beta = 5(\beta > \sigma + 1), a = 50$, the solution $u(x,t) \to \infty$ locally neighbourhood of point $x = 0.5$, when $t \to T_* < \infty$ (for finite value of T_*, this is local ”blow up” solution).

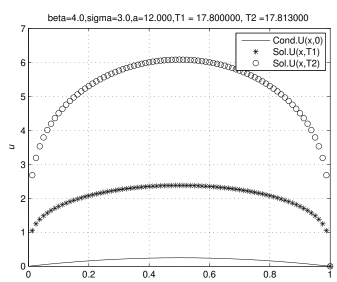


Fig. 2.2 $U \to \infty$ for $x \in (0,1)$, $\beta = 4, \sigma = 3, a = 12, T_* = 17.813001$

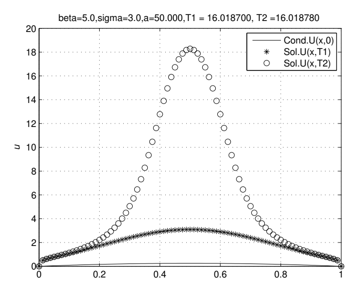


Fig. 2.3 $U \to \infty$ for $x = 0.5$, $\beta = 5, \sigma = 3, a = 50, T_* = 16.018781$

From following MATLAB **m.file ”nellabDS”** we can obtain the solution with the solver ”ode15s” by $\beta = 5$, $\sigma = 3$, $a = 15$.

```
%solutions of ODEs system  $dU/dt=AU^{sigma1}+ a U^{beta},
%t=Tb, A =WDW is the matrix with the best appoximation,
% sigma1=sigma+1
function nellabDS(N)
sigma1=4;beta=5;a=15;Tb=20;L=1;
N1=N+1;N4=N/2+1; x=linspace(0,L,N1)';h=L/N;N2=N-1;
W=sqrt(2/N)*sin(pi/N*(1:N2)'*(1:N2));
A=-W*diag((pi/L*(1:N2)').^ 2)*W;
x=x(2:N);
y0=x.*(L-x);
options=odeset('RelTol', 1.0e-7);
[T,Y]=ode15s(@SIST,[0 Tb],y0,options,A,sigma1,beta,a);
plot([0;x;L],[0;Y(end,:)';0],'k*','MarkerSize',8)
grid on
title(sprintf('time = %8.6f Y(Tb,N/2))=%9.7f '. . .
,T(end),Y(end,N4)))
xlabel('x'), ylabel('u')
figure
plot(T(:),Y(:,N4), 'LineWidth', 3)
title(sprintf('DV lab.aproks.DS,N=%3.0f . . .
time = %8.6f Y(Tb,N/2))=%9.7f ',N,T(end),Y(end,N4)))
xlabel('t'), ylabel('u')
function F=SIST(t,y,A,sigma1,beta,a)
F=A*(y.^ { sigma1})+a*y.^ {beta};
```

The results of the calculations with the operator " nellabDS(20)" are represented in the Figs. 2.4, 2.5 ($U_{max} = 2021.33$).

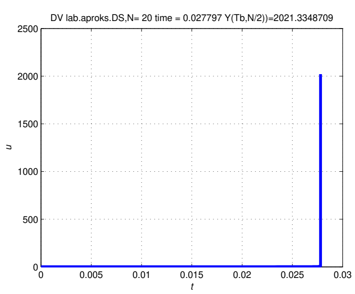


Fig. 2.4 $max(U)$ depending on t by $\beta = 5, \sigma = 3, a = 15, T_* = 0.027797$

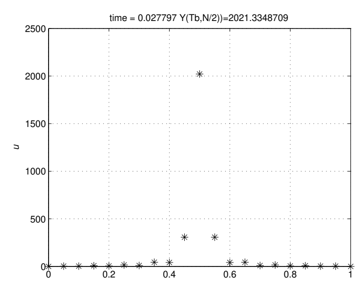


Fig. 2.5 U depending on x by $\beta = 5, \sigma = 3, a = 15, t = T_*$

2.4 Dynamic of magnetic droplet: A. Cebers, H. Kalis, 2013 [75]

The droplet's axial tangent angle β can be described by a nonlinear PDE parabolic type with the BC of first kind [11]

$$\frac{\partial \beta}{\partial t} = \frac{\partial^2}{\partial x^2} F(\beta) + \omega\tau, \beta(0,t) = \beta(L,t) = 0, \beta(x,0) = \beta_0(x), \quad (2.7)$$

where Bm is the magnetic Bond number, $\omega\tau$ is the frequencies,
the nonlinear function $F(\beta) = \frac{1}{Bm}\beta + sin(2\beta)$ is not monotonic for Bm larger than $\frac{1}{2}$ (see Fig. 2.6).
The problem (2.7) is noncorrect for $F'(\beta) \leq 0$ and the solution is discontinuity for $\beta > \beta_*$,
where $F'(\beta_*) = 0$. For steady state, equation (2.7) with boundary condition posseses a simple solution
$F(\beta) = \frac{\omega\tau}{2}(1 - x^2)$ (see Fig. 2.7).
The figures are obtained with following MAPLE file **cebst.mws**:
> restart: b:=1.5: a:=5.: c:=1.:
plot(u/b+sin(2*u)*c,u=0..3,thickness=3,color=black);
with(plots): fsolve(u/b+sin(2*u)=1., u=0..1);
implicitplot(u/b+sin(2*u)*c=a/2.*(1-x*x), x=-1..1, u=0..8, thickness=3, color=black);

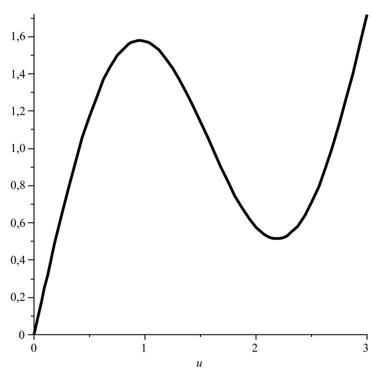


Fig. 2.6 Function $F(\beta)$by $Bm = 1.5, \omega\tau = 5$

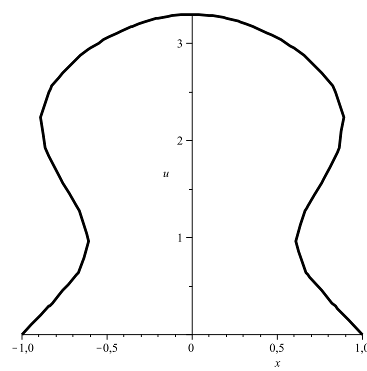


Fig. 2.7 Stationary solutions $F(\beta)$ by $Bm = 1.5, \omega\tau = 5$

The MATLAB m.file **ceblabDS** is following

```
%PDE system dU/dt=wt+AF(U),F(U)=U/Bm +sin(2*U)
%Tb-end time, A =-WDW  transf. x=l+1
function ceblabDS(N)
Tb=3;L=2;wt=5;Bm=1.5;
M1=10;N1=N+1;N4=N/2+1; x=linspace(0,L,N1)';h=L/N;N2=N-1;M=M1-1;
t=linspace(0,Tb,M1); tau= Tb/M;
```

```
W=sqrt(2/N)*sin(pi/N*(1:N2)'*(1:N2));
A=-W*diag((pi/L*(1:N2)').^2)*W; %FDSES
%A=-W*diag(4/(h^2)*(sin(pi/L*h/2*(1:N2)').^2))*W;% FDS
x=x(2:N);
y0=zeros(N2,1);
options=odeset('RelTol', 1.0e-7);
[T,Y]=ode15s(@SIST,[0,Tb],y0,options,A,wt,Bm);% Mat.solvers
%[T,Y]=ode23s(@SIST,[0,Tb],y0,options,A,wt,Bm);
%[T,Y]=ode23t(@SIST,[0,Tb],y0,options,A,wt,Bm);
%[T,Y]=ode23tb(@SIST,[0,Tb],y0,options,A,wt,Bm);
%[T,Y]=ode45(@SIST,[0,Tb],y0,options,A,wt,Bm);
%[T,Y]=ode23(@SIST,[0,Tb],y0,options,A,wt,Bm);
%[T,Y]=ode113(@SIST,[0,Tb],y0,options,A,wt,Bm);
ym=max(Y(end,:));
figure,plot(T(:),Y(:,N4))% Max dep. on t
grid on
title(sprintf('dep on t,N=%3.0f,wt=%2.0f,Bm = %4.3f, . . .
Y(Tb,N/2))=%9.7f ',N,wt,Bm,Y(end,N4)))
xlabel('\itt'), ylabel('\itu')
K=length(T);
X11=ones(K,1)*x';Y11=T*ones(1,N2);
figure, surfc(X11,Y11,Y(:,1:N2)) % sace graf.
colormap(hsv)
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('Surf.,tNr.=%4.1f,wt=%3.0f, Bm=%4.3f',K,wt,Bm))
% Begin of animation
for k=1:M
options=odeset('RelTol', 1.0e-7);
[T,Y]=ode15s(@SIST,[tau*(k-1), tau*k],y0,options,A,wt,Bm);
y0=Y(end,:);
T1(k)=T(end);
X1(k,:)=Y(end,:);
end
for k=1:M
figure,plot([0;x;L],[0;X1(k,:)';0],'ko')% max dep. on x
grid on
title(sprintf('dep. on x,wt=...
%2.0f,time = %8.6f, Bm=%4.3f ',wt,T1(k),Bm))
xlabel('\itx'), ylabel('\itu')
M2(k)=getframe;% animation
end
movie(M2,10)
for k=1:M
X=X1(k,:);
cx(1)=0;cy(1)=0; for k1=2:N2
    cx(k1)=cx(k1-1)+h/2*(cos(X(k1-1))+cos(X(k1)));
    cy(k1)=cy(k1-1)-h/2*(sin(X(k1-1))+sin(X(k1)));end
% the actual shape of a droplet
c1=0;c2=0;
for k1=2:N2
    c1=c1+h/2*(cx(k1-1)+cx(k1));
    c2=c2+h/2*(cy(k1-1)+cy(k1));
end % the centre of gravity
```

```
cx=cx-c1/2; cy=cy-c2/2;
figure,plot(cx,cy,'-', 'LineWidth',4)% trajekt.
title(sprintf('Figures,wt=%3.0f,Bm=%4.3f',wt,Bm))
M2(k)=getframe;
end
movie(M2,10)
function F=SIST(t,y,A,wt,Bm)
F=wt+A*(y/Bm+sin(2*y));
```

In this programm the actual shape of a droplet in the plane $\bar{x},\bar{y}$ found by numerical integration (trapezoid formula) of the differential equations

$$\frac{d\bar{x}}{dx} = \cos(\beta), \frac{d\bar{y}}{dx} = -\sin(\beta),$$

with the centre of gravity. We can work out the filmed procedure by following MATLAB operators

```
mov=avifile('wwtren1.avi','compression','Cinepak');
for k0=dK:dK:K0
X=Y(k0,:);
cx(1)=0;cy(1)=0;
for k1=2:N2
    cx(k1)=cx(k1-1)+h/2*(cos(X(k1-1))+cos(X(k1)));
    cy(k1)=cy(k1-1)-h/2*(sin(X(k1-1))+sin(X(k1)));
end
c1=0;c2=0;
for k1=2:N2
    c1=c1+h/2*(cx(k1-1)+cx(k1));
    c2=c2+h/2*(cy(k1-1)+cy(k1));
end
cx=cx-c1/2; cy=cy-c2/2;
hf=figure;
plot(cx(:),cy(:))
axis([-1 1 -0.6 0.6])
print(hf,'-dbmp16m','temp.jpg');
[temp color]=imread('temp.jpg') ;
mov = addframe(mov,im2frame(temp,color));
close(hf);
end
mov = close(mov);
```

where $K = length(T), dK = fix(K/M), K0 = dK * M$.
The solution are obtained with MATLAB operator **ceblabDS(40)** by $L = 2, Bm = 1.5, \omega\tau = 5, t_f = 3$ (see the Figs. 2.8-2.15).

2.5 The hysteresis: A. Cebers, H. Kalis, 2012 [26]

As a result the existence of the multiple stationary states of the droplet in definite ranges of the frequency of rotating field and therefore hysteresis phenomena are predicted.

2.5.1 Mathematical model

In [11] is general form of PDE (2.7) consider

$$\omega\tau = \frac{\partial\beta}{\partial t} - \frac{\partial^2 F(\beta)}{\partial l^2} + \varepsilon\frac{\partial^5\beta}{\partial^4 l\partial t}, \tag{2.8}$$

where ε is a small coefficient (about $10^{-4}, l = x$).
The regularization term in (2.8) is added from physical considerations.
The equation (2.8) is supplemented by boundary conditions corresponding to the absence of normal forces and torques at the ends of the droplet.
The last term in the equation (2.8) is used for the regularization of the numerical calculations.
By setting $\varepsilon = 0$ we obtain the following problem (2.7), where $\beta(l,0) = \Theta_0(l)$ is the initial conditions in the time $t = 0$.

2.5.2 Solution of the problem

In [11] the numerical solution of (2.8) is obtained by an implicit scheme with the spatial derivatives approximated with central differences. The nonlinear term with the function $F(\beta)$ is resolved by Newton iterations at each time step. Numerical method for the reduced problem (2.7) shows instabilities.
The stationary solution $\beta_s(l)$ with the boundary conditions $\beta(\pm 1,t) = 0$ is $F(\beta_s) = 0.5\omega\tau l(2-l)$.
The maximal value β_m is the solution of the transcendental equation $F(\beta_m) = 0.5\omega\tau$.
The solution $(\beta(l,t) \geq 0)$ is symmetrical with a respect to $l = 1$:

$\beta(1-l,t) = \beta(1+l,t), l \in (0,1)$ or $\frac{\partial\beta(1,t)}{\partial l} = 0$.
The angle β as function of the arc length variable l is discontinuous for $\omega\tau = 2F(\beta_0)$, where
β_0 are the roots of equation $F'(\beta) = 0$ (the local maxima or minimum of the function $F(\beta)$).
The values $w_c = (\omega\tau)_0 = 2F(\beta_0)$, define the critical frequencies.

Discontinuity of β variation along the droplet at $Bm > 0.5$ is described by the modified functions $F(\beta)$ defined as follows.

For the path with increasing frequency of rotating field (direct function) $F(\beta) = F(u)$ is defined as follows (see Fig. 2.30 for $Bm = 1.5$):
1) $F(u) = \frac{1}{Bm}u + \sin(2u), u \in [0, u_1]$,
where $u_1 = \frac{\pi}{2} - 0.5\arccos(0.5/Bm)$, is the first local maxima of function $F(u)$ or the solution of the equations $F'(u_1) = \frac{1}{Bm} + 2\cos(2u_1) = 0$,
2) $F(u) = F(u_1) = F_1, u \in [u_1, u_2]$,
where u_2 is the solution of the transcendental equation $F(u) = F_1$ at the interval $(u_1, u_3), u_3 = \frac{3\pi}{2} - 0.5\arccos(0.5/Bm)$,
3) $F(u) = \frac{1}{Bm}u + \sin(2u), u \in [u_2, u_3]$,
4) $F(u) = F(u_3) = F_3, u \in [u_3, u_4]$, where u_4 is the solution of the transcendental equation $F(u) = F_3$ at the interval $(u_3, u_5), u_5 = \frac{5\pi}{2} - 0.5\arccos(0.5/Bm)$,
5) $\cdots$.
Therefore in the segment $[u_{2k-1}, u_{2k}], k = 1, 2, \cdots$ the function $F(u)$ is replaced with line segment $F(u) = F(u_{2k-1}) = F_{2k-1}$,
where $u_{2k-1} = \frac{(2k-1)\pi}{2} - 0.5\arccos(0.5/Bm)$ are the local maxima of the function $F(u)$.
The ends of the segment u_{2k-1}, u_{2k} satisfy following conditions:
$u_{2k-1} = u_1 + (k-1)\pi; u_{2k} = u_2 + (k-1)\pi, k = 2, 3, \cdots$.
The maximal value of $F(u_{2k-1})$is equal $F_{2k-1} = F_1 + (k-1)\frac{\pi}{Bm}$, where $F_1 = F(u_1)$.
From $F'(u_{2k}) = F'(u_2), F'(0) = 2 + \frac{1}{Bm}$ follows that $F'(u_{2k}) \leq F'(0)$.
The critical frequencies $w_c(k)$ are defined by $w_c(k) = 2F_{2k-1}, k = 1, 2, \cdots$

For the path with decreasing frequency of the rotating field reverse function $F(\beta) = f(u)$ is defined as follows
(see Fig. 2.31 for $Bm = 1.5$):
1) $f(u) = \frac{1}{Bm}u + \sin(2u), u \in [0, v_2]$, v_2 is the solution of the transcendental equation

$f(u) = f_1$ at the interval $(0, u_1)$, where $f_1 = f(v_1)$ is the value of the minimal value of function $f(u)$,
the local minimum is the solution of the equations $f'(u) = 0$ in the segment $[u_1, u_2]$,
2) $f(u) = f(v_1) = f_1, u \in [v_2, v_1]$,
3) $f(u) = \frac{1}{Bm}u + \sin(2u), u \in [v_1, v_4]$, where v_4 is the solution of the transcendental equation
$f(u) = f_3$ at the interval (u_2, u_3), where $f_3 = f(v_3)$ is the value of the local minimum of function $f(u)$
or the solution of the equations $f'(u) = 0$ in the segment $[u_3, u_4]$,
4) $f(u) = f(v_3) = f_3, u \in [v_4, v_3]$,
5) $\cdots$.
Therefore in the segment $[v_{2k}, v_{2k-1}], k = 1, 2, \cdots$
the function $f(u)$ is replaced with line segment $f(u) = f(v_{2k-1}) = f_{2k-1}$,
where $v_{2k-1} = \frac{(2k-1)\pi}{2} + 0.5\arccos(0.5/Bm)$
are the local minimum of the function $f(u)$. The ends of the segment v_{2k-1}, v_{2k} satisfy following conditions:
$v_{2k-1} = v_1 + (k-1)\pi; v_{2k} = v_2 + (k-1)\pi, k = 2, 3, \cdots, v_1 = \pi - u_1$.
The minimal value of $f(u_{2k-1})$is equal $f_{2k-1} = f_1 + (k-1)\frac{\pi}{Bm}$, where $f_1 = f(v_1)$.
The critical frequencies $w_c(k)$ in this case are defined with the expression $w_c(k) = 2f_{2k-1}, k = 1, 2, \cdots$ The stationary shapes constructed according to modified functions are shown in Fig. 2.32 (direct function) and Fig. 2.33 (reverse function).
It is interesting to remark that the curvature of the droplet between the seps of the tangent angle has opposite signs for the cases of direct and reverse functions.
From $\frac{\partial^2 F(u)}{\partial l^2} = F''(u)(\frac{\partial u}{\partial l})^2 + F'(u)\frac{\partial^2 u}{\partial l^2} = -\omega\tau$
follows that $\frac{\partial^2 u}{\partial l^2} \le 0$ if $F''(u) \ge 0$.

In the segments $\omega\tau \in [2F_{2k-1}, 2f_{2k-1}], k = 2, 3, \cdots$ we have two stationary solutions, but
in the segments $\omega\tau \in (2F_{2k-1}, 2f_{2k+1}), k = 2, 3, \cdots$ the solution is unique ($F'(u) \ge 0$).
An example, at $Bm = 1.5$ (see Figs. 2.30, 2.31) we have
following maximal value $\max(\beta_s(l)) = \beta_s(1)$ for different value of $\omega\tau$:
$\omega\tau = 2(0.4078; 2.6858), \omega\tau = 3(0.7531, 2.9128)$,

$\omega\tau = 2F_1(0.9485, 2.9448)$;
$\omega\tau = 1(0.1910), \omega\tau = 4(3.1062), \omega\tau = 5(3.2955)$,
$\omega\tau = 8(6.2122), \omega\tau = 12(9.3180), \omega\tau = 15(9.4940)$.

The modified functions $F(\beta)$ are monotonic $(0 \leq F'(\beta) \leq 2 + \frac{1}{Bm})$ and we can obtain that for fixed time t the solution $\beta(l,t)$ is quadratically integrable together with their first order generalized partial derivatives with a respect to l.

If $t = 0, \Theta_0 = 0$, then from (2.8) follows that $\frac{\partial \beta(l,0)}{\partial t} = \omega\tau > 0$ and the function β is increasing in time.

For modified function $F(\beta)$ we can prove that the weak solution for fixed t is bounded in the norm of Sobolev space $\overset{0}{W}{}_2^1$ and the problem (2.8) is uniquely soluble. Similarly results can be obtained for equation (2.7).

The problems (2.7, 2.8) are solved by the MATLAB "solver ode15s" with relative error 10^{-7} (RelTol=10^{-7}), using the method of lines and finite difference method for the approximation of the spatial derivatives. We consider the uniform grid in the space $l_j = jh, j = \overline{0,N}, Nh = \bar{L}$.

Using the finite differences of second order approximation for partial derivatives of second and fourth order with respect to l, from (2.8) with boundary conditions on the ends of the droplet $\beta = 0$ we obtain the initial value problem for the system of nonlinear ordinary differential equations (ODEs) of the first order in the following matrix form

$$\begin{cases} (E + \varepsilon B)\dot{U}(t) + AF(U(t)) = G, \\ U(0) = 0, \end{cases} \tag{2.9}$$

where E is the unit matrix of $N-1$ order,
A is the standard 3-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^2}\{-1; 2; -1\}$ approximating the derivative $-\frac{\partial^2}{\partial l^2}$,
B is the 5-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^4}\{1; -4; 6; -4; 1\}$ approximating the derivative $\frac{\partial^4}{\partial l^4}$,
(the first and last elements of matrix B are $B(1,1) = B(N-1, N-1) = 5/h^4$, by using following finite difference expressions $u_1(t) = u_N(t) = 0, u_0(t) = -u_2(t), u_{N+1} = -u_{N-1}$ for approximation of the boundary conditions of the equation (2.8))

$U(t), \dot{U}(t), U_0, F(U), G$ are the column-vectors of $N-1$ order with the elements $u_j(t) \approx \beta(l_j,t)$,
$\dot{u}_j(t) \approx \frac{\partial \beta(l_j,t)}{\partial t}, u_j(0) = 0, f_j \approx F(u_j(t)), g_j = \omega\tau, j = \overline{1,N-1}$.

It is obvious that $B = A^2$. In book [28] for solving physically unstable retrospective problem of the linear heat transfer equation at PDE (2.7) is added the regularization term $\varepsilon \frac{\partial^4 \beta}{\partial^4 l}$ or $\varepsilon BU(t)$ at the ODEs system (2.9).
From (2.9) we obtain the initial value problem at $\varepsilon = 0$.

2.5.3 Numerical results

Constructed direct and reverse functions allow us to calculate the dynamics of shapes corresponding to the path with increasing the frequency of rotating field staring from straight configuration and the reverse path starting from the deformed shape calculated by using direct function.
The dynamics of the tangent angle and corresponding shapes formed in direct path for different frequencies of rotating field are shown in Figs. 2.34-2.41.
We remark the formation of highly spiralized shapes at large frequencies of the rotating field (see Figs. 2.39, 2.41).

Relaxation of the tangent angle and the droplet shape obtained using the reverse function is shown in Figs. 2.16-2.23.
Figs. 2.17, 2.20 show a step-like behavior of the maximal tangent angle during the relaxation to the straight configuration which characterizes the disappearance of the jumps of tangent angle.

Main result of the paper is shown in Figs. 2.25-2.29. In this case shape obtained by reverse function straing from the stationary configuration obtained by using direct function at higher frequency is different from the configuration obtained at the same frequency by direct path.
So shapes obtained at $\omega\tau = 2$ by direct and reverse paths are different (Fig. 2.24).
The same is valid for $\omega\tau = 6$ (Fig. 2.26) and $\omega\tau = 10$ (Fig. 2.28).
The tangent angle for these frequencies is shown in Figs. 2.25, 2.27, 2.29.
Direct function $F(u)$ at $Bm = 1.5$ and with the following numerical

values are shown in Fig. 2.30:
$u_1 = 0.9553, u_2 = 2.9448, u_3 = 4.0969, u_4 = 6.0864, u_5 = 7.2385, u_6 = 9.2280, F_1 = 1.5797, F_3 = 3.6741, F_5 = 5.7685, F'(0) = 2.6667$.
In Fig.2.30 with the fixed points $\odot$ are denoted the values of $\omega\tau$ with the coordinates $(\beta_s(1), \omega\tau/2)$.
Reverse function $F(u)$ at $Bm = 1.5$ and with the following numerical values are shown in Fig. 2.31:
$v_1 = 2.1863, v_2 = 0.1968, v_3 = 5.3279, v_4 = 3.3384, v_5 = 8.4695, v_6 = 6.4800, f_1 = 0.5147, f_3 = 2.6091, f_5 = 4.7035$.
In the fixed points $\odot$ are the values of $\omega\tau$ with the coordinates $(\beta_s(1), \omega\tau/2)$.

Thus the considered model predicts multiple stationary states of the droplet in definite ranges of the frequency of rotating field.
It would be interesting to confirm this prediction in experiment. Here we should remark that available experiments [16] indeed show the formation of the shapes with discontinuity of tangent angle which breaks at places with large curvature.
The breaking phenomenon is not described by the present model.

2.5.4 MATLAB programm

```
%system ODE dU/dt=wt+AF(U)+ae B1 dU/dt,
%F(U)=U/Bm +sin(2*U) with Matlab solvers
%Tb-end -time, A =-WDW finite-difference matrix of FDS or FDSES
%transf.x=l+1 [0,2], hom. BDs,
%dynamic of filament (0=>10=>12=>10,N=100)
function cebexp1(N)
Tb=6;L=2;wt=10;Bm=1.5;ae=10^(-4);%epsilon in ODEs, initial wt
M1=4;N1=N+1;N4=N/2;
 x=linspace(0,L,N1)';h=L/N;N2=N-1;M=M1-1;
bb=0:0.01:10;F=bb/Bm+sin(2*bb); % for F(u) plot
figure,plot(bb,F,'k-','LineWidth',2)
grid on
title(sprintf('The function F,Bm = %4.3f',Bm))
xlabel('\it \beta'), ylabel('F')
W=sqrt(2/N)*sin(pi/N*(1:N2)'*(1:N2));% matrix of eigenvectors
%A=-W*diag((pi/L*(1:N2)').^2)*W; %FDSES
A=-W*diag(4/(h^2)*(sin(pi/L*h/2*(1:N2)').^2))*W;% FDS of O(h^2)
x=x(2:N);
y0=zeros(N2,1);%initial cond.for direct path wt=0=>10
B1=zeros(N2,N2);
B1=B1+6*diag(ones(N2,1))-4*diag(ones(N2-1,1),-1)- ...
```

```
4*diag(ones(N2-1,1),1)+...
diag(ones(N2-2,1),2)+diag(ones(N2-2,1),-2);
B1(1,1)=5;B1(N2,N2)=5;
B1=B1/(h^4);% matrix of 4-order deivatives or B1=A^2
E1=eye(N2);
options=odeset('RelTol', 1.0e-7);
[T,Y]=ode15s(@SIST,[0,Tb],y0,options,A,wt,Bm,B1,E1,h,ae);
% wt=0=>10
%[T,Y]=ode45(@SIST,[0,Tb],y0,options,A,wt,Bm,B1,E1,h,ae);
ym=max(Y(end,:))
K=length(T)
figure,plot(T(:),Y(:,N4),'k-','LineWidth',2)% Max val.dep. t
grid on
title(sprintf('Beta-max on t,wt=%2.0f,Bm =%2.1f,. . .
eps=%5.2d,ymax=%5.4d',wt,Bm,ae,ym))
xlabel('\itt'), ylabel('\it \beta_{max}')
figure,plot([0;x;L]',[0;Y(end,:)';0]','k-.','LineWidth',2)
axis([0 2 0 6])
xlabel('l'), ylabel('\beta')
title(sprintf('Beta ,wt=%2.0f,Bm=%2.1f,eps=%5.2d,. . .
 ymax=%5.4d',wt,Bm,ae,ym))
y0=Y(end,:)';wt=12;% next wt and new in-cond. wt=10=>12
y00(1,:)=y0;% values of beta(x,Tb) by wt=10 (direct path)
[T,Y]=ode15s(@SIST,[0,Tb],y0,options,A,wt,Bm,B1,E1,h,ae);
%  wt=10=>12
%[T,Y]=ode45(@SIST,[0,Tb],y0,options,A,wt,Bm,B1,E1,h,ae);
ym=max(Y(end,:))
y0=Y(end,:)';wt=10;% next wt and new in-cond.for wt=12=>10
[T,Y]=ode15s(@SIST,[0,Tb],y0,options,A,wt,Bm,B1,E1,h,ae);
y00(2,:)=Y(end,:);% values of beta(x,Tb) by wt=10 (opposite)
%figure,plot(T(:),Y(:,N4),'k-','LineWidth',2)% Max depends on t
%grid on
%title(sprintf('Beta-max on t,wt=%2.0f,Bm =. . .
 %3.1f,eps=%5.2d,ymax=%5.4d',wt,Bm,ae,ym))
%xlabel('\itt'), ylabel('\it \beta_{max}')
hold on
plot([0;x;L]',[0;Y(end,:)';0]','k--','LineWidth',2)
axis([0 2 0 10])
xlabel('l'), ylabel('\beta')
legend('Direct','Opposite')
title(sprintf('Beta ,wt=%2.0f,Bm=%2.1f,eps=%5.2d',wt,Bm,ae))
K=length(T)
X11=ones(K,1)*x';Y11=T(1:K)*ones(1,N2);
figure,surfc(X11,Y11,Y(1:K,1:N2)) % 3D graphics
colormap(hsv)
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('Beta surf.,tf=...
%4.3f,wt=%3.0f, Bm=%4.3f',Tb,wt,Bm))
%begin of dynamics
dK=fix(K/M);K0=dK*M;YY=zeros(K0,N1);XX=zeros(1,N1);
CX=zeros(K0,N2);CY=zeros(K0,N2);
dK0=fix(dK/2);
t00=T(1),t0=T(dK0),t1=T(dK),t2=T(2*dK),t3=T(K0)
```

```
%for k0=[1,dK,2*dK,K0]
%YY(k0,:)=[0;Y(k0,:)';0];% for dynamics
%X=Y(k0,:);
for k0=1:2
YY(k0,:)=[0;y00(k0,:)';0];
X=y00(k0,:);
cx(1)=0;cy(1)=0;
for k1=2:N2
    cx(k1)=cx(k1-1)+h/2*(cos(X(k1-1))+cos(X(k1)));
    cy(k1)=cy(k1-1)-h/2*(sin(X(k1-1))+sin(X(k1)));
end
c1=0;c2=0;
for k1=2:N2
    c1=c1+h/2*(cx(k1-1)+cx(k1));
    c2=c2+h/2*(cy(k1-1)+cy(k1));
end
cx=cx-c1/2; cy=cy-c2/2;
CX(k0,:)=cx'; CY(k0,:)=cy';
end
figure,plot(CX(1,:)',CY(1,:)','k-.','LineWidth',3)
hold on
plot(CX(2,:)',CY(2,:)','k--','LineWidth',3)
%plot(CX(dK,:)',CY(dK,:)','k--','LineWidth',1.5)
%hold on
%plot(CX(2*dK,:)',CY(2*dK,:)','k-','LineWidth',1.5)
%hold on
%plot(CX(K0,:)',CY(K0,:)','k-.','LineWidth',3)
axis([-1 1 -0.6 0.6])
%legend('t=0.0','t=0.4','t=1.08','t=6.0')
legend('Direct','Opposite')
xlabel('x'), ylabel('y')
%title(sprintf('Dynamic,Bm=%3.1f,wt=%2.0f,. . .
eps=%5.2d,ymax=%5.4',Bm,wt,ae,ym))
title(sprintf('Droplets,Bm=...
%3.1f,wt=%2.0f,eps=%5.2d',Bm,wt,ae))
%XX=[0;x;0]; % for dynamics
%figure,plot(XX',[0;Y(1,:)';0]','k--','LineWidth',3)
%hold on
%plot(XX',[0;Y(dK0,:)';0]','k--','LineWidth',1)
%hold on
%plot(XX',YY(dK,:)','k--','LineWidth',1.5)
%hold on
%plot(XX',YY(2*dK,:)','k-','LineWidth',1.5)
%hold on
%plot(XX',YY(K0,:)','k-.','LineWidth',3)
%axis([0 2 0 20])
%legend('t=0','t=0.2','t=0.4','t=1.08','t=6.0')
%xlabel('l'), ylabel('\beta')
title(sprintf('Beta dynamics,wt=%3.0f,. . .
Bm=%2.1f,eps=%5.2d,ymax=%5.4d',wt,Bm,ae,ym))
function F=SIST(t,y,A,wt,Bm,B1,E1,h,ae)
F=inv((E1+B1*ae))*(wt+A*(y/Bm+sin(2*y)));% with  epsilon
```

2.6 Ill-posed problem: H. Kalis, S. Rogovs et al., 2015 [76]

Many applied problems formulated as inverse problems of mathematical physics belong to the class of problems that are ill-posed in the classical sense. An inverse problem assumes a direct problem that is a well-posed problem of mathematical physics.
If we know completely a physical device, we have a classical mathematical
description of this device including uniqueness, stability and existence of a solution of the corresponding mathematical problem. But if one of the parameters desribing this device is to be found from experimental data, then we arrive at an inverse problem.
For their approximate solution regularization methods (approximation by welll-posed problems) are widely used. The theorie of stable numerical solutions of ill-posed problems using regularization was developed in the 1950-1960 s John and Tikhonov.

For inverse problems for time-dependent equations the generalized inverse method or quasi-reversibility (R. Lattes and J-L. Lions [28]) of the backward parabolic equation is also used. In book [28] for solving
physically unstable retrospective problem (ill-posed time reverse problem for backward parabolic equation) of the linear heat transfer equation is added the regularization term $\varepsilon \frac{\partial^4 u}{\partial x^4}$,
where $u = u(x,t)$ is the solution of the heat transfer equation with homogenous boundary condition (BCs) of the first kind and ε is small coefficient.

In paper [27] the backward parabolic equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, t \in (0,T)$ is regularizing by pseudo-parabolic equation $\frac{\partial}{\partial t}(u_\varepsilon + \varepsilon \frac{\partial^2 u_\varepsilon}{\partial x^2}) = \frac{\partial^2 u_\varepsilon}{\partial x^2}$.
When approximate solving ill-posed problems the choice of regularization parameter ε must correspond to the amount of error in the input data. Here we merely construct stable computational algorithms for ill-posed time-dependent problems and the influence of the regularization parameter only on the stability of the corresponding difference scheme. Inverse problems for partial differential equations and their methods of regularization are considered in the book by V. Isakov [29].

The nonlinear heat transfer ill-posed problem with the heat conductivity in form of trigonometrical function was derived in [26]. Its

numerical solution was regularized with two methods: by introducing the differential operator of higher order in the following form $\varepsilon\frac{\partial^5 u}{\partial^4 x \partial t}$ and constructing monotonous continuous functions.
Here similar simple nonlinear ill-posed problem with heat conductivity in form of power function is consider.

2.6.1 Mathematical model

Similarly [26] for numerical simulation the magnetic droplet dynamics in a rotating field we consider simplest nonlinear problem of heat transfer partial differential equation (PDE) in following form:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 F(u(x,t))}{\partial x^2} - \varepsilon\frac{\partial^5 u(x,t)}{\partial^4 x \partial t} + \\ g_0, x \in (0,L), t \in (0,T), \\ u(0,t) = u(L,t) = 0, \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} t \in [0,T), \\ u(x,0) = u_0(x), x \in [0,L] \end{cases} \tag{2.10}$$

where $F(u) = 0.4u^3 - 1.8u^2 + 2.4u$ is the nonlinear function-the polynomial of the third order,
ε is a small coefficient, $g_0 = const \geq 0$ is the constant heat source term, T, L are the final time for stationary solution and the length,
$u_0(x)$ is continuously differentiable function with $u_0(0) = u_0(L) = 0$.
In [26] the nonlinear function is in the form $F(u) = \frac{1}{Bm}u + \sin(2u)$, where Bm is the magnetic Bond number, $g = \omega\tau$, ω - angular frequency, τ - time scale, ε is the parameter for the regularization equal about 10^{-4} obtained from physical considerations .
The function $F(u)$ is not monotonic with $F'(1) = F'(2) = 0, F(1) = 1(max), F(2) = 0.8(min)$ (see Fig. 2.42).

The last term in the equations (2.10) is used for the regularization of the numerical calculations.
By setting $\varepsilon = 0$ we obtain the ill-posed problem.
In the numerical experiments by changing the right side constant (increasing or decreasing) g_0 we can take $u_0(x)$ equal to the stationary solution obtained at the previous asssigned value of g_0.

If $F(u) = gu, g = const > 0$, then we have the linear problem of heat tranfer equation with the constant source term g_0. If $g_0 = 0$, then we have the following simple linear problem for homogenous heat transfer equation:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = g\frac{\partial^2 u(x,t)}{\partial x^2}, x \in (0,L), t \in (0,T), \\ u(0,t) = u(L,t) = 0, t \in [0,T), \\ u(x,0) = u_0(x), x \in [0,L] \end{cases} \tag{2.11}$$

Here $u_0(x)$ is continuously differentiable function.

In book [28] the method of quasi-reversibility has been suggested for solving physically unstable retrospective problem of the linear homogenous heat transfer equation (ill-posed time reverse problem). The problem is replaced by regularized "higher"-order equation with the added regularization term $\varepsilon\frac{\partial^4 u}{\partial^4 x}$.

The retrospective or reverse in the time problem for linear homogenous backward heat tranfer equation is in following form:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = g\frac{\partial^2 u(x,t)}{\partial x^2}, x \in (0,L), t \in (T,0), \\ u(0,t) = u(L,t) = 0, t \in [T,0], \\ u(x,T) = u_T(x), x \in [0,L], \end{cases} \tag{2.12}$$

where the initial function $u_T(x), u_T(0) = u_T(L) = 0$ by $t = T$ is given the final data. This function also can be obtained by solving t he direct problem (2.11) for $t \in [0,T]$.

The obtained solution $u(x,0)$ for the backward parabolic problem (2.12) for $t = 0$ need compare with the initial function $u_0(x)$. The inverse (retrospective) problem (2.12) is ill-posed as it is unstable with respect to relative small perturbations of the initial data.

Using the **Fourier's series**
$u(x,t) = \sum_{k=1}^{\infty} a_k(t)w_k(x), w_k(x) = \sqrt{2/L}\sin\frac{k\pi x}{L}$
we obtain that the Fourier coefficients of the solution are
$a_k(t) = \exp(g\lambda_k(T-t))a_{Tk}$, where $a_{Tk} = \int_0^L u_T(x)w_k(x)dx, \lambda_k = (\frac{k\pi}{L})^2$.
The coefficients $a_k(0)$ are uniquely determined from the relations
$a_k(0) = \exp(g\lambda_k T)a_{Tk}$. These relations show that for any u_T a solution $u(x,t)$ does not exist, and when it does, it is exponentially unstable:
for u that is the k-th term of the sum $u(x,t) = \sum_{k=1}^{\infty} a_k(0)\exp(-g\lambda_k t)a_{Tk}w_k(x)$ with
$a_k(0) = \varepsilon\exp(g\lambda_k T)$ we have for the norm for this term in $L_2[0,L]$ - $||u(0)|| = \varepsilon\exp(g\lambda_k T)$, while $||u_T|| = \varepsilon$.

The ill-posed problem (2.12) we can regularizated in following form:

$$\begin{cases} \frac{\partial u_\varepsilon(x,t)}{\partial t} = g\frac{\partial^2 u_\varepsilon(x,t)}{\partial x^2} + \varepsilon_1 \frac{\partial^4 u_\varepsilon(x,t)}{\partial x^4} - \varepsilon_2 \frac{\partial^6 u_\varepsilon(x,t)}{\partial x^6}, x \in (0,L), t \in (T,0), \\ u_\varepsilon(0,t) = u_\varepsilon(L,t) = 0, \frac{\partial^2 u_\varepsilon(0,t)}{\partial x^2} = \frac{\partial^2 u_\varepsilon(L,t)}{\partial x^2} = 0, \\ \frac{\partial^4 u_\varepsilon(0,t)}{\partial x^4} = \frac{\partial^4 u_\varepsilon(L,t)}{\partial x^4} = 0, \\ u_\varepsilon(x,T) = u_T(x), x \in [0,L], \end{cases} \tag{2.13}$$

where $\varepsilon_1 \geq 0, \varepsilon_2 \geq 0$ are small coefficients.
In book [28] is analysed the solution with $\varepsilon_2 = 0$.
Then $a_k(t) = \exp(g\lambda_k - \varepsilon_1 \lambda_k^2)(T-t))a_{Tk}$ and the series for $u(l,t)$ is convergent in $L_2(0,2)$ for any u_T in this space and any $t < T$.
Moreover, when ε_1 goes to 0, the regularized solutions are convergent to the solutions u of the initial problem (2.12) [29].
If $\varepsilon_2 \neq 0$ then $a_k(t) = \exp(g\lambda_k - \varepsilon_1 \lambda_k^2 - \varepsilon_2 \lambda_k^3)(T-t))a_{Tk}$ and we have exponentially stable solution when $\varepsilon_1 \lambda_1 + \varepsilon_2 \lambda_1^2 \geq g$.

For the comparison the values of the maximal error $\delta = max|u_\varepsilon(x,0) - u_0(x)|$ we consider also following regularization problem:

$$\begin{cases} \frac{\partial u_\varepsilon(x,t)}{\partial t} = g\frac{\partial^2 u_\varepsilon(x,t)}{\partial x^2} + \varepsilon_1 \frac{\partial^4 u_\varepsilon(x,t)}{\partial^4 x} - \varepsilon_2 \frac{\partial^5 u_\varepsilon(x,t)}{\partial x^4 \partial t}, x \in (0,L), t \in (T,0), \\ u_\varepsilon(0,t) = u_\varepsilon(L,t) = 0, \frac{\partial^2 u_\varepsilon(0,t)}{\partial x^2} = \frac{\partial^2 u_\varepsilon(L,t)}{\partial x^2} = 0, t \in [T,0], \\ u_\varepsilon(x,T) = u_T(x), x \in [0,L]. \end{cases} \tag{2.14}$$

Then $a_k(t) = \exp((g\lambda_k - \varepsilon_1 \lambda_k^2)/(1 + \varepsilon_2 \lambda_k^2))(T-t))a_{Tk}$ and we have exponentially stable solution when $\varepsilon_1 \lambda_1 \geq g$.

Using the simple regularization from H. Gajevski, K. Zacharias [27] we have for problem (2.12) following problem ($g > 0$):

$$\begin{cases} \frac{\partial u_\varepsilon(x,t)}{\partial t} = g\frac{\partial^2 u_\varepsilon(x,t)}{\partial x^2} - \varepsilon \frac{\partial^3 u_\varepsilon(x,t)}{\partial x^2 \partial t}, x \in (0,L), t \in (T,0), \\ u_\varepsilon(0,t) = u_\varepsilon(L,t) = 0, u_\varepsilon(x,T) = u_T(x), x \in [0,L], \end{cases} \tag{2.15}$$

$a_k(t) = \exp(\frac{-g\lambda_k}{1-\varepsilon\lambda_k} t)a_k(0)$ and we have exponentially stable solution for $g < 0$ when $\varepsilon_1 \lambda_1 \geq 1$.

We can consider also parabolic equation direct in the time $\bar{t} = T - t$. Then we have ill-posed problem which differs from the standart parabolic equation only in the sign of the derivatives in space $\frac{\partial}{\partial t} = -\frac{\partial}{\partial \bar{t}}$, [28].
In this case we have ill-posed direct problem with the negative coefficients g of the heat conductivity. Then we obtain from direct problem

(2.15) $a_k(t) = \exp(\frac{-g\lambda_k}{1-\varepsilon\lambda_k}t)a_k(0)$ and we have exponentially stable solution for $g < 0$ when $\varepsilon_1\lambda_1 \geq 1$.

$a_k(t) = \exp(\frac{g\lambda_k}{1-\varepsilon\lambda_k}(T-t))a_{Tk}$ and we have exponentially stable solution when $\varepsilon_1\lambda_1 \geq 1$.

2.6.2 Some theoretical estimations

Using estimation [29]

$$||u(t)|| \leq ||u(0)||^{1-t/T}||u(T)||^{t/T}. \tag{2.16}$$

then the stability for the class of bounded solutions follows from this estimate. Corresponding estimation can not be obtained from the problem (2.14. This estimate remained for the direct ill-posed problem obtained from (2.13) with the transformation $\bar{t} = T - t$.
Using the **regularization** (2.13) with $\varepsilon_2 = 0$ for the direct nonlinear problem (2.10) we have the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(g(u)\frac{\partial u}{\partial x}\Big) - \varepsilon_1\frac{\partial^4 u}{\partial x^4} + g_0 \tag{2.17}$$

and following integral identity

$$\frac{\partial}{2\partial t}\int_0^L u^2 dx + \int_0^L g(u)\Big(\frac{\partial u}{\partial x}\Big)^2 dx + \varepsilon_1\int_0^L \Big(\frac{\partial^2 u}{\partial x^2}\Big)^2 dx = g_0\int_0^L u dx, \tag{2.18}$$

where $g(u) = F'(u) = 1.2u^2 - 3.6u + 2.4, -0.3 \leq g(u), g'(u) = 2.4u - 3.6$.
For **investigating the solvability** of the corresponding initial-value problem in the Sobolev space $\overset{0}{W}{}_2^2$
for the weak solution and for obtaining the apriori estimations for fixed t we need determine the parameter ε_1 from following inequality:
$\int_0^L g(u)\Big(\frac{\partial u}{\partial x}\Big)^2 dx + \varepsilon_1\int_0^L\Big(\frac{\partial^2 u}{\partial x^2}\Big)^2 dx \geq k_0\int_0^L\Big(\frac{\partial u}{\partial x}\Big)^2 dx$, or $\varepsilon_1 \geq \kappa = \max I(u)$,
where $I(u) = \int_0^L((k_0 - g(u))(\frac{\partial u}{\partial x})^2)dx / \int_0^L(\frac{\partial^2 u}{\partial x^2})^2 dx, u \in \overset{0}{W}{}_2^2$.

From $\frac{dI(u+\varepsilon\phi)}{d\varepsilon} \to 0, \varepsilon \to 0$($\phi$ is arbitrary function $\in \overset{0}{W_2^2}$) follows the integral equation

$$-2\int_0^L \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 \phi}{\partial x^2}dx + \frac{1}{\kappa}\int_0^L \Big(-g'(u)\Big(\frac{\partial u}{\partial x}\Big)^2\phi + 2(k_0 - g(u))\frac{\partial u}{\partial x}\frac{\partial \phi}{\partial x}\Big)dx = 0,$$

or

$$\int_0^L \Big(\frac{\partial^4 u}{\partial x^4} + \frac{1}{\kappa}\Big(0.5g'(u)\Big(\frac{\partial u}{\partial x}\Big)^2 + (k_0 - g(u))\frac{\partial^2 u}{\partial x^2}\Big)\Big)\phi dx = 0.$$

Since arbitrary ϕ is arbitrary for fixed t the nonlinear differential equation follows

$$\frac{\partial^4 u}{\partial x^4} + \frac{1}{\kappa}\Big(0.5g'(u)\Big(\frac{\partial u}{\partial x}\Big)^2 + (k_0 - g(u))\frac{\partial^2 u}{\partial x^2}\Big) = 0. \tag{2.19}$$

The **numerical solution** by $L = 2$ with Matlab solver "bvp4c" (5 boundary conditions:
$u(0,t) = u(L,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} = 0, \frac{\partial u(0,t)}{\partial x} = b_c > 0$) is $k_0 = 1.29, \kappa = 0.0051, b_c = 0.55$ (for this values k_0 and b_c the minimal value of κ is obtained).
If $g(u) = -|g| = const < 0$ then we have $\frac{k_0+|g|}{\kappa} = \lambda_1$, where $\lambda_1 = \frac{\pi^2}{L^2}$ is the first eigenvalues of the differential operator $-\frac{\partial}{\partial x}$ for homogenous BCs. Therefore,

$$\kappa = \frac{(k_0 + |g|)L^2}{\pi^2}.$$

Using Matlab solver "bvp4c" with $g = -1, k_0 = 0, L = 2$ from (2.19) get $\kappa = 0.4053 \approx \frac{4}{\pi^2}$.

For the **Fourier's series** $u(x,t) = \sum_{k=1}^{\infty} a_k(t)w_k(x)$, we obtain for constant function g,
$\frac{da_k(t)}{dt} = -g\lambda_k a_k(t) - \varepsilon_1(\lambda_k)^2 a_k(t) + b_k, b_k = g_0\int_0^L w_k(x)dx = g_0\frac{L}{k\pi}(1 - (-1)^k)$,
or $a_k(t) = \exp(\rho_k t)a_k(0) + \frac{b_k}{\rho_k}(\exp(\rho_k t) - 1)$,
where $\rho_k = -g\lambda_k - \varepsilon_1(\lambda_k)^2, a_k(0) = \int_0^L u(x,0)w_k(x)dx$.
We have bounded solution for $g = -|g| < 0$, when $\rho_k \leq 0$ or $\varepsilon_1 \geq$

$\max \frac{|g|}{\lambda_k} = \frac{|g|}{\lambda_1} = \frac{L^2|g|}{\pi^2}$.

From (2.18) using Hölder's inequality $|\int_0^L u(x,t)dx| \leq \sqrt{L}||u(t)||$ follows the inequality

$$\frac{d}{2dt}||u(t)||^2 + k_0||u_x(t)||^2 \leq \sqrt{L}g_0||u(t)||. \quad (2.20)$$

Using Friedrichs inequality [29] $||u(t)||^2 \leq \frac{L^2}{\pi^2}||u_x(t)||^2$ we obtain
$\frac{d||u(t)||}{dt} + \frac{k_0\pi^2}{L^2}||u(t)|| \leq \sqrt{L}g_0$ or
$||u(t)|| \leq ||u(0)||\exp(-\frac{k_0\pi^2}{L^2}t) + \sqrt{L}g_0\frac{L^2}{k_0\pi^2}(1-\exp(-\frac{k_0\pi^2}{L^2}t))$.
Here $||u(t)|| = (\int_0^L u(x,t)^2dx)^{1/2}$, $||u_x(t)|| = (\int_0^L(\frac{\partial u(x,t)}{\partial x})^2dx)^{1/2}$.
For the stationary solutions $u_s(x)$ follows the estimation $||u_s|| \leq C_s$, where $C_s = \sqrt{L}g_0\frac{L^2}{k_0\pi^2}$.
Using Matlab by $N = 100, L = 2(\varepsilon_1 = 0.001$, see chapter 3) we obtain for $g_0 = 5 : ||u_s|| = 2.21(C_s = 2.23, \kappa = 0.0051, k_0 = 1.29)$.

Using the **regularization** (2.13) with $\varepsilon_1 = 0$ for the direct nonlinear problem (2.10) we have the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(g(u)\frac{\partial u}{\partial x}\Big) + \varepsilon_2\frac{\partial^6 u}{\partial x^6} + g_0 \quad (2.21)$$

and following integral identity

$$\frac{\partial}{2\partial t}\int_0^L u^2dx + \int_0^L g(u)\Big(\frac{\partial u}{\partial x}\Big)^2dx + \varepsilon_2\int_0^L\Big(\frac{\partial^3 u}{\partial x^3}\Big)^2dx = g_0\int_0^L udx. \quad (2.22)$$

For **obtaining the apriori estimations** we need determine the parameter ε_2 from following inequality: $\varepsilon_2 \geq \kappa = \max I(u)$,
where $I(u) = \int_0^L\Big((k_0 - g(u))(\frac{\partial u}{\partial x})^2\Big)dx / \int_0^L\Big(\frac{\partial^3 u}{\partial x^3}\Big)^2dx, u \in \overset{0}{W}{}_2^2$.
From $\frac{dI(u+\varepsilon\phi)}{d\varepsilon} \to 0, \varepsilon \to 0$ follows the nonlinear differential equation in the form:

$$\frac{\partial^6 u}{\partial x^6} - \frac{1}{\kappa}\Big(0.5g'(u)\Big(\frac{\partial u}{\partial x}\Big)^2 + (k_0 - g(u))\frac{\partial^2 u}{\partial x^2}\Big) = 0. \quad (2.23)$$

The **numerical solution** by $L=2$ with Matlab solver "bvp4c" (7 BCs: $u(0,t)=u(L,t)=\frac{\partial^2 u(0,t)}{\partial x^2}=\frac{\partial^2 u(L,t)}{\partial x^2}=\frac{\partial^4 u(0,t)}{\partial x^4}=\frac{\partial^4 u(L,t)}{\partial x^4}=0, \frac{\partial u(0,t)}{\partial x}=b_c$) is $k_0=1.0, \kappa=0.0041, b_c=0.7$.
If $g(u)=-|g|=const<0$ then we have $\frac{k_0+|g|}{\kappa}=\lambda_1^2$. Therefore,

$$\kappa=\frac{(k_0+|g|)L^4}{\pi^4}.$$

Using Matlab solver "bvp4c" with $g=-1, k_0=0, L=2$ from (2.23) get $\kappa=0.1643\approx\frac{16}{\pi^4}$.

From (2.22) using Hölder's inequality and Friedrichs inequality we obtain
the inequality (2.20) and previous estimates for $||u(t)||$ and $||u_s||$.
Using the **Fourier's series** we obtain for constant function g,
$\frac{da_k(t)}{dt}=-g\lambda_k a_k(t)-\varepsilon_2(\lambda_k)^3 a_k(t)+b_k$, $b_k=g_0\frac{L}{k\pi}(1-(-1)^k)$,
or $a_k(t)=\exp(\rho_k t)a_k(0)+\frac{b_k}{\rho_k}(\exp(\rho_k t)-1)$,
where $\rho_k=-g\lambda_k-\varepsilon_2(\lambda_k)^3, a_k(0)=\int_0^L u(x,0)w_k(x)dx$.
We have bounded solution for $g=-|g|<0$, when $\rho_k\leq 0$ or $\varepsilon_2\geq \max\frac{|g|}{\lambda_k^2}=\frac{|g|}{\lambda_1^2}=\frac{L^4|g|}{\pi^4}$.

Using the **regularization** (2.13) with $\varepsilon_1\neq 0, \varepsilon_2\neq 0$ for the direct nonlinear problem (2.10) we have the PDE

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(g(u)\frac{\partial u}{\partial x}\right)-\varepsilon_1\frac{\partial^4 u}{\partial x^4}+\varepsilon_2\frac{\partial^6 u}{\partial x^6}+g_0 \tag{2.24}$$

and following integral identity

$$\begin{cases} \frac{\partial}{2\partial t}\int_0^L u^2 dx+\int_0^L g(u)\left(\frac{\partial u}{\partial x}\right)^2 dx+\varepsilon_1\int_0^L\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx+ \\ \varepsilon_2\int_0^L\left(\frac{\partial^3 u}{\partial x^3}\right)^2 dx=g_0\int_0^L u dx. \end{cases} \tag{2.25}$$

For **obtaining the apriori estimations** we need determine the parameter ε_2 from previous inequality: $\varepsilon_2\geq\kappa$,
where the parameter κ can be obtained from the equation (2.23). We have the inequality (2.20), where k_0 is replaced with $k_0+\varepsilon_1$.

2.6.3 Appoximations and solution of the problems

Analysed the nonstationary solution of the problem (2.10) the limit case - the stationary solution is consider.

The stationary solution of the problem (2.10).

The stationary solution $u_s(x)$ of the problem (2.10) can be obtained from following transcendental equation $F(u_s(x)) = 0.5g_0x(L-x)$ by fixed values of g_0 and $x \in (0,L)$. The maximal value $u_m = u_s(x)$ is the solution of the transcendental equation $F(u_m) = \frac{L^2}{4}0.5g_0$.

The solution $(u(x,t) \geq 0)$ is symmetrical with a respect to $x = L/2:$ $u(L/2-x_1,t) = u(L/2+x_1,t), x_1 \in (0,L/2)$ or $\frac{\partial u(L/2,t)}{\partial x} = 0$.

The $u(x,t)$ as function of the variable x is discontinuous for $g_0 = \frac{4}{L^2}2F(u_*)$, where $u_* = 1, u_* = 2$ are the roots of equation $F'(u_*) = 0$ (the local maxima or minimum of the function $F(u)$). The ill-posed problem (2.10) with $\varepsilon = 0$ similarly [26] get also the regularization of the otherwise using the modifying of the function $F(u)$ in such a way, that in the intervals where the function is with the derivative $F'(u) < 0$ the function is replaced with constant value. There are two possible variations depending on the behaviour of the value g_0 (direct for reverse f

functions for increasing or decreasing g_0, see Figs. 2.43, 2.44).

The **stationary solutions** with one jump are shown in Fig. 2.45 (direct function) and Fig. 2.46 (reverse function) for $g_0 = 3$. For the stationary solutions depending on the value of g_0 we can obtain one or two solutions. The modified functions F, f are continous and monotonous with discontinous first derivatives.

This stationary solutions can be obtained also with numerical simulation as the limit of the nonstationary solution of (2.10) with large time moment.

Metod of lines and finite differences approximations.

The problems (2.10-2.14) are solved numerically using the method of lines and 3 way finite difference methods for the approximation of spatial derivatives: local approximation with finite differences in uniform grid (LAU) and global approximation with derivatives matrices in nonuniform (GAN) and uniform (GAU) grids.

1. For **local approximation** LAU we consider the uniform grid in the space $x_j = jh, j = \overline{0,N}, Nh = L$.

Using the finite differences of second order approximation for partial derivatives of second and fourth order with respect to x, we obtain from (2.10) the Cauchy problem for the system of nonlinear ODEs of the first order in the following matrix form

$$\begin{cases} (E+\varepsilon B)\dot{U}(t)+AF(U(t))=G, t\in(0,T), \\ U(0)=U_0, \end{cases} \tag{2.26}$$

where E is the unit matrix of $N-1$ order, A is the standar 3-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^2}\{-1;2;-1\}$ approximating the derivative of the second order $-\frac{\partial^2}{\partial x^2}$, $B=A^2$ is the 5-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^4}\{1;-4;6;-4;1\}$, approximating the derivative $\frac{\partial^4}{\partial x^4}$ with second order, the first and last elements of matrix B are $B(1,1)=B(N-1,N-1)=5/h^4$, using the approximation of the BCs
$\frac{\partial^2 u(0,t)}{\partial x^2}=\frac{\partial^2 u(L,t)}{\partial x^2}=0$ with second order (we use following finite difference expressions $u_0(t)=u_N(t)=0, u_{-1}(t)=-u_1(t), u_{N+1}(t)=-u_{N-1}(t)$),
$U(t), \dot{U}(t), U_0, F(U), G$ are the column-vectors of $N-1$ order with the elements $u_j(t)\approx u(x_j,t),\ \dot{u}_j(t)\approx\frac{\partial u(x_j,t)}{\partial t}, u_j(0)=u_0(x_j), f_j\approx F(u_j(t)), g_j=g_0, j=\overline{1,N-1}$.

Using the matrix form $A+\frac{h^2}{12}B$ for **approximation with fourth order** of derivative of the second order [3] we can obtain the following problem:

$$\begin{cases} (E+\varepsilon B)\dot{U}(t)+(A+\frac{h^2}{12}B)F(U(t))=G, \\ U(0)=U_0, \end{cases} \tag{2.27}$$

Concerning the R.Lattes and J.L.Lions regularization we can consider the following initial value problem:

$$\begin{cases} \dot{U}(t)+(A+\frac{h^2}{12}B)F(U(t))+\varepsilon_1 BU(t)-\varepsilon_2 CU(t)=G, \\ U(0)=U_0 \end{cases} \tag{2.28}$$

or

$$\begin{cases} \dot{U}(t)+AF(U(t))+\varepsilon_1 BU(t)-\varepsilon_2 CU(t)=G, \\ U(0)=U_0, \end{cases} \tag{2.29}$$

where $B=A^2, C=A^3$.

Using regularization from [27] (2.15) we have following problem:

$$\begin{cases} (E-\varepsilon A)\dot{U}(t)+AF(U(t))=G, \\ U(0)=U_0. \end{cases} \quad (2.30)$$

For the problem (2.27) we have the linear system of ODEs (2.26) with $F(U)=gU, \varepsilon=0, G=0$. In this case the solution can be obtained in the matrix form $U(t)=exp(-Agt)U_0$ and $U(T)=U_T=exp(-AgT)U_0$, where U_T is the column-vector of $N-1$ order with the elements $u_T(x_j)$.

2. For **global approximations** GAN we consider nonuniform grid with the grid points of the roots of the Chebyshev polynomials of the second kind

$$x_j=0.5L(1-\cos(\pi j/N)), \quad j=\overline{0,N}. \quad (2.31)$$

Using this grid points we can approximate the derivatives
$\frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^4}{\partial x^4}$, $\frac{\partial^6}{\partial x^6}$
with matrix $\bar{D}, \bar{D}^2, \bar{D}^4, \bar{D}^6$ of derivatives in the form

$$u_h'=\bar{D}u_h, u_h''=\bar{D}^2u_h, u_h^4=\bar{D}^4u_h, u_h^6=\bar{D}^6u_h, \quad (2.32)$$

where $u_h=(u_0,u_1,\ldots,u_N)$, $u_h'=(u_0',u_1',\ldots,u_N')$, etc.
are the column-vectors of the coresponding values u_j depending on t: $u_j \approx u(x_j,t)$, $u_j' \approx \frac{\partial u(x_j,t)}{\partial x}$, etc.
From the Lagrange interpolation follows, that the elements of matrix $\bar{D}$ are in the form

$$d_{j,k}=\frac{dl_k(x_j)}{dx}, j,k=\overline{0,N}, \quad (2.33)$$

where $l_k(x)=\frac{\omega(x)}{\omega'(x_k)(x-x_k)}$ are the elementary Lagrange multiplyers, $\omega=\prod_{k=0}^{N}(x-x_k)$.
For this nonuniform grid the interpolation error is small.
The determinants of derivatives matrix $\bar{D}^2$ are equal to zero
(this matrix are singular). Therefore to need decrease the orders of this matrix by deleting the first columns and corresponding rows. Then we have the matrix $A=-\bar{D}^2$ and $A^2=(-\bar{D}^2)^2$ with $N-1$ order.
Similarly we can consider the global approximation GAU in uniform grid.

Using the finite differences of second order approximation for partial derivatives of sixth order with respect to x and $A^3=(-\bar{D}^2)^3$, we obtain from (2.13) the initial value problem for the system of linear ODEs in the following form

$$\begin{cases} \dot{U}_\varepsilon(t) + gAU_\varepsilon(t) - \varepsilon_1 A^2 U_\varepsilon(t) - \varepsilon_2 A^3 U_\varepsilon(t) = 0, t \in (T,0) \\ U_\varepsilon(T) = U_T, \end{cases} \tag{2.34}$$

or

$$\begin{cases} \dot{U}_\varepsilon(t) + g(A + \frac{h^2}{12}B)U_\varepsilon(t) - \varepsilon_1 BU_\varepsilon(t) - \varepsilon_2 CU_\varepsilon(t) = 0, t \in (T,0) \\ U_\varepsilon(T) = U_T, \end{cases} \tag{2.35}$$

where $B = A^2$ and $C = A^3$ for LAU is the 7-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^6}\{-1;6;-15;20;-15;6;-1\}$ approximating the derivative $-\frac{\partial^6}{\partial x^6}$ with the second order, the first and last elements of matrix C are $C(1,1) = C(N-1,N-1) = 14/h^4, C(1,2) = C(N-1,N-2) = C(2,1) = C(N-2,N-1) = -14/h^4$
the approximation of the boundary conditions $\frac{\partial^4 u_\varepsilon(0,t)}{\partial x^4} = \frac{\partial^4 u_\varepsilon(L,t)}{\partial x^4} = 0$ with the second order (we use following finite difference expressions $u_0(t) = u_N(t) = 0, u_{-1}(t) = -u_1(t), u_{-2}(t) = -u_2(t), u_{N+1}(t) = -u_{N-1}(t), u_{N+2}(t) = -u_{N-2}(t),$), $U_\varepsilon(t), \dot{U}_\varepsilon(t), U_T$
are the column-vectors of $N-1$ order with the elements $u_{\varepsilon j}(t) \approx u_\varepsilon(x_j,t)$, $\dot{u}_{\varepsilon j}(t) \approx \frac{\partial u_\varepsilon(x_j,t)}{\partial t}, u_{Tj} = u_T(x_j), j = \overline{1,N-1}$.

For the nonlinear direct problem (equation 2.24) we have following initial value problem for the system of nonlinear ODEs :

$$\begin{cases} \dot{U}_\varepsilon(t) + AF(U_\varepsilon(t)) + \varepsilon_1 A^2 U_\varepsilon(t) - \varepsilon_2 A^3 U_\varepsilon(t) = G, t \in (0,T) \\ U_\varepsilon(0) = U_0. \end{cases} \tag{2.36}$$

For the **bounded solution** of (2.34) can be obtained the estimate (2.16). For the discrete functions v,u we define the scalar product $(v,u) = h\sum_{j=1}^{N-1} v_j u_j, (v,u] = h\sum_{j=1}^{N} v_j u_j$ and difference operators $\Lambda u_j = u_{x,\bar{x}_j} = \frac{1}{h^2}(u_{j-1} - 2u_j + u_{j+1}), \Lambda u = -Au$ (for the approximation of the second order derivatives) and $u_{\bar{x}_j} = \frac{1}{h}(u_j - u_{j-1}), u_{x_j} = \frac{1}{h}(u_{j+1} - u_j)$
(for the approximation of the first order derivatives). Then for $u_0 = u_N = 0$ follows $(\Lambda u, u) = -(u_{\bar{x}}, u_{\bar{x}}]$ [3].
Similarly $(\Lambda^2 u, u) = (\Lambda u_{\bar{x}}, \Lambda u_{\bar{x}}]$,$(\Lambda^3 u, u) = (\Lambda^2 u_{\bar{x}}, \Lambda^2 u_{\bar{x}}]$ if $\Lambda u_0 = \Lambda u_N = \Lambda^2 u_0 = \Lambda^2 u_N = u_0 = u_N = 0, u_{-1} = -u_1, u_{-2} = -u_2, u_{N-1} =$

$-u_{N+1}, u_{N-2} = -u_{N+2}$.
Then the ODEs (2.24) we can rewriten in following form:

$$\dot{u} = g\Lambda u + \varepsilon_1 \Lambda^2 u - \varepsilon_2 \Lambda^3 u.$$

For the squared norm $f(t) = ||u(t)||^2 = (u(t), u(t))$ we can proved that the logarithm $K(t) = \ln(f(t))$ is a convex function or $K''(t) \geq 0$.
By using the differential equation we obtain $f' = 2(u, \dot{u}) = 2(u, g\Lambda u + \varepsilon_1 \Lambda^2 u - \varepsilon_2 \Lambda^3 u) = -2(g(u_{\bar{x}}, u_{\bar{x}}] + \varepsilon_1(\Lambda u_{\bar{x}}, \Lambda u_{\bar{x}}] - \varepsilon_2(\Lambda^2 u_{\bar{x}}, \Lambda^2 u_{\bar{x}}])$.
Further,
$f'' = -4(g(u_{\bar{x}}, \dot{u}_{\bar{x}}] + \varepsilon_1(\Lambda u_{\bar{x}}, \Lambda \dot{u}_{\bar{x}}] - \varepsilon_2(\Lambda^2 u_{\bar{x}}, \Lambda^2 \dot{u}_{\bar{x}}]) =$
$4(g(\Lambda u, \dot{u}) + \varepsilon_1(\Lambda^2 u, \dot{u}) - \varepsilon_2(\Lambda^3 u, \dot{u})) =$
$4(g\Lambda u + \varepsilon_1 \Lambda^2 u - \varepsilon_2 \Lambda^3 u, \dot{u}) = 4(\dot{u}, \dot{u}) = 4||\dot{u}||^2$.
Now, we have $f'' f - (f')^2 = 4||u||^2 ||||\dot{u}||^2 - 4(u, \dot{u})^2 \geq 0$ according to the discrete Schwarz inequality.
Therefore, $K(t) \leq (1 - t/T)K(0) + t/TK(T)$, $f(t) \leq f(0)^{1-t/T} f(T)^{t/T}$ and we obtain the estimate (2.16).

The solution of (2.35)can be obtained in the following matrix form $U_\varepsilon(t) = exp(C_\varepsilon(t - T)U_T$ and

$$U_\varepsilon(0) = exp(-TC_\varepsilon)U_T = exp(-T(C_\varepsilon + A))U_0, \tag{2.37}$$

where $C_\varepsilon = -gA + \varepsilon_1 A^2 + \varepsilon_2 A^3$. Similarly we obtain from (2.14) the initial value problem for the system of linear ODEs in the following

$$\begin{cases} (E + \varepsilon_2 A^2)\dot{U}_\varepsilon(t) + gAU_\varepsilon(t) - \varepsilon_1 A^2 U_\varepsilon(t) = 0, t \in (T, 0) \\ U_\varepsilon(T) = U_T. \end{cases} \tag{2.38}$$

In this case the matrix $C_\varepsilon = (E + \varepsilon_2 A^2)^{-1}(-gA + \varepsilon_1 A^2)$ in the solution (2.37).

2.6.4 The spectral representation for LAU and discrete Fourier methods

The solution of the corresponding discrete spectral problem $Aw^k = \mu_k w^k, k = \overline{1, N-1}$ for matrix A obtained with LAU is orthonormed eigenvectors $w^k((w^k, w^m) = \sum_{j=1}^{N-1} w_j^k w_j^m = \delta_{k,m})$, with the elements $w_j^k = \sqrt{\frac{2}{N}} \sin \frac{\pi jk}{N}$, $j = \overline{1, N-1}$ (elements of the symmetri-

cal matrix W), and eigenvalues $\mu_k = \frac{4}{h^2}\sin^2\frac{k\pi}{2N}, k=\overline{1,N-1}$ [3].
In the matrix form we get

$$AW = WD, WW = E, W^{-1} = W, A = WDW,$$

where the elements of the diagonal matrix D is $d_k = \mu_k$.
The representations for matrix $A = WDW$ is equivalent with the standard 3-diagonal matrix of $N-1$ order with the elements $\frac{1}{h^2}\{-1;2;-1\}$ approximating the derivative of the second order $-\frac{\partial^2}{\partial x^2}$.
For **the difference scheme with exact spectrum** (FDSES) [26] the matrix A is replaced with the matrix WDW,
where the diagonal matrix D contains the first $N-1$ eigenvalues $\lambda_k = (\frac{k\pi}{L})^2$ of the differential operator $(-\frac{\partial^2}{\partial l^2})$. For FDS the elements of the diagonal matrix D is $d_k = \mu_k$.

For any matrix function $p(A)$ follows that $p(A) = Wp(D)W$. Then $A^2 = WD^2W, A^3 = WD^3W$ and from (2.37)
we get $U_\varepsilon(0) = Wexp(-T(D_\varepsilon + gD))WU_0$,
where $D_\varepsilon = -gD + \varepsilon_1 D^2 + \varepsilon_2 D^3$ for the problem (2.35) and $D_\varepsilon = (E + \varepsilon_2 D^2)^{-1}(-gD + \varepsilon_1 D^2)$ for the problem (2.38).
In the limit case $((\varepsilon_1, \varepsilon_2) \to 0)$ follows that $U_\varepsilon(0) \to U_0$.
This is only theoretical result for exact date without
the computer errors for numerical calculations.

Using the transformation $V = WU, V_\varepsilon = WU_\varepsilon (U = WV, U_\varepsilon = WV_\varepsilon)$ or $U_\varepsilon(t) = \sum_{k=1}^{N-1} v_{\varepsilon k}(t)w^k$ we obtain $V_\varepsilon(0) = exp(-T(D_\varepsilon + gD))V_0$
or $v_{\varepsilon k}(0) = exp(-T(d_{\varepsilon k} + gd_k))v_k(0), k = \overline{1,N-1}$, where $v_k, v_{\varepsilon k}$ are the elements of vectors V, V_ε, $d_{\varepsilon k} = -gd_k + \varepsilon_1 d_k^2 + \varepsilon_2 d_k^3$ for the problem (2.34) and $d_{\varepsilon k} = (1 + \varepsilon_2 d_k^2)^{-1}(-gd_k + \varepsilon_1 d_k^2)$ for the problem (2.38).
Therefore for the problem (2.35) $v_{\varepsilon k}(0) = exp(-T(\varepsilon_1 d_k^2 + \varepsilon_2 d_k^3))v_k(0)$ and for the problem (2.38) $v_{\varepsilon k}(0) = exp(-T((1 + \varepsilon_2 d_k^2)^{-1}(\varepsilon_1 d_k^2 + \varepsilon_2 d_k^3))v_k(0)$.
Similarly the analytical solutions of the problems (2.34, 2.38) are $(U_\varepsilon = WV_\varepsilon)$:

$$v_{\varepsilon k}(t) = exp(d_{\varepsilon k}(t-T))v_{Tk}, k = \overline{1,N-1}, \tag{2.39}$$

or $V_\varepsilon(t) = exp(D_\varepsilon(t-T))V_T, U_\varepsilon(t) = Wexp(D_\varepsilon(t-T))WU_T$,
where v_{Tk} is the element of vector $V_T = WU_T$ or $V_T = \sum_{k=1}^{N-1} u_{Tk}w^k$.
For the error $\delta = ||U_\varepsilon(0) - U_0|| = max|u_{\varepsilon k}(0) - u_k(0)|$

we have following estimate:

$$\delta \leq ||exp(-T(D_\varepsilon + gD)) - E||||U_0||.$$

For minimal error

$$|exp(-T\min_k(d_{\varepsilon k} + gd_k)) - 1|, (\min_k(d_{\varepsilon k} + d_k) = d_{\varepsilon 1} + d_1),$$

we need choose the parameters $\varepsilon_1^2 + \varepsilon_2^2 > 0$ that the numerical process is stable. Here

$$d_{\varepsilon 1} + gd_1 = \begin{cases} d_1^2(\varepsilon_1 + \varepsilon_2 d_1), & \text{for}(4.5) \\ d_1^2(\varepsilon_1 + \varepsilon_2 d_1)/(1 + \varepsilon_2 d_1^2), & \text{for}(4.7) \end{cases}$$

For the finite difference scheme with the exact spectrum (FDSES) or discrete Fourier method the diagonal matrix D contains the first $N-1$ eigenvalues $d_k = \lambda_k = (k\pi/L)^2$, $k=\overline{1,N-1}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$ correspondly (the eigenvectors remained).
For fixed eigenvector $U_0 = w^m, 1 \leq m \leq N-1$ we have
$U_T = exp(-d_m T)w^m$, $U_\varepsilon(0) = exp(-T(d_{\varepsilon m} + gd_m))U_0$
and the error $\delta_m = ||U_\varepsilon(0) - U_0|| = ||U_0(exp(-T(d_{\varepsilon m} + gd_m)) - 1)||$.

For the time reverse discrete problems (2.34,2.35) with $g = const > 0$ we analogously obtain that
$\varepsilon_1 \geq \frac{g}{\mu_1}$ or $\varepsilon_1 \geq \frac{g(1+\mu_1 h^2/12)}{\mu_1}$, if $\varepsilon_2 = 0$
and $\varepsilon_2 \geq \frac{g}{\mu_2}$ or $\varepsilon_2 \geq \frac{g(1+\mu_1 h^2/12)}{\mu_2}$, if $\varepsilon_1 = 0$.

2.6.5 Some numerical results

In the following examples we are shown the accuracy of the 4 way finite difference approximations (FDA) for boundary- value problems (BVP) of simple ODEs by $L = 2, N = 10$:
1) BVP $u''(x) = 1, u(0) = u(L) = 0$, exact solution- $u(x) = x^2/2 - x$, FDA $AU = -e_1$ with maximal errors- $8.3 * 10^{-16}$ (GAN), $4.2 * 10^{-15}$ (GAL), $3.3 * 10^{-17}$ (LAU) ,
2) BVP $u^{(4)}(x) = 1, u(0) = u(L) = 0, u^{(2)}(0) = u^{(2)}(2) = 0$, exact solution- $u(x) = x^4/24 - x^3/6 + x/3$, FDA $A^2U = e_1$ with maximal errors- $7.5 * 10^{-15}$ (GAN), $5.5 * 10^{-14}$ (GAL), $4.2 * 10^{-4}$ (LAU) ,
3)BVP $u^{(6)}(x) = 1, u(0) = u(L) = 0, u^{(2)}(0) = u^{(2)}(2) = 0, u^{(4)}(0) =$

$u^{(4)}(2) = 0$, exact solution- $u(x) = x^6/720 - x^5/120 + x^3/18 - 2 * x/15$, FDA $A^3U = -e_1$ with maximal errors- $1.5 * 10^{-13}$ (GAN), $1.1 * 10^{-12}$ (GAL), $1.4 * 10^{-3}$ (LAU).
Here e_1 is the column-vector of the $N-1$ order with ones.

The problems (2.26-2.34) are solved numerically by the MATLAB with relative error 10^{-7} (RelTol=10^{-7}).

Some numerical results for ill-posed problem for nonlinear heat transfer equation.
Constructed direct and reverse functions allow us to calculate the solution corresponding to the path with increasing the source g staring from the initial condition $u_0 = 0$ and the reverse path starting from the the initial condition u_0 calculated by using direct function. The numerical simulations are carried out by integating the system of ODEs (2.26) with two way:
1) $\varepsilon \neq 0$ (an example $\varepsilon = 10^{-4}$),
2) $\varepsilon = 0$ by using the modified functions F, f.
The obtained results in either case are consistent . Some numerical results obtained for different g_0 is shown in Fig. 2.47. Here for $\varepsilon = 10^{-4}$) the first stationary solution is obtained for $g_0 = 1.6$ with decreasing the source from $g_0 = 3(U_0(x)$ is equal to the stationary solution obtained with increasing the frequency from $g_0 = 0$ to $g_0 = 3$ by $T = 6$), the second stationary solution is obtained by $g_0 = 1.6$ with increasing the source from $g_0 = 0$ ($U_0(x) = 0$) to $g = 1.6$. Similarly the solution obtained at $g_0 = 1.6$ by direct and reverse paths are different.
Here is two stacionary solutions at $g_0 = 1.6$ obtained by $T = 6$ in following way:
1) we use the direct path (function) with $U_0(x) = 0$ and obtain the stationary solutions by $g_0 = 1.6$ and $g_0 = 3$,
2) we use the reverse path (function) with $U_0(x)$ equal to stationary solution obtained with direct function by $g_0 = 3$.
Using the MATLAB the stationary solutions $u(x)$ obtained at $g_0 = 1.6$ by direct and reverse paths (the initial condition is the stationary solutions calculated on the direct path at $g_0 = 3$) are different. See also the dynamic of the solution from $g_0 = 3$ to $g_0 = 1.6$ (Figs. 2.47, 2.48).

The stationary solutions for $g_0 = 3, u(1) = 2.861$ are shown in the Figs. 2.45, 2.46.
Using (2.29, 2.30) and (2.26) by decreasing the source from $g_0 = 2$ to $g_0 = 1.62$ (the reverse path) are obtained following results for max

value M_s of the stationary solutionave (the exact value is $M_s = 2$):
1) for (2.29) $\varepsilon_2 = 0$ -FDSES: $M_s = 2.074, \varepsilon_1 = 3 * 10^{-5}$; FDS: $M_s = 1.995, \varepsilon_1 = 4.6 * 10^{-5}$,
2) for (2.29) $\varepsilon_1 = 0$ -FDSES: $M_s = 2.012, \varepsilon_2 = 4.8 * 10^{-9}$; FDS: $M_s = 2.052, \varepsilon_2 = 1.2 * 10^{-8}$,
3) for (2.26) -FDSES and FDS: $M_s = 1.990, \varepsilon = 6.0 * 10^{-3}$,
4) for (2.30) -FDSES and FDS: $M_s = 2.224, \varepsilon = 7.0 * 10^{-1}$,
For (2.27, 2.28) we obtain similar results.

We can use the transformation with the inverse function $u = F^{-1}(v)$, where $F(u)$ is the modified function. Then we have following initial-boundary-value problem

$$\begin{cases} \frac{\partial v}{\partial t} = F'(u)(\frac{\partial^2 v}{\partial x^2} + \omega\tau), \\ v(0,t) = v(2,t) = 0, t \in (0,T), v(x,0) = 0, x \in [0,2], \end{cases} \tag{2.40}$$

where $F'(u) = 1.2u^2 - 3.6u + 2.4$ and $u = u(x,t)$ is the solution of the cubic equation $u^3 - 4.5u^2 + 6u - 2.5v = 0$. Using Cardano formula we obtain for $v \in [0, 0.8] \bigcup [1, \infty)$ one real root in the form
$u = 1.5 + v1 + v2, v1 = ((10v - 9)/8 + 0.625\sqrt{v3})^{1/3}, v2 = ((10v - 9)/8 - 0.625\sqrt{v3})^{1/3}, v3 = 4v^2 - 7.2v + 3.2$.
We can obtain the maximal value of the stationary solution from this root, where $v = \omega\tau/2$. The nonstationary solutions $v(x,t)$ are obtained at $g_0 = 1.6$ (see Fig. 2.49).

2.7 Two coaxial cylinders: A. Gedroics et al., 2010 [77]

A large number of papers in the time period of 1970 -1990 are devoted to blow-up phenomena in quasilinear parabolic equations [5].
In this paper the 1-D initial - boundary problem for nonlinear PDEs in the polar coordinates with radial symmetry of blow-up regimes

$$\frac{\partial u(r,t)}{\partial t} = \frac{\partial}{r\partial r}(\lambda r \frac{\partial (u(r,t))^{\sigma+1}}{\partial r}) + a(u(r,t))^{\beta},\ r \in [r_0, R], t > 0, \tag{2.41}$$

by $\sigma \geq 0, \beta > 0, \lambda > 0, a \geq 0$ and with conditions $u(r_0,t) = u(R,t) = 0, u(r,0) = u_0(r) \geq 0$ is considered.
We study the behaviour of solutions(2.41) at the time and also when $t \to \infty$, depending on the parameters $\sigma, \beta, \lambda, a$.

In [38] is solved the equation (2.41) in one layer of Cartesian coordinates.

Let the cylindrical domain $\{(r,\phi,z) : r_0 < r < R, 0 \le \phi \le 2\pi, -\infty < z < \infty\}$ contain thermal conducting material,
where r_0, R are the radiuses of the coaxial cylinders. The surfaces of these cylinders are with constant temperature $u = 0$.
The 2D domain (r,ϕ) with thickness $l = R - r_0$ is multilayer media Ω of $\bar{N}$ layers
$\Omega = \{(r,\phi) : r \in \Omega_k, k = \overline{1,\bar{N}}, 0 \le \phi \le 2\pi\}$,
where each layer is in the form
$\Omega_k = \{(r,\phi) : r_{k-1} \le r \le r_k, 0 \le \phi \le 2\pi\},\ r_{\bar{N}} = R$.
In the 2D case we shall consider the initial - boundary value problem for solving the temperature
$u = u(r,\phi,t) \ge 0$ from the following nonlinear heat transfer PDEs:

$$\frac{\partial u}{\partial t} = \lambda \Delta g(u) + a f(u),\ r \in \Omega, \phi \in [0,2\pi], t > 0, \tag{2.42}$$

where in every layer $\lambda > 0$ is the piece-wise constants coefficient of heat conductivity, $a > 0$ is the constant parameter,
$g(s)$ is nonlinear convex continuously differentiable function with $\partial g/\partial s = g'(s) > 0, s \in [0,\infty]$,
$f(s)$ is nonlinear convex continuous source function with $f''(s) \ge 0$,
Δ is Laplace operator,
$\Delta g = r^{-1}\frac{\partial}{\partial r}(r\frac{\partial g}{\partial r}) + r^{-2}\frac{\partial^2 g}{\partial \phi^2}$.
If $u = g'(u) = 0$ by $r = r_0; r = R$ then the solution of this problem is not classical [3].
For power functions $g(s) = s^{\sigma+1}, f(s) = s^{\beta}, \sigma > 0, \beta > 1, \beta \ge \sigma + 1$.
If the initial condition $u(r,\phi,0) = u_0(r)$ depends only on r then the solution $u(r,t)$ is with the radial symmetry (2.41).
In every layer Ω_k the functions g, f, u and λ are in the form $g(u_k), f(u_k)$, $u_k, \lambda_k,\ k = \overline{1,\bar{N}}$.
We have the continuity conditions on the interior surfaces
$r = r_k, k = \overline{1,\bar{N}-1} (g_r = \frac{\partial g}{\partial r} = g'\frac{\partial u}{\partial r})$
$u_k(r_k,\phi,t) = u_{k+1}(r_k,\phi,t), \lambda_k g_r(u_k(r_k,\phi,t)) = \lambda_{k+1} g_r(u_{k+1}(r_k,\phi,t))$,
and boundary conditions on the exterior surfaces $r = r_0,\ r = r_{\bar{N}} = R$
$u_1(r_0,\phi,t) = u_{\bar{N}}(R,\phi,t) = 0$.
For the initial condition by $t = 0$ we give $u_k(r,\phi,0) = u_{0,k}(r,\phi), k =$

$\overline{1,\bar{N}}$,
where $u_{0,k}(r,\phi)$ is continuous function in every layer.

2.7.1 Some theoretical aspects in one and two layers by radial symmetry

In one layer similarly [3] we consider the following spectral problem for Laplace operator by radial symmetry:
$\lambda \frac{\partial}{\partial r}(r\frac{\partial \psi}{\partial r}) + r\mu\psi = 0, \psi(r_0) = \psi(R) = 0,$
where μ are the eigenvalues.
The solution of this problem is in the form:
$\psi_k(r) = Y_0(b_k r) - \gamma_k J_0(b_k r), \gamma_k = \frac{Y_0(b_k r_0)}{J_0(b_k r_0)}, k = 1,2,3,...$
where $b_k = \sqrt{\mu_k/\lambda}, J_0, Y_0$ are the Bessel functions for zero order of the first and the second kind.
The eigenvalues μ_k satisfy following transcendent equations:
$J_0(b_k r_0)Y_0(b_k R) - J_0(b_k R)Y_0(b_k r_0) = 0.$
From the norm for the first eigenfunction $\psi_1^*(r) = \frac{\psi_1(r)}{I_0}, I_0 = \int_{r_0}^R r\psi_1(r)dr$ follows that $\int_{r_0}^R r\psi_1^*(r)dr = 1$.
Multiplying the equation (2.42) with function $r\psi_1^*(r)$ and integrating it by parts twice we get
$\frac{dE}{dt} = a(\psi_1^*, f) - \mu_1(\psi_1^*, g),$
where $(\psi_1^*, f) = \int_{r_0}^R r\psi_1^*(r)f(u(r,t))dr, (\psi_1^*, g) = \int_{r_0}^R r\psi_1^*(r)g(u(r,t))dr,$
$E(t) = \int_{r_0}^R r\psi_1^*(r)u(r,t)dr \geq 0$, if $u(r,t) \geq 0$,
$E_0 = E(0) = \int_{r_0}^R r\psi_1^*(r)u_0(r)dr \geq 0$.
Similarly [5] we can prove following **theorem**:
If the initial function $u_0(r)$ satisfies the inequality $\mu_1 g(E_0) < af(E_0)$ then the solutions $u(r,t)$ are **unbounded in the time** and exist finite value of $T \leq T_*$ when $\max u(r,t) \to \infty$ if $t \to T_*$, where $T_* = f(E_0)(af(E_0) - \mu_1 g(E_0))^{-1} \int_{E_0}^\infty \frac{ds}{f(s)} < \infty$.
For the power functions $g(u) = u^{\sigma+1}, g'(u) = (\sigma+1)u^\sigma, f(u) = u^\beta$, $\beta \geq 1, \sigma \geq 0$ follows $u(r,t) \geq 0$ for all $t \geq 0$, if $u_0(r) \geq 0$, and for $\beta \geq \sigma + 1$ if $\mu_1 E_0^{\sigma+1} < aE_0^\beta$ then we get

$$T_* = E_0^{-\sigma}(aE_0^{\beta-\sigma-1} - \mu_1)^{-1}/(\beta - 1) < \infty. \quad (2.43)$$

If $\beta = \sigma + 1$ and $\mu_1 < a$ then the fixed time moment $T_* < \infty$ when $E(T_*) = \infty$ (the "blow up" phenomena) is

$$T_* = \frac{1}{\sigma(a-\mu_1)} E(0)^{-\sigma}. \tag{2.44}$$

If $\beta < \sigma + 1$ then we obtain a priori estimation in the form
$E(t) \leq E(0) + cat$ $(c = \frac{\sigma+1-\beta}{\sigma+1}(\frac{(\sigma+1)\mu_1}{a\beta})^{\beta/(\beta-\sigma-1)})$,
and **the solution is bounded for finite value of** $t < \infty$.

Similarly for $\bar{N} = 2, r_1 = H < R, r_2 = R$, we consider the following spectral problem
$\lambda \frac{\partial}{\partial r}(r \frac{\partial \psi}{\partial r}) + r\mu\psi = 0, \psi_1(r_0) = \psi_2(R) = 0,$
where
$\psi(r) = \{\psi^1(r)(r_0 \leq r \leq H); \psi^2(r)(H \leq r \leq R)\}, \lambda = \{\lambda_1; \lambda_2\},$
$\psi^1(H) = \psi^2(H), \lambda_1 \partial\psi^1(H)/\partial r = \lambda_2 \partial\psi^2(H)/\partial r.$
The solution of this problem is in the form:
$\psi_k^1(r) = \gamma_k(Y_0(b_k^1 r_0)J_0(b_k^1 r) - J_0(b_k^1 r_0)Y_0(b_k^1 r)),$
$\psi_k^2(r) = Y_0(b_k^2 R)J_0(b_k^2 r) - J_0(b_k^2 R)Y_0(b_k^2 r), k = 1, 2, 3, ...$
where $b_k^1 = \sqrt{\mu_k/\lambda_1}, b_k^2 = \sqrt{\mu_k/\lambda_2},$
$\gamma_k = \frac{Y_0(b_k^2 R)J_0(b_k^2 H) - J_0(b_k^2 R)Y_0(b_k^2 H)}{J_0(b_k^1 H)Y_0(b_k^1 r_0) - J_0(b_k^1 r_0)Y_0(b_k^1 H)},$
J_0, Y_0 are the Bessel functions for zero order of the first and the second kind.
The eigenvalues μ_k satisfy following transcendental equations

$$\begin{cases} \lambda_1 b_k^1 (Y_0(b_k^2 R)J_0(b_k^2 H) - J_0(b_k^2 R)Y_0(b_k^2 H))(Y_0(b_k^1 r_0)J_1(b_k^1 H) - \\ J_0(b_k^1 r_0)Y_1(b_k^1 H)) - \lambda_2 b_k^2 (Y_0(b_k^1 r_0)J_0(b_k^1 H) - \\ J_0(b_k^1 r_0)Y_0(b_k^1 H))(Y_0(b_k^2 R)J_1(b_k^2 H) - J_0(b_k^2 R)Y_1(b_k^2 H)) = 0, \end{cases} \tag{2.45}$$

where J_1, Y_1 are the Bessel functions of the first order of the first and the second kind.
From the norm of the first eigenfunction
$\psi_1^*(r) = \frac{\psi_1(r)}{I_0}, \ \psi_1^* = \{\psi_1^{1*}; \psi_1^{2*}\}, \psi_1 = \{\psi_1^1; \psi_1^2\}$
$I_0 = \int_{r_0}^R r\psi_1(r)dr = \int_{r_0}^H r\psi_1^1(r)dr + \int_H^R r\psi_1^2(r)dr,$
follows that $\int_{r_0}^R r\psi_1^*(r)dr = \int_{r_0}^H r\psi_1^{1*}(r)dr + \int_H^R r\psi_1^{2*}(r)dr = 1.$
Multiplying the equation (2.42) by function $r\psi_1^*(r)$ and integrating it by parts twice both integrals we obtain the expressions (2.43, 2.44).

2.7.2 Methods of lines and FDS for the one and two layers

For numerical calculation **in the one layered** domain ($\lambda_k = \lambda, u_k = u$) we consider uniform grid with additional grid points ($r_k = kh + r_0, k = \overline{0,N}, Nh = R - r_0$). We consider two cases: the radial symmetry and the 2-D problem in the space with coordinates (r, ϕ). For solving the equation (2.42) with radial symmetry (1-D problem in the space) we use the method of lines to reduce the nonlinear heat transfer problem to initial value problem for system of nonlinear ODEs of the first order. For the 2-D problem we obtain the stationary solutions using the vector finite difference scheme with circulant matrix.

In the 1-D case from (2.42) we can be directly obtain the system of nonlinear ODEs with the second order of approximation in the space in the following form
$\dot{u}(r_k,t) = \frac{\lambda}{h^2}(\frac{r_{k+0.5}}{r_k} g(u(r_{k+1},t)) - 2g(u(r_k,t)) + \frac{r_{k-0.5}}{r_k} g(u(r_{k-1},t))) + af(u(r_k,t))$,
where $k = \overline{1, N-1}, g(u(r_N,t)) = g(u(r_0,t)) = 0$.

We can rewrite this system in the following matrix form

$$\dot{U} = \frac{\lambda}{h^2} AG + F, \tag{2.46}$$

where A is the standard 3-diagonal matrix of $N-1$ order with the non zero elements $a_{k,k} = -2, a_{k,k+1} = \frac{r_{k+0.5}}{r_k}, a_{k-1,k} = \frac{r_{k-0.5}}{r_k}$,
$G, F, \dot{U}$ are the column-vectors of $N-1$ order with elements $g_k = g(u(x_k,t)), f_k = af(u(x_k,t)), \dot{u}_k = \dot{u}(x_k,t), k = \overline{1,N}$.

In the 2-D case using the powers function and the transformation $V(t,r,\phi) = u^{\sigma+1}(t,r,\phi)$ and the method of stationarity for the equation (2.42) we approximate the derivative $\frac{\partial V}{\partial t}$ by the discrete difference $(V_{i+1}(r,\phi) - V_i(r,\phi))/\tau$, where $i = 0, 1, ..., I$,
τ is the parameter of iterations. The number of iterations I is determined from following conditions:
$\max |V_{I+1}(r,\phi) - V_I(r,\phi)| \leq \varepsilon$, where ε is the desirable precision.
In this case for each iteration we can rewrite the heat transfer problem in the following form

$$\begin{cases} (V_{i+1}(r,\phi)-V_i(r,\phi))/\tau = \lambda\Delta V_{i+1}(r,\phi)+a(V_i(r,\phi))^{\alpha}, i=0,1,...,I, \\ V_0(r,\phi)=u_0(r,\phi), V_i(r_0,\phi)=V_i(R,\phi)=0, V_i(r,\phi+2\pi)=V_i(r,\phi), \end{cases} \tag{2.47}$$

where $\alpha=\frac{\beta}{\sigma+1}$, the function $u_0(r,\phi)=V_0(r,\phi)$ is the initial condition for the iterations.
We consider an uniform grid $((N+1)\times M)$:
$\omega_h=\{(r_k,\phi_j), r_k=r_0+kh, \phi_j=jh_\phi\}, k=\overline{0,N},\ j=\overline{1,M}, r_0+Nh=R, Mh_\phi=2\pi.$
The equation(2.47) in the grid points (r_k,ϕ_j) is replaced by vector difference equations of second order approximation in 5- point stencil:

$$A_kV_{i+1,k-1}-C_kV_{i+1,k}+B_kV_{i+1,k+1}+F_{i,k}=0, V_{i+1,0}=V_{i+1,N}=0, \tag{2.48}$$

where $V_{i,k},F_{i,k}$ are column-vectors with components $v_{i,k,j}\approx V_i(r_k,\phi_j)$, $f_{i,k,j}=a(v_{i,k,j})^{\alpha}+v_{i,k,j}/\tau, k=\overline{1,N-1}, j=\overline{1,M},$ A_k,B_k,C_k are the circulant symmetrical matrices with M-order:
$A_k=[a_{k,1},0,0,...,0], B_k=[b_{k,1},0,0,...,0], C_k=[c_{k,1},c_{k,2},0,0,....,0,c_{k,2}],$
where $a_{k,1}=\frac{r_{k-0.5}}{r_kh^2}$, $b_{k,1}=\frac{r_{k+0.5}}{r_kh^2}$,
$c_{k,2}=c_{k,M}=-\frac{1}{r_k^2h_\phi^2}$, $c_{k,1}=a_{k,1}+b_{k,1}-2c_{k,2}+\tau^{-1}$.
Using special arithmetical operations with circulant matrices the finite vector difference scheme (2.48) is solved by the Gauss elimination method.

Similarly we consider **the two layered domain** $(\Omega_1,\Omega_2,\lambda_1\neq\lambda_2)$ and uniform grid in every layer with grid points
$(r_k=kh+r_0, k=\overline{0,N}, Nh=R-r_0, Kh+r_0=H=r_1<R=r_2.)$

In the 1-D case we obtain following system of nonlinear ODEs with the second order of approximation in the space
$\dot{u}_1(r_k,t)=\frac{\lambda_1}{h^2}(\frac{r_{k+0.5}}{r_k}g(u_1(r_{k+1},t))-2g(u_1(r_k,t))+\frac{r_{k-0.5}}{r_k}g(u_1(r_{k-1},t)))+af(u_1(r_k,t)), k=\overline{1,K-1};$
$\dot{u}_1(r_k,t)=\frac{1}{h^2}(\lambda_2\frac{r_{k+0.5}}{r_k}(g(u_2(r_{k+1},t))-g(u_2(r_k,t)))-\lambda_1\frac{r_{k-0.5}}{r_k}(g(u_1(r_k,t))-g(u_1(r_{k-1},t)))))+af(u_1(r_k,t)), k=K;$
$\dot{u}_2(r_k,t)=\frac{\lambda_2}{h^2}(\frac{r_{k+0.5}}{r_k}g(u_2(r_{k+1},t))-2g(u_2(r_k,t))+\frac{r_{k-0.5}}{r_k}g(u_2(r_{k-1},t)))+af(u_2(r_k,t)), k=\overline{K+1,N-1},$
where $g(u_2(r_N,t))=g(u_1(r_0,t))=0$.

We can rewrite this system in the following matrix form (2.46,$\lambda=1$),
where A is the block matrix of $N-1$ order with two blocks of 3-

diagonal matrix form of $K-1$ and $N-K$ orders.
In the 2-D case similarly (2.47) follows the heat transfer problem in iterations form

$$\begin{cases} (V_{i+1}^m(r,\phi) - V_i^m(r,\phi))/\tau = \lambda_m \Delta V_{i+1}^m(r,\phi) + a(V_i^m(r,\phi))^\alpha, i = 0,1,...,I, \\ V_0^m(r,\phi) = u_0(r,\phi), V_i^1(r_0,\phi) = V_i^2(R,\phi) = 0, V_i^m(r,\phi + 2\ pi) = V_i^m(r,\phi), \\ V^1(H,\phi) = V^2(H,\phi), \lambda_1 \partial V^1(H,\phi)/\partial r = \lambda_2 \partial V^2(H,\phi)/\partial r, \end{cases} \tag{2.49}$$

where V^1, V^2 are corresponding the solutions in the domains $\Omega_1, \Omega_2, m = 1,2$.
The heat transfer equation(2.49) in the uniform grid (r_k, ϕ_j) can be rewritten in the matrix-vector form (2.48).

2.7.3 Some examples and numerical results

The numerical experiment for the linear equation (2.42)
with $g = u, \sigma = 0, f = sin(\phi), a = 3, \lambda_1 = 1; 100, \lambda_2 = 1$ and $u_0(r,\phi) = (R-r)(r-r_0) \geq 0, R = 1, r_0 = 0.2, H = 0.6$
is compared with the following stationary analytical solution
$u_1(r,\phi) = (C_1 r + C_2 r^{-1} - r^2/\lambda_1$;
$u_2(r,\phi) = (C_3 r + C_4 r^{-1} - r^2/\lambda_2$, where C_1, C_2, C_3, C_4 are constants.
For the radial symmetry case is used also nonlinear test with $\beta = 0$.
The stationary solution is in the following form:
$u_1(r) = (-0.25ar^2/\lambda_1 + C_1 \ln r + C_2)^\alpha, u_2(r) = (-0.25ar^2/\lambda_2 + C_3 \ln r + C_4)^\alpha$,
where $\alpha = \frac{1}{\sigma+1}$.
The numerical results are agreed with 4 decimal signs with respect to analytical solutions.
From the numerical results follows that the minimal value of error is by $N = M$ and further the calculations are produced by different value of σ, β and $N = M = 80, \varepsilon = 10^{-4}$.

In the Figs. 2.50-2.57 we can see the four type solutions (radial symmetry) for three time moments $(t = 0, t = T1, t = T2 > T1)$, depending on the parameters
σ, β, a and with $\lambda_1 = \lambda_2 = 1, \mu_1 = 14.5615$ (in one layer), $\lambda_1 = 100, \lambda_2 = 1, \mu_1 = 59.2001$;
$\lambda_1 = 1, \lambda_2 = 100, \mu_1 = 58.9950$; (in two layers):
1) $\sigma = 3, \beta = 5, a = \mu_1$, the stationary solution $u_{st}(r)$ is zero (Figs.

2.50, 2.51),
2) $\sigma = 3, \beta = 4, a = \mu_1$, $u_{st}(r) \neq 0$ if $t \to T_{st} < \infty$, (Figs. 2.52, 2.53),
3) $\sigma = 3, \beta = 4, a = 60, a > \mu_1$, $u(r,t) \to \infty$ globally for all r
when $t \to T_* < \infty$ for $T_* = 2.822468$ (theoretical value $T_{*,*} = 3.3308$) in one layer,
$T_* = 268.9988($ $T_{*,*} = 293.8056)$ in two layers) (Figs. 2.54, 2.55),
4) $\sigma = 3, \beta = 5, a = 500$, in two layers $u(r,t) \to \infty$ locally,
when $t \to T_* < \infty$, for $T_* = 32.44096$ of point $r = 0.75$ if $\lambda_1 = 100, \lambda_2 = 1$ and for $T_* = 14.46177$ of point
$r = 0.25$ if $\lambda_1 = 1, \lambda_2 = 100$ (Figs. 2.56, 2.57).

If $\beta < \sigma + 1$, then for all $a > 0$ we have by $t \to T_{st} < \infty$
the stationary solution $u_{st}(r)$.
If $a = 0$, then for all σ $u_{st}(r) = 0$.
If $\beta = \sigma + 1$, then for all $a < \mu_1$ $u_{st}(r) = 0$.
If $a = \mu_1$, then the convergence to stationary solution is very fast in the time. If $a > \mu_1$, then the solutions is unbounded in the time
$t \geq T_*$ in all interval $r \in (r_0, R)(T_*$ is finite value, see the theoretical estimation (2.43)).

If $\beta > \sigma + 1$, then we have "blow up" phenomena by sufficient large value of $E(0)$ or a, when the solution tends to infinity locally in small neighbourhood of interior point in segment $[r_0, R]$.
From the theoretical estimation $Q = aE(0)^{\beta-\sigma-1}/\mu_1 > 1$ follows that the solution is unbounded in the time $t \geq T_* < \infty$, where $T_* = E_0^{-\sigma}(aE_0^{\beta-\sigma-1} - \mu_1)^{-1}/(\beta - 1)$.
The behaviour of the solution for $\sigma + 1 \leq \beta$ we can see in the table (T_* is numerical value,T_{**} is theoretical value).

Table 2.1 The values of T_* (numerical value) and T_{**} (theoretical value) by $a = 120$

σ	β	T_*	T_{**}	Q	T_*	T_{**}	Q
1	2	0.06822	0.0729	8.2405	0.13804	0.1464	2.0270
1	3	0.70117	3.6554	1.0722	$ust = 0$	$ust = 0$	0.2277
2	3	0.23933	0.2801	8.2405	0.57820	0.6514	2.0270
3	4	1.09944	1.4354	8.2405	3.1775	3.8654	2.0270
4	5	5.61870	8.2744	8.2405	19.3153	25.8023	2.0270

For the 2D case $(\beta = 4, \sigma = 3, a = \mu_1)$ the stationary solution is independing on the azimuthal coordinate ϕ (Figs. 2.58, 2.59).
This pictures are obtained by $\tau = 0.01; 0.001; 0.0005$ and $I = 20; 40; 70$.

2.8 The special methods: H. Kalis, M. Kokainis etc., 2015 [78]

In this section the finite difference scheme (FDS) for local approximating periodic function's derivatives in a $2n+1, n \geq 1$ point stencil is studied, obtaining higher order accuracy approximation. This method in the uniform grid with N mesh points is used to approximate the differential operator of the second and the first order derivatives in the space, using the multi-point stencil.
The described methods are applicable for various mathematical physics problems involving periodic boundary conditions(PBC).
The solutions of some linear problems for parabolic type partial differential equations (PDE) with PBCs are obtained, using the method of lines (MOL) to approach the PDEs in the time and the discretization in space applying the FDS of different order of the approximation and finite difference scheme with exact spectrum (FDSES).
These methods are compared with the global approximations method [42],which is based on using differentiation matrices (DMs) for derivatives on uniform grid with trigonometric interpolation.
Here we show that the approximation FDSES is equivalent to spectral differentiation matrix based on trigonometric (Fourier) interpolant.
In the last three decades the concept of a differentiation matrix (DM) to be a very useful tool in he numerical solution of differential equations is developed. DMs are derived from the spectral collocation or pseudospectral method for solving differential equations of boundary value type [43], [45], [44], [42].
In the spectral collocation method the unknown solution is expanded as a global interpolant, such a trigonometric or polynomial interpolant.
The DMs are based on Chebyshev, Fourier, Hermite and other interpolants.
Spectral DMs for problems with PBCs are based on Fourier interpolant. In other methods, such as finite elements or finite differences, the expansion involves local iterpolants such a piecewise polynomials.

In practice the accuracy of the spectral method is superior:
for problems with smooth solutions convergence rates of $O(e^{-cN})$ are archieved (c=const > 0) [44], [41],[42] .
In contrast, finite elements or finite differences on 3-point stencil yield convergence rates that are only algebraic in N, typically $O(N^{-2})$.

The spectral collocation method for solving differential equations is based on weighted interpolants of the form [45]:

$$f(x) \approx P_{N-1} = \sum_{j=1}^{N} \frac{\alpha(x)}{\alpha(x_j)} \Phi_j(x) f(x_j).$$

Here $\{x_j\}_{j=1}^N$ is set of distinct interpolation nodes (grid points), $\alpha(x) > 0$ is a weight function and the set of interpolation functions
$\{\Phi_j(x)\}_{j=1}^N$ satisfies $\Phi_j(x_k) = \delta_{jk}$ (the Kronecker delta), $f(x_k) = P_{N-1}(x_k), k = \overline{1,N}$.
For Chebyshev, Hermite and other interpolant the interpolating functions $\Phi_j(x)$ are polynomials of degree $N-1$.
For nonpolynomial cases there are trigonometric interpolants. The collocation derivative operator is generated by taking m-order derivatives of the interpolants and evaluting the result at the nodes $\{x_k\}$.
The derivative operator may be represented by matrix $D^{(m)}$ (DM) with entries $D_{kj}^{(m)} = \frac{d^m}{dx^m}[\frac{\alpha(x)\Phi_j(x)}{\alpha(x_j)}]_{x=x_k}$.
The numerical differentiation process may therefore be performed as the matrix-vector product $F^{(m)} = D^{(m)}F$,
where $F, F^{(m)}$ are the vectors of function values at the nodes. When solving differential equations the derivatives are approximated by this discrete derivative operators.
The matrix-vector product can be computed using the Fast Fourier Transform (FFT)
in $O(NlogN)$ operation rather than the $O(N^2)$ operations than the direct computation.
For boudary value problems with the PBCs the DMs matrlces $D^{(m)}$ are circulant and there can be given with first row.
Concerning higher derivatives often the second- and higher-derivative DMs are equal to the first-derivative matrix raised to the appropriate power.

Finite difference methods are important for approximating differential operators and solving various ordinary and partial differential equations numerically. Probably the most casual are the second order accurate FDS for approximation of the first and second order derivatives in a uniform 3- point stencil.
In this paper more accurate methods for approximation of the first and

second order derivatives in a uniform multi-point $2n+1, n \geq 1$ stencil are investigated. The algebraic convergence rate is $O(N^{-n})$.

We define the FDSES [39], where the finite difference matrix A is represented in the form $A = WDW^*$,
W, W^* are the complex and conjugate-complex matrices of finite difference eigenvectors,
D is diagonal matrix of the discrete eigenvalues and the elements of diagonal matrix D are replaced with the first N eigenvalues from the differential operator.
For PBCs this method is equivalent to the spectral method with DMs based on trigonometric interpolant.
In the first publication about FDSES [40] the finite differences with the second order of approximation in the uniform grid are used for the approximation of the second order derivative in the space segment $x \in [0, L]$ with the homogeneous boundary conditions of first kind.
Special numerical algorithms are developed for solving 1-D and 2-D problems of the second order ordinary (ODE) and partial differential (PDE) equations with PBC.

The linear heat transfer equations with variable coefficients can be written in the following form:
$u_t(x,t) = k(x)(u(x,t))_{xx} + p(x)(u(x,t))_x + q(x)u(x,t) + f(x), u(x,0) = u_0(x)$,
where $k(x), p(x), q(x), u_0(x)$ are real functions,
$x \in (0, L), t > 0$ are the space and time variables, L is the period, u=u(x,t) is the unknown function (for ODE we have boundary value problem with $u = u(x)$).
Note, that the similar system of PDE is considered, where k, p, q are matrices, u is column-vector.
The heat transfer problem are solved numerically using the method of lines and two way finite difference methods for the approximation of spatial derivatives:
local approximation with finite differences in uniform grid (FDS , FDSES) and global approximation with differentiation matrices.

For **local approximation** we have the discrete equations ($x_j = jh, Nh = L, j = \overline{1,N}$) as a system of ODEs in following form:
$\dot{U} = (KA_{2,n})(U) + (PA_{1,n})(U) + Q(U) + F, U(0) = U_0$,
where $A_{1,n}, A_{2,n}$ are N-th order circulant matrices, $U = U(t)$,
$F = F(t), \dot{U} = \dot{U}(t), U_0$ are the column-vectors of the N-th order with elements $u_j(t), f(x_j(t)), u_t(x_j,t), u_0(x_j)$

K,P,Q are N-th order diagonal matrices with elements $k(x_j),p(x_j),q(x_j)$ (in the case of the constant matrices k,p,q we have Kronecker tensor products $k\bigotimes A_{2,n}, p\bigotimes A_{1,n}, q\bigotimes I, I$ is N-order unit matrix).

Unlike other boundary condition types, PBC allows to freely increase approximation order by increasing the stencil of grid points. For example, $2n+1$ points stencil need to use additional discrete conditions of periodicity $u_k = u_{N+k}, k=\overline{-n,n}$. Thus can be obtained algorithms with higher order precision (different order FDS).

The second advantage of PBC is the fact that circulant matrices can be defined with the first row only. For such matrices it is easy to do arithmetic operations in shorter computation time. Also it is possible to get the inverse matrix analytically.

As another advantage one can mention the simple solution of the spectral problem. Orthonormal eigenvectors w_k, w_k^* with the elements
$w_{k,j} = \sqrt{N^{-1}}\exp(2\pi ijk/N)$,
$w_{k,j}^* = \sqrt{N^{-1}}\exp(-2\pi ijk/N), i=\sqrt{-1}, k; j=\overline{1,N}$
do not depend on the elements of circulant matrix.
By solving the discrete spectral problem, we can express the matrix $A_{m,n}$ in the form $A_{m,n} = WD_{m,n}W^*$, where W is the complex matrix which consists of the eigenvectors in it's columns, W^*
is the conjugate transpose of W and $D_{m,n}$- diagonal matrix with the eigenvalues $\mu_{m,k}$ of matrix$A_{m,n}$ on the diagonal (FDS),
$m=1;2, k=\overline{1,N}$.
Eigenvalues are obtained analytically for every multi-points stencil.

2.8.1 *The special numerical methods for approximations derivatives*

To determine the special numerical methods in the linear case the differential and discrete problems with constant coefficients can be solved analytically in different way:
1) the solution of the differential problem can be obtained with

- the complex Fourier series using the orthonormed eigenfunctions $w_k(x) = \sqrt{L^{-1}}\exp(2\pi ikx/L), w_k^*(x) = \sqrt{L^{-1}}\exp(-2\pi ikx/L)$ and eigenvalues $\lambda_{1,k} = i2\pi k/L$(for first derivative), $\lambda_{2,k} = -(2\pi k/L)^2$ (for second derivative), $k=\overline{-\infty,\infty}$ (the first N

eigenvalues coinside with the differentiation matrices, obtaining with trigonometric interpolant and FDSES),
- real Fourier series using the functions $\sin(2\pi kx/L), \cos(2\pi kx/L)$, $k = \overline{1,N}$,

2) the solution of the discrete problem can be obtained with

- the tranformation $V = WU^*$ by reducing the vector-problem to scalar-sapereted problem with the discrete eigenvalues,
- the complex discrete Fourier series for vector components $u_j = u(x_j), f_j = f(x_j)$ using the discrete orthonormal eigenvectors w_k, w_k^* and eigenvalues
  $\mu_{m,k}$ of matrix $A_{m,n}, m = 1,2, k = \overline{1,N}$,

- the real discrete Fourier series for vector components u_j, f_j using trigonometrical functions $\sin(2\pi kj/N), \cos(2\pi kj/N), k = \overline{1,N/2}$ (in this case the real discrete Fourier expression from matrix $A_{m,n}$ spectral representation is obtained).

For formed complex FDSES in the diagonal matrices $D_{m,n}$ elements d_k, the discrete eigenvalues $\mu_{m,k}$ are replaced
with the first N eigenvalues $\lambda_{m,k}, m = 1;2$ in special way (N even):
1) $d_k = \lambda_{2,k}, k = \overline{1,N/2}, d_{k+N/2} = \lambda_{2,N/2-k}$,
$k = \overline{1,N/2-1}, d_N = 0$,
2)$d_k = \lambda_{1,k}, k = \overline{1,N/2-1}, d_{k+N/2} = -\lambda_{1,N/2-k})$,
$k = \overline{1,N/2-1}, d_{N/2} = 0, d_N = 0$ (see Fig. 2.60).
In Figs. 2.60, 2.61 by $N = 80, L = 1$ for FDS are represented the discrete eigenvalues for $-u''$, the imaginary parts of the discrete eigenvalues for u' by different values of $n = 1;2;3;4;30$ and coresponding first $N = 80$ modified continues values for FDSES.
For N odd we obtain:
1) $d_k = \lambda_{2,k}, k = \overline{1,(N-1)/2}$,
$d_{k+(N-1)/2} = \lambda_{2,(N-1)/2-k+1}, k = \overline{1,(N-1)/2}, d_N = 0$),
2)$d_k = \lambda_{1,k}, k = \overline{1,(N-1)/2}$,
$d_{k+(N-1)/2} = -\lambda_{1,(N-1)/2-k+1}), k = \overline{1,(N-1)/2}, d_N = 0$).

For variable coefficients k, p, q the numerical solutions of the discrete heat transfer problem are obtained with Matlab ODEs solver "ode15s", using the spectral representation of matrices $A_{m,n}, m = 1;2$ (FDS, FDSES and versions of differentiation matrices).

2.8.2 Global approximations with DMs

For **global approximations** in uniform grid
$x_j = jh, Nh = L, j = \overline{1,N}$
we can use the elements $d_{k,j}^{(1)}, d_{k,j}^{(2)}$ for the differentiation matrices $D^{(1)}, D^{(2)}$, obtained with trigonometrical interpolation in segment $[0,2\pi]$ [41], [42]:
1) N-even,
$d_{k,j}^{(1)} = 0.5(-1)^{k-j}\cot(\frac{\pi}{N}(k-j)), (k \neq j), d_{k,k}^{(1)} = 0,$
$d_{k,j}^{(2)} = -0.5(-1)^{k-j}\csc^2(\frac{\pi}{N}(k-j)), (k \neq j),$
$d_{k,k}^{(2)} = -\frac{N^2+2}{12};$
2) N-odd,
$d_{k,j}^{(1)} = 0.5(-1)^{k-j}\csc(\frac{\pi}{N}(k-j)), (k \neq j), d_{k,k}^{(1)} = 0,$
$d_{k,j}^{(2)} = -0.5(-1)^{k-j}\csc(\frac{\pi}{N}(k-j))\cot(\frac{\pi}{N}(k-j)),$
$(k \neq j), d_{k,k}^{(2)} = \frac{1-N^2}{12},$
(in this case $D^{(2)} = (D^{(1)})^2$ and $D^{(m)} = (D^{(1)})^m, m > 1$) [42].
For segment [0,L] the differentiation matices
$A_1 = D^{(1)}\frac{2\pi}{L}, A_2 = D^{(2)}\frac{4\pi^2}{L^2}$ are circulant matrices for FDSES:
$A_m = WD_mW^*, m = 1,2,$ where D_1, D_2 are diagonal matrices with the elements
$\lambda_{1,k} = i\frac{2\pi k}{L}, \lambda_{2,k} = \lambda_{1,k}^2 = -(\frac{2\pi k}{L})^2, k = \overline{1,N}.$
In Fig. 2.62 by $N = 80, L = 1$ are represented coresponding first and second derivatives from $exp(\sin(4\pi x)$,
obtained with the differentiation matrix A_1, A_2 from the trigonometrical interpolant (maximal errors: 1.65 e-12, 2.39 e-10).

2.8.3 Comparison between DMs and FDSES methods

For circulant matrix $A = [a_1, a_2, \dots, a_{N-1}, a_N]$ the eigenvalues can be obtained from following expression:

$\lambda_k = \sum_{j=1}^{N} a_j \exp(2\pi ik(j-1)/N), k = \overline{1,N}$.
For DM $D^{(1)}$ and N even we have $a_j = 0.5(-1)^{1-j}\cot(\pi(1-j)/N), a_1 = 0$.
Therefore $\lambda_k = i\sum_{j=1}^{M}(-1)^{1-j}\cot(\frac{\pi j}{N})\sin(\frac{2\pi jk}{N})$,
$M = N/2, k = \overline{1,N}$.
We can see that the eigenvalues are
$\lambda_k = ik, k = \overline{1,M-1}, \lambda_M = 0$,
$\lambda_{k+M} = -i(M-k), k = \overline{1,M-1}, \lambda_N = 0$
and
we obtain with multiply λ_k by $\frac{2\pi}{L}$ the eigenvalues for diagonal matrix $D_{1,n}$ from FDSES.
Similarly for N odd follows:
$a_j = 0.5(-1)^{1-j}\csc(\pi(1-j)/N)$,
$\lambda_k = i\sum_{j=1}^{M}(-1)^{1-j}\csc(\frac{\pi j}{N})\sin(\frac{2\pi jk}{N})$,
$M = (N-1)/2, k = \overline{1,N}$.
The eigenvalues are
$\lambda_k = ik, k = \overline{1,M}, \lambda_{k+M} = -i(M-k+1)$,
$k = \overline{1,M}, \lambda_N = 0$
and wilh multiply λ_k by $\frac{2\pi}{L}$ we obtain the eigenvalues of FDSES.

2.8.4 Local approximations with FDS in multi-points stencil

We start with describing methods for higher order accuracy approximation of a smooth from the space $C^{2n+2}[0;L]$ function in an interval $[0;L]$.
Consider a uniform grid $x_j = jh,\ j = \overline{0,N}$, where $Nh = L$. Let n be natural number, satisfying $2n+1 \leq N$.

The spectral decomposition of the matrix $A_{m,n}$ representation for FDS and FDSES. Let $\mu_{m,j}$ be the j-th eigenvalue of the matrix $A_{m,n}$ representation for FDS, but w_j – the corresponding eigenvector. It follows from the properties of circulant matrices consists of N components $w_{j,r},\ r = \overline{1,N}$, where
$w_{j,r} = \frac{1}{\sqrt{N}}\exp(\frac{2\pi ijr}{N}) =$
$\frac{1}{\sqrt{N}}(\cos(\frac{2\pi jr}{N}) + i\sin(\frac{2\pi jr}{N}))$,
but eigenvalues can be found as $\mu_{m,j} = \frac{1}{h^m}\nu_j$.
We can find that [46]:

1) $\nu_j = \cos\alpha_j \sum_{k=1}^{n} \frac{4^k \cdot k}{(2k)!}((k-1)!)^2 \sin^{2k-1}\alpha_j$ (m=1),
2) $\nu_j = -2\sum_{k=1}^{n} \frac{4^k}{(2k)!}((k-1)!)^2 \sin^{2k}\alpha_j$ (m=2),
where $\alpha_j = \frac{\pi j}{N}$.

The matrix $W = (w_1, \ldots, w_N)$ is a symmetric unitary matrix and the matrix $A_{m,n}$ can be factored as $A_{m,n} = W D_{m,n} W^*$.
Periodic function's of first and second order derivatives (in $2n+1$ stencil) can be approximated by finite differences with $O(h^{2n})$, given by the following circular matrices:

$$A_{1,n} = \frac{1}{h}[0 D_1^{1,n} \ \ldots \ D_n^{1,n} \ 0 \ \ldots \ 0 \ -D_n^{1,n} \ \ldots, \ -D_1^{1,n}],$$

$$A_{2,n} = \frac{1}{h^2}[D_0^{2,n} \ \ldots \ D_n^{2,n} \ 0 \ \ldots \ 0 \ D_n^{2,n} \ \ldots \ D_1^{2,n},]$$

respectively, where $D_k^{1,n} = (-1)^{k+1}\frac{(n!)^2}{k\,(n-k)!\,(n+k)!}$;
$D_k^{2,n} = \frac{2}{k} D_k^{1,n}, \ k = \overline{1,n}, D_0^{2,n} = -2\sum_k^1 n D_k^{2,n}$.
We have spectral representation $A_{1,n} = W D_{1,n} W^*$; $\quad A_{2,n} = W D_{2,n} W^*$,
where $D_{1,n}, D_{2,n}$ are diagonal matrices with entries
$\mu_{1,j} = \frac{i}{h}\cos(\alpha_j)\sum_k^1 n Q_{k,1} \sin^{2k-1}(\alpha_j)$,
$\mu_{2,j} = -\frac{2}{h^2}\sum_k^1 n Q_{k,2} \sin^{2k}(\alpha_j)$ on the diagonal,
$Q_{k,1} = \frac{4^k \cdot k}{(2k)!}((k-1)!)^2, Q_{k,2} = \frac{Q_{k,1}}{k}$.

For FDSES we can replace the discrete eigenvalues $\mu_{m,j}, j = \overline{1,N}$ with the first N continues eigenvalues $\lambda_{m,j}$ in special way (see Fig. 2.60).

2.8.5 Solving ODEs and PDEs

The described methods of derivative approximation, like other finite difference approximation, can be applied to estimate function's derivatives, solve ordinary and partial difference equations, etc.

2.8.6 Solving ODEs with constant coefficients

The described finite differences can be used to solve numerically ODEs in form

$$\begin{cases} u''(x) + p \cdot u'(x) + q \cdot u(x) = f(x), x \in (0, L), \\ u(0) = u(L),\ u'(0) = u'(L), \end{cases} \tag{2.50}$$

where p and q are constants (if $q = 0$, then $\int_0^1 f(x)dx = 0$.
For the **discrete problem** the finite difference equation is
$A_{2,n}U + pA_{1,n}U + qU = F$,
where U, F are the column-vectors of N order.
Using spectral decomposition of $A_{2,n}$ and $A_{1,n}$ and let $\tilde{U} = W^* U$, $\tilde{F} = W^* F$, then, since $W^*W = E$, we get
$(D_{2,n} + pD_{1,n} + qI)\tilde{U} = \tilde{F}$.
Observe that $D_{2,n} + pD_{1,n} + qI$ is a diagonal matrix with the elements $\mu_{2,k} + p\mu_{1,k} + q, k = \overline{1,N}$,
hence now we can easily find the vector $\tilde{U}$ and then the vector $U = W\tilde{U}$.

Similarly we can obtain the solution use the **complex discreate Fourier method**:
$u_j = \sum_{k=1}^{N} a_k w_{k,j}, f_j = \sum_{k=1}^{N} b_k w_{k,j}$,
where $w_{k,j} = \sqrt{\frac{1}{N}} \exp(2\pi ikj/N)$,
$w^*_{k,j} = \sqrt{\frac{1}{N}} \exp(-2\pi ikj/N) = w_{N-k,j}$,
$(w_k, w_m*) = \sum_{j=1}^{N} w_{k,j} w^*_{m,j} = \delta_{k,m}$, are the orthonormed eigenvectors,
$b_k = (f, w_k*)$.

Then for the unknown coefficients a_k we get $a_k = b_k/\mu_k$,
where $\mu_k = \mu_{2,k} + p\mu_{1,k} + q$ (if $q = 0$ then $b_N = \sqrt{\frac{1}{N}} \sum_{j=1}^{N} f_j = 0$).

Example 1: $p = 3, q = -1, f(x) = \cos(4\pi x) - 3\sin(28\pi x), \quad L = 1$.
The exact solution is $u(x) = u_1(x) + 3u_2(x)$
where $u_1(x) = \frac{-(16\pi^2+1)\cos(4\pi x) + 12\pi\sin(4\pi x)}{(16\pi^2+1)^2 + (12\pi)^2}$;
$u_2(x) = \frac{84\pi\cos(28\pi x) + (784\pi^2+1)\sin(28\pi x)}{(784\pi^2+1)^2 + (84\pi)^2}$.

Several maximal error of solutions were computed corresponding to different n values, namely
$n = 1$ (the standard FDS in 3-point stencil), $n = 7$, $n = 15$ and FDSES.

For $N = 35$ we have: $5.3e-4(n=1), 2.24e-05(n=15)$, $4.57e-06(n=15), 1.5e-15$ (FDSES)).
The spectral method for $N \geq 28$ is exact, because of linear combination for functions $\sin(p\pi x), \cos(p\pi x), p \leq 28$.

2.8.7 Solving ODEs with variable coefficients

For functions $k = 1, p = p(x), q = q(x)$ the finite difference equation (the linear system of algebraic equations) is
$A_{2,n}U + (P * A_{1,n})U + QU = F$,
where $P, Q)$ are N- order diagonal matrices with corresponding elements $p_j = p(x_j), g_j = q(x_j)$.
The matrices $A_{2,n}, A_{1,n}$ we can formed using the spectral decomposition $A_{2,n} = WD_{2,n}W^*, A_{1,n} = WD_{1,n}W^*$ in two way:
1) for the multi-point stencil FDS diagonal matrices $D_{2,n}, D_{1,n}$ with elements $\mu_{2,k}, \mu_{1,k}$,
2) for the FDSES diagonal matrices $D_{2,n}, D_{1,n}$ with elements $\lambda_{2,k}, \lambda_{1,k}, k = \overline{1,N}$ in special way.

We can find the vector U in the form $U = A^{-1}F$ or in Matlab $U = A\backslash F$, where $A = A_{2,n} + P * A_{1,n} + Q$.

Example 2: $p(x) = 4k_0\pi\cos(2k_0\pi x)$,
$q(x) = -(2k_0\pi)^2(\sin(2k_0\pi x) - \cos^2(2k_0\pi x)$,
$f(x) = f_1(x)f_0(x)$, $f_0(x) = \exp(-\sin(2k_0\pi x))$,
$f_1(x) = \cos(4\pi x)$.
We can see, that function $v(x) = u(x)/f_0(x)$ is solution from ODEs of constant coefficients $v''(x) = f_1(x), x \in (0,1)$ with periodic BCs.
This solution is in following fom: $v(x) = -\frac{\cos(4\pi x)}{16\pi^2} + C$, where C is arbitrary constant.
The exact solution is
$u(x) = -\dfrac{\cos(4\pi x)}{16\pi^2} f_0(x) + Cf_0(x)$, where from $u(0) = 0$ follows that $C = \frac{1}{16\pi^2}$.
In Tab.2.2 are rpresented the maximal errors of solutions by $N = 20, 40, 80, 100, 160; k_0 = 2, 8, 14$, obtained with global (FDSES= DM) and local approximations for different n (1;2;10).

The solutions by $N = 80, k_0 = 8, n = 1; 2$ are represented in Figs. 2.63, 2.64.

Table 2.2 The maximal errors obtained with global (FDSES= DMs) and local approximations for different n

k_0	N	$FDSES$	$n=1$	$n=2$	$n=10$
2	20	$4.5e-6$	$1.1e-2$	$2.1e-3$	$3.3e-5$
2	40	$2.2e-11$	$2.5e-3$	$1.6e-4$	$1.1e-8$
8	80	$2.6e-6$	$7.1e-3$	$1.3e-3$	$4.1e-5$
8	100	$4.7e-8$	$7.0e-2$	$9.4e-3$	$1.0e-5$
14	160	$3.0e-7$	$1.4e-2$	$3.0e-3$	$7.7e-5$

2.8.8 Solving linear PDEs with variable coefficients

For the heat transfer equation, using method of lines (MOL) we obtain the linear system of ODEs:
$\dot{U}(t) = (K * A_{2,n})U(t) + (P * A_{1,n})U(t) + QU(t) + F(t), U(0) = U_0$,
where U, F, U_0 are N-order column-vectors with the elements
$u_j(t) \approx u(x_j,t), f_j(t) = f(x_j,t), u_j(0) = u_0(x_j)$, K, P, Q are N-order diagonal matrices with the elements $k(x_j), p(x_j), q(x_j), j = \overline{1,N}$.
Formed the matrices with the spectral decomposition we obtain FDS and FDSES approximations for the linear system of ODEs.
If the cefficients k, p, q are costants, then solution we can obtain analytically, using Fourier methods. For variable coefficients Matlab solver "ode15s" is used.

Example 3: Using example 2 with $k(x) = 1, f(x,t) = -f_1(x)f_0(x), L = 1, u_0(x) = \frac{1}{16\pi^2}f_0(x), k_0 = 2, N = 20$
we obtain the stationary solution by $t = 0.1$ with 42 time step for FDSES (max. eror $4.0e-06$),
FDS $(n \geq 7)$ (max. eror $2.0e-05$)and with 45 time step for FDS $(n = 1)$, (max. eror $3.0e-03$).

In Figs. 2.65, 2.66 are represented the stationary solutions by $k_0 = 14, n = 1; 2, N = 160$ obtaining with FDS and FDSES (t=0.1). In Figs. 2.67, 2.68 are represented the solutions u(x,t) by $k_0 = 8; 14, N = 80; 160$ obtaining with FDSES (t=0.04).

The function $v(x,t) = u(x,t)/f_0(x)$ is solution from the heat transfer equation $v(x,t)_t = v(x,t)_{xx} - f_1(x)$ with periodic BCs.
The exact solution by $v(x,0) = \frac{1}{16\pi^2}$ is
$v(x,t) = f_1(x)(\exp(-16\pi^2 t) - 1)/(16\pi^2) + \frac{1}{16\pi^2} \to \frac{1-f_1(x)}{16\pi^2}$ if $t \to \infty$.

We can solved numerically also PDEs with coefficients $k(x,t), p(x,t), q(x,t)$ depending on x,t.
Example 4: for $k(x,t) = \exp(-t\sin(2\pi x))/(4\pi^2)$,
$p = q = 0$,
$f(x,t) = \exp(t\sin(2\pi x))\sin(2\pi x) - t^2\cos^2(2\pi x) + t\sin(2\pi x)$
we have following discrete problem with Matlab operators:
$\dot{U}(t) = (ES1.^t * A_{2,n}/(4\pi^2))U(t) + F(t), U(0) = 1$,
where $ES1$ is N-order diagonal matrix with elements
$ES_j^{-1} = \exp(-\sin(2\pi x_j))$,
$F(t) = (ES.^t).*S + S*t - (C*t).^2$,
S,C are N-order column-vectors with the elements $\sin(2\pi x_j), \cos(2\pi x_j), x_j = jL/N$.
The exact solution is $u(x,t) = \exp(t\sin(2\pi x))$.
The maximal errors of solutions by $N = 20, t = 1$ are:
$0.0077(n=1)$; $4.9e-04(n=2)$; $1.7e-05(n=4)$; $\mathbf{5.2e-06}(n \geq 6)$ and for FDSES.

2.8.9 Solving nonlinear PDEs

We shall consider the initial - boundary value problem for solving the following nonlinear heat transfer equation:
$u_t(x,t) = g(u(t,x))_{xx} + f(u(t,x))$,
where $g(u) = u^{\sigma+1}, f(u) = au^{\beta}$ are power functions with $a > 0, \beta \geq 1, \sigma \geq 0, u(x,t) \geq 0, u_0(x) \geq 0$.
In paper [5] by $(a = 1)$ is proved with the first kind boundary conditions that
1) by $\beta < \sigma + 1$ exists global bounded solution for all t,
2) by $\beta \geq \sigma + 1$ exists global bounded solution for sufficient small $||u_0||$,
but for larger $||u_0||$, exists finite value of time T_* when $u(x,t) \to \infty$ if $t \to T_*$.
The initial value problem for ODEs is in the form
$\dot{U} + AG = F, U(0) = U_0$,
where G,F are the vectors-column of N order with elements $g_k = g(u(x_k,t)), f_k = af(u(x_k,t)), k = \overline{1,N}$.
The numerical experiment with $L = 1$ and $u_0(x) = x(1-x) \geq 0$,
is produced by MATLAB solver "ode23s" by first kind boundary con-

ditions [1].
For $a = 5, \sigma = \beta = 3, (\beta < \sigma + 1), t = 10$,
$N = 6, 10, 20$ are obtained following maximal error using FDS and FDSES methods:
1) $N = 6 - -0,0125(FDS), 0.0011(FDSES)$;
2) $N = 10 - -0.0046(FDS), 0.0003(FDSES)$;
3) $N = 20 - -0.0013(FDS), 0.0001(FDSES)$.
Example 5: in the Figs. 2.69, 2.70, 2.71, 2.72
are represented 4 type solutions by $u_0(x) = \sin^{100}(\pi x), N = 50, \sigma = 3$ for periodical boundary conditions obtained:
1) $\beta = 5, a = 100$, the solution is "blow up" locally by $T_* = 5.481136$,
2) $\beta = 4, a = 100$, the solution is "blow up" globally by $T_* = 0.2020261$,
3)$\beta = 5, a = 1$, the solutions tends to zero, if $t \to \infty$,
4) $\beta = 4, a = 0.01$, the solutions tends to the stationary limit.

2.8.10 Solving the nonlinear system of heat transfer equations

We consider the nonlinear system of M-heat transfer equations in the following form:
$\mathbf{u}_t = K(g_1(\mathbf{u}))_{xx} + P(g_2(\mathbf{u}))_x + \mathbf{f}$,
with the PBCs, $g_1(\mathbf{u}) = \mathbf{u}^\alpha, g_2(\mathbf{u}) = \mathbf{u}^\beta$
are the vector power functions, K is the positive definete M-order matrix with the elements $k_{m,s}, m, s = \overline{1,M}$, with different positive eigenvalues $\mu_K > 0$,
P is the real M-order matrix with the elements $p_{m,s}, m, s = \overline{1,M}$, with different real eigenvalues
μ_P; $\mathbf{u}(x,0) = \mathbf{u}_0(x), \mathbf{u}(x,t), \mathbf{u}_0(x)$ – are periodic functions- column-vectors of the M-order with elements $u^m(x,t), u^m(x,0), m = \overline{1,M}$.

The discrete equations are in the form
$\dot{\mathbf{U}} = (K \bigotimes A_{2,n})\mathbf{U}^\alpha + (P \bigotimes A_{1,n})\mathbf{U}^\beta + F$,
where $\mathbf{U}(t), \mathbf{U}(0), \mathbf{F}(t)$ are MN column-vectors with the elements $u_j^m(t), u_j^m(0), f_j^m, m = \overline{1,M}, j = \overline{1,N}$.
Example 6: $M = 2, L = 1, u_0^1(x) = b_{1,1}\sin(2\pi x) + b_{1,2}\cos(2\pi x)$,
$u_0^2(x) = b_{2,1}\sin(2\pi x) + b_{2,2}\cos(2\pi x)$,
$f^1(x,t) = a_{1,1}(t)\sin(2\pi x) + a_{1,2}(t)\cos(2\pi x)$,
$f^2(x,t) = a_{2,1}(t)\sin(2\pi x) + a_{2,2}(t)\cos(2\pi x)$,
$b_{1,1} = 0, b_{1,2} = 1, b_{2,1} = -1, b_{2,2} = 0$,

$a_{1,1} = 5, a_{1,2} = 10, a_{2,1} = -10, a_{2,2} = -5,$
and we can consider 2 matrices

$$\tilde{B} = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}, \tilde{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

Matrices

$$K = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}, P = \begin{pmatrix} 2 & -5 \\ 1 & -4 \end{pmatrix}$$

are with the eigenvalues $\lambda_K = (1;5), \lambda_P = (1;-3)$.
We have following maximal and minimal values of solutions U^1, U^2 for $t = 0.1$;
$\alpha = \beta = 3$ (solution tends to stationary for small time t=0.1(see in the Figs. 2.73-2.74 the solution by t=0.1 and maximal and minimal values depending on t)
If $N = 40$ then we have for max-min values of $\pm U^1, \pm U^2$:
$0.51212; 0.42610 (n = 1)$,
$0.51234; 0.42653 (n = 2), 0.51286; 0.42679 (n = 3)$,
$0.51308; 0.42701 (n = 4), 0.51320; 0.42719 (FDSES)$.
If N=80, then by FDSES and FDS $n = 4$): 0.51323; 0.42725.

2.9 The combustion proceses: H. Kalis, U. Strautins etc., 2019 [79]

The experimental study of the effect of co-firing on the main gasification and combustion characteristics was carried out by varying the propane supply and additional heat input into the pilot device with estimation the effect of co-firing on the thermal decomposition of wheat straw pellets, the formation, ignition and combustion of volatiles (CO, H_2). The mathematical model is developed using the environment of MATLAB package with account variations of supply the heat energy and combustible volatiles (CO, H_2) into the bottom of the combustor. The dominant exothermal chemical reactions were used to evaluate the effect of co-firing on the main combustion characteristics and composition of the products CO_2 and H_2O. The results prove that the additional heat from the propane flame allows control the thermal decomposition of straw pellets, the formation, ignition and combustion

of volatiles and the development of combustion dynamics, thus completing the combustion of biomass and leading to cleaner heat energy production.
For mathematical modelling the 1D distribution of axial component of velocity w, density ρ, mass fraction for different species and temperature T has been calculated with Matlab routine "pdepe" .
We use the mathematical model with following exothermic chemical raction for obtaining the final products H_2O, CO_2:
$C_3H_8 + 5O_2 \to 3CO_2 + 4H_2O$(one reaction with 4 species).
The 1D distribution of axial component of velocity $w = u_z/U_0, (U_0 = 0.1m/s)$, density $\rho/\rho_0, (\rho_0 = 1kg/m^3)$, temperature $T/T_0, (T_0 = 300K)$ and mass fractions for species
$C_k, k = \overline{1,K}, K = 4, C_1(C_3H_8), C_2(O_2), C_3(CO_2), C_4(H_2O)$,
has been calculated of the nonlinear parabolic type system of PDEs, depending on time $t/t_0, (t_0$=1s) and axial coordinate $x = z/z_0, (z_0 = 0.1m)$ with Matlab routine "pdepe" for 7 unknown functions.
At the inlet of the combustor $x = 0 : T = T_0, u_z = U_0, \rho = \rho_0$ in physical experiment obtained averaging values of concentration C_3H_8, O_2 as the mass fraction of reactants C_1, C_2 are used for the mathematical model,
the summ of reactants is equal 1, but of products is zero.
The production rate for k-th species can be written in following form [69]

$$\Omega_k = \sum_{j=1}^{J} [(\nu''_{j,k} - \nu'_{j,k}) R_j(T) \prod_{n=1}^{K} (\frac{\rho C_n}{m_n})^{\nu'_{jn}}], k \in [1, K],$$

where J is number of reactions, $R_j(T)$ is rate constant modified with Arrhenius temperature dependence for the forward path of chemical reaction $R_j(T) = A'_j T^{\beta_j} \exp(E_j/RT)$,
A'_j are the reaction-rate pre-exponential factors, R = 8.314 [J/(mol K] is the universal gas constant,$\nu''_{j,k}, \nu'_{j,k}$ are the corresponding stoichiometric coefficients of k-th species appering as a product and reactant in j-th reaction,
β_j is the order for temperature, $m_n[g/m^3]$ is the molecular weight of species C_k :
$O_2 = 32, CO_2 = 44, H_2O = 18, C_3H_8 = 44.$
In the equation for the mass fractions of concentation C_k the source term is $m_k \Omega_k/\rho [1/s]$, but in the equation for temperature

$\frac{1}{m\rho c_p}\sum_{k=1}^{K} h_k m_k \Omega_k [K/s]$, where $m = \frac{1}{K}\sum_{k=1}^{K} m_k$
is averaged molecular weight of mixture, $c_p = 1000[J/(kgK)]$ is the specific heat at a constant pressure, $h_k[kJ/mol]$ is the specific enthalpy of k-thspecies:
$O_2 = 0, CO_2 = -394, H_2O = -242, C_3H_8 = -105.$
For mathematical modelling the 1D compressible reacting swirling flow and density we considered a two parabolic type partial differential equations (PDEs) in following dimensionless form:

$$\begin{cases} \frac{\partial \rho}{\partial t} + M(\rho) + \rho\frac{\partial w}{\partial x} = e\frac{\partial^2 \rho}{\partial x^2}, \\ \frac{\partial w}{\partial t} + M(w) = -\frac{\partial p}{\rho \partial x} + Re^{-1}\frac{\partial^2 w}{\partial x^2}, \end{cases} \tag{2.51}$$

where $M(s) = w\frac{\partial s}{\partial x}, s = \rho, w, C_k, Re = U_0 z_0 \rho_0/\mu = 1000$ is the Reynolds number, $\mu = 510^{-6}[kg/ms]$ is the viscosity,$e = 10^{-5}$ and the value of Re are the factors of the artificial viscosity for approximation the density and velocity eguations. For the dimensionless pressure p we use a model for perfect gas: $p = \rho T$.
The boundary conditions at inlet (x=0): $\rho = w = T = 1$.
These values are used for initial conditions by t=0. In outlet the zero derivatives conditions are used. The numerical results depending on (x,t) are obtained for $x \in [0,2], t \in [0,t_f], t_f = 1; 10$.

2.10 The reactions: $C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$

For mathematical modelling of one reaction for four chemical species and for the temperature, we have the following equations

$$\begin{cases} \frac{\partial T}{\partial t} + M(T) = \frac{P_0}{\rho}\frac{\partial^2 T}{\partial x^2} + q_1\rho^5 A_1 T^{\beta_1} C_1 C_2^5 \exp(-\frac{\delta_1}{T}), \\ \frac{\partial C_1}{\partial t} + M(C_1) = P_1\frac{\partial^2 C_1}{\partial x^2} - \rho^5 A_1 T^{\beta_1} C_1 C_2^5 \exp(-\frac{\delta_1}{T}), \\ \frac{\partial C_2}{\partial t} + M(C_2) = P_2\frac{\partial^2 C_2}{\partial x^2} - 5\rho^5 A_1 T^{\beta_1} m_2/m_1 C_1 C_2^5 \exp(-\frac{\delta_1}{T}), \\ \frac{\partial C_3}{\partial t} + M(C_3) = P_3\frac{\partial^2 C_3}{\partial x^2} + 3\rho^5 A_1 T^{\beta_1} m_3/m_1 C_1 C_2^5 \exp(-\frac{\delta_1}{T}), \\ \frac{\partial C_4}{\partial t} + M(C_4) = P_4\frac{\partial^2 C_4}{\partial x^2} + 4\rho^5 A_1 T^{\beta_1} m_4/m_1 C_1 C_2^5 \exp(-\frac{\delta_1}{T}), \end{cases} \tag{2.52}$$

where $\beta_1 = 0, Q_1 = (m_1 h_1 + 5m_2 h_2 - 3m_3 h_3 - 4m_4 h_4)/(m_1 m), A_1 = A_1' \rho_0^5 z_0/(U_0 m_2^5)$ is the scaled reaction-rate pre-exponential factor, $A_1' =$

$14[m^{15}/mol^5 s], E_1 = 120000$ [73]. The boundary conditions for C_1, C_2 at the inlet (x=0) are
$\mathbf{C_{10} = \overline{0.1, 0.7}(C_3H_8), C_{20} = 1 - C_{10}(O_2)}$ (see Tab. 2.3 for $t_f = 10$). From the other algorithm it follows that: $\frac{C_1}{C_2} = \frac{44}{5*32} = \frac{11}{40}, C_1 + C_2 = 1$ or $C_{10} = 0.22, C_{20} = 0.78$. The maximum values of MT, Mw, wend,

Table 2.3 The values of Mw , wend, MT, Tend, C1end(C_3H_8), C2end(O_2), C3end (CO_2), C4end (H_2O), depends on C_{10}

C_{10}	Mw	$wend$	MT	$Tend$	$C1end$	$C2end$	$C3end$	$C4end$
0.1	4.48	1.61	4.78	2.11	0.00	0.54	0.30	0.16
0.2	**6.64**	**1.97**	**8.48**	**3.67**	0.01	0.09	**0.59**	**0.32**
0.3	6.41	1.93	7.91	3.52	0.12	0.04	0.54	0.30
0.4	5.64	1.87	7.02	3.27	0.25	0.04	0.46	0.25
0.5	4.71	1.81	6.23	3.04	0.37	0.04	0.38	0.21
0.6	4.50	1.73	5.21	2.82	0.50	0.04	0.30	0.16
0.7	2.83	1.61	4.17	2.59	0.63	0.04	0.22	0.12

Tend, CO_2, H_2O are obtained for $C_{10} = 0.2$ when the propane mass fraction at the outlet x=2, C1end=0.004. For $C_{10} = 0.1$, C1end=0, but the mass fractions of O_2 C2end=0.54 (the mass fraction for propane at the inlet is to small).
Figs. 2.75-2.80 illustrate the development the temperature, axial velocity and concentrations in the time ($t_f = 10$) and space for $C_1 = C_{10} = 0.4, C_2 = C_{20} = 0.6$.

2.11 The qypsum products: A. Aboltins, I. Kangro etc., 2020 [80]

At first, we study the heat transfer problem for two layers for gypsum board products -
gypsum plate and gypsum carton plate with different density exposed to fire. For studying the heat transfer we have to solve the system of 2 nonstationary partial differential equations (PDEs) (with heat diffusion coefficients depending on temperature) expressing the rate of change of temperature in every layer.
The approximation of corresponding initial-boundary value problem (IBVP) of this system is based on the conservative averaging method (CAM) by using special splines with hyperbolic type functions. This

procedure allows to reduce the 2-D heat transfer IBVP described by a system of 2 PDEs to initial value problem for a system of 2 ordinary differential equations (ODEs) of the first order.
The second problem under question is combustion process with Arrhenius kinetics using single step chemical reactions. The exothermic chemical reactions are modelling by single step of fuel and oxidant, at the inlet the constant axial velocity is given. Numerical solution with Matlab routines "pdepe"and " bvp4c" is obtained.

2.11.1 The mathematical model and formulation of the gypsum board problem

We consider gypsum board material with two layered plates in x-direction : gypsum plate (0.0525 [m]) with density $300[\frac{kg}{m^3}]$ and gypsum carton plate(0.0125 [m]) with density $1000[\frac{kg}{m^3}]$. The gypsum plate on one border is heated with temperature $T_l = 20 + 345 lg(8t + 1)[^0C]$, where t is the time in minutes.The domain Ω consists of two layer medium:

$$\Omega_i = \{(x,y,z) : x \in (x_{i-1}, x_i), y \in (-\infty, \infty), z \in (-\infty, \infty)\}, i = \overline{1,2},$$

where $H_i = x_i - x_{i-1}$ is the height of layer $\Omega_i, x_0 = 0, x_1 = H_1 = 0.0525, x_2 = L = H_1 + H_2 = 0.0650, H_2 = 0.0125$.
The heat conduction PDE is the following form:

$$c_{pi}(T_i)\rho_i \frac{\partial T_i}{\partial t} = \frac{\partial}{\partial x}(K_i(T_i)\frac{\partial T_i}{\partial x}); x \in [x_{i-1}, x_i], i = \overline{1,N}, t > 0, \quad (2.53)$$

where c_{pi} are the specific heat, K_i, ρ_i are the heat conductivity and the density of the gypsum material.
We assume that the coefficients c_{pi}, K_i in the PDEs are dependent of the temperature T_i.
In two layers (N=2) we get the system of two PDEs

$$\begin{cases} D_1(T)\frac{\partial^2 T_1(x,t)}{\partial x^2} = \frac{\partial T_1(x,t)}{\partial t}, \\ D_2(T)\frac{\partial^2 T_2(x,t)}{\partial x^2} = \frac{\partial T_2(x,t)}{\partial t}, \end{cases} \quad (2.54)$$

where $D_i(T) = \frac{K_i(T_i)}{\rho_i c_{pi}(T_i)}, i = 1,2$ are thermal difussion coefficients depending on T_i.

In [71], [72] there are obtained, that the coefficients of the specific heat c_p and thermal conductivity K depend on temperature T $(T = u)$ in the following way: K decreases from value $0.24[\frac{W}{m^oC}]$ at $u = 20[^oC]$ to 0.12 at $u = 200$, then K increases depending on u to value 0.24; coefficient c_p in this heat intervals is increasing from $1000[\frac{J}{kg^oC}]$ at $T = 20[^oC]$ to 8000 and then decreasing to 1000.
For approximation we use the cubic spline interpolation, see Figs. 2.81, 2.82. For the initial condition for $t = 0$ we give $T_1(x,0) = T_2(x,0) = T_0$, where $T_0 = 20[^oC]$.
We use following boundary and continuous conditions:

$$\begin{cases} D_1(T)\frac{\partial T_1(0,t)}{\partial x} - \alpha(T_1(0,t) - T_a) = 0, T_2(L,t) = T_b + T_l(t), \\ T_1(x_1,t) = T_2(x_1,t), D_1(T)\frac{\partial T_1(x_1,t)}{\partial x} = D_2(T)\frac{\partial T_2(x_1,t)}{\partial x}, \end{cases} \tag{2.55}$$

where α are the constant mass transfer coefficients,
$T_l(t) = 345lg(8t+1)$ in minute, $T_a = T_b = 20[^oC]$.
CAM procedure allows to reduce the problem to a initial problems for system of ODEs. Using averaged method with hyperbolic type splines with respect to x we have
$T_i(x,t) = T_{i,v}(t) + m_i(t)\frac{0.5H_i \sinh(a_i(x-\overline{x_i}))}{\sinh(0.5a_iH_i)} + e_i\frac{\cosh(a_i(x-\overline{x_i}))-A_i}{8\sinh^2(0.25a_iH_i)}$,
where $T_{i,v}(t) = \frac{1}{H_i}\int_{x_{i-1}}^{x_i} T_i(x,t)dx, \overline{x_i} = (x_{i-1}+x_i)/2, x \in [x_{i-1}, x_i]$,
$A_i = \frac{sinh(0.5a_iH_i}{0.5*a_iH_i}, i = 1;2$.
We can see if parameters $a_1 > 0, a_2 > 0$ tend to zero then the limit is the integral parabolic spline (A.Buikis). The unknown functions $m_i(t), e_i(t)$, can be determined from boundary conditions.
Follows the nonlinear system of ODEs

$$\begin{cases} \dot{T}_{1v} = b_{11}T_{1v} + b_{12}T_{2v} + f_1 + p_1(t), \\ \dot{T}_{2v} = b_{21}T_{1v} + b_{22}T_{2v} + f_2 + p_2(t), \\ T_{1v}(0) = T_{2v}(0) = T_0. \end{cases} \tag{2.56}$$

2.11.2 The mathematical model of the combustion process

The combustion process with temperature $T[K]$ and simple exothermic chemical reaction by first-order Arrhenius kinetics with mass fraction C of the reactant in the time t in the z-plane with the length $z_0 = 0.1[m]$ is modelling.
Let $T_0 = 300[K]$, $\rho_0 = 1[\frac{kg}{m^3}], C = C_0 = 1$ the initial temperature, nominal density, mass fraction for concentration of fuel and axial velocity with uniform streem $U_0 = 0.01[\frac{m}{s}]$ at the inlet $z = 0$.
We analyze the nonstationary physical model for simple chemical reaction with temperature and 2 reaction-diffusion equations:

$$\begin{cases} \frac{\partial T}{\partial t} + u_z \frac{\partial T}{\partial z} = \frac{\lambda}{\rho c_p} \frac{\partial^2 T}{\partial z^2} + \frac{\tilde{B}}{c_p} A' \, C \exp(-\frac{E}{RT}), \\ \frac{\partial C}{\partial t} + u_z \frac{\partial C}{\partial z} = D \frac{\partial^2 C}{\partial z^2} - A' \, C \exp(-\frac{E}{RT}), \end{cases} \tag{2.57}$$

where $D = 5.10^{-5}[\frac{m^2}{s}]$ is the molecular diffusivity,
$\lambda = 5.10^{-5}[\frac{J}{s,m K}]$ - the thermal conductivity, $c_p = 1000[\frac{J}{kg K}]$ - the specific head at constant pressure,
$\tilde{B} = 1.5\ 10^6[\frac{J}{kg}], A' = 10^4[\frac{1}{s}], E = 2.510^4[\frac{J}{mol}]$ are the specific heat relese, the reaction-rate pre-exponential factor and the activation energy, R- the universal gas constant.
The equations were put in the dimensionless form scaling all the lengths to z_0, the density to ρ_0, the velocities u_z to U_0, the temperature to T_0, the special heat release $\tilde{B}$ to $\frac{c_p}{T_0}$, the reaction-rate pre-exponential factor A' to $\frac{U_0}{z_0}$, the activation energy E to $\frac{R}{T_0}$.
Following parameters are used:
$Pe = \frac{z_0 U_0}{D}$, $Le = \frac{\lambda}{c_p D \rho_0}$ - Peclet and Lewis numbers, $P_1 = \frac{Le}{Pe}, P_2 = \frac{1}{Pe}$,
$\beta = \frac{\tilde{B}}{c_p T_0}$; $\delta = \frac{E}{R T_0}$ -the scaled heat-release and activation-energy.
For the dimensionless parameters $t, x = \frac{z}{z_0}, w = \frac{u_z}{U_0}$ we have following equations

$$\begin{cases} \frac{\partial T}{\partial t} + w \frac{\partial T}{\partial x} = \frac{P_1}{\rho} \frac{\partial^2 T}{\partial x^2} + \beta A \, C \exp(-\frac{\delta}{T}), \\ \frac{\partial C}{\partial t} + w \frac{\partial C}{\partial x} = P_2 \frac{\partial^2 C}{\partial x^2} - A \, C \exp(-\frac{\delta}{T}). \end{cases} \tag{2.58}$$

We use following boundary conditions:
1) at the outlet $x = x_0 - \ \frac{\partial s}{\partial x} = 0, s = T; C$,
2) at the inlet $x = 0 - \ \rho = 1, T = C = 1$.

The approach seeks the steady solution as the time asymptotic limit of the solutions of the unsteady equations.

2.11.3 Some numerical results of calculation of gypsum boards

The results of calculations are obtained by MATLAB. We use the discrete values $x_j = jh j = \overline{0,N_x}, N_x\ h = L, t_n = n\tau, n = \overline{0,N_t}, N_t\tau = t_f, N_t = 100, T_0 = 20[^0C]$,
$T_a = 20^0C, T_b = 20 + 345lg(8t+1)[^0C], t \in [0,t_b]$. The stacionary solutions are obtained in the time t_f with the maximal temperature $678.43[^0C]$.
The results of calculation for $t_f = 2000[s], t_b = 600[s]$ are represented in the Figs. 2.83, 2.84,
$T(0,t_f) = 139.94, T(H_1,t_f) = 150.42, T(L,t_f) = 678.43, T_{1v} = 139.08$, $T_{2v} = 387.10$ (the temperature in the first layer of gypsum plate is nearly constant).
The parameters $a_1 = 20, a_2 = 10$ in the spline functions are obtained for minimal value of maximal error for averaging values.
For numerical experiment we use also backward orentation: for gypsum plate $H_2 = 0.0525m$ with density $\rho_2 = 300\frac{kg}{m^3}$, gypsum carton plate $H_1 = 0.0125[m]$ with density $\rho_1 = 1000[\frac{kg}{m^3}], a_1 = 10, a_2 = 20$.
The results of calculation are represented in the Figs. 2.85, 2.86,
$T(0,t_f) = 100.27, T(H_1,t_f) = 479.73, T(L,t_f) = 678.43, T_{1v} = 283.88, T_{2v} = 569.15$, (the temperature for x=0 in this case is smaller as in the direct orientation). In the Fig. 2.87 we can see good coincidence with experimental results $(T = u)$ obtained in [70].

2.11.4 Some numerical results with reaction-diffusion equations for combustion process

The minimum value of flow density, maximum values of the temperature, the reaction rate $R^* = A.C.\exp(-\delta/T)$ are calculated. The influence of the molecular diffusivity and thermal conductivity on the main characteristics of the undisturbed flame flow is obtained for $\beta = 5$ (see Tab. 2.4).

These results show that at the constant molecular diffusivity D the decreasing of the thermal conductivity $\lambda(P_1 = 0.01, Le = 0.1)$ leads to an increasing of the reaction rate and maximal values of the temperature
but at the constant thermal conductivity λ the decreasing of molecular diffusivity $D(P_2 = 0.01, Le = 10)$ results in an increasing of maximal density and in a decreasing of temperature and reaction rate.

Table 2.4 The values of MaxR , Min ρ, Max T depends on P_1, P_2

P_1	P_2	$MaxR^*$	$Min\rho$	$MaxT$
0.01	0.01	98.71	0.0286	6.000
0.10	0.01	61.22	0.0425	**2.467**
0.01	0.10	401.2	0.0228	**15.81**
0.10	0.10	298.8	0.0305	5.994

For fixed values of velocities $w = 1$ and $P_1 = P_2 = 0.1$ the heat-reaction problem are solved numerically using finite differences approximation in two way: $\rho = 1/T$ (small Mah numbers for compressible fluid) and $\rho = 1$ (incompressible flow).
If $\rho = 1/T$ then we have for maximal and averaged values of temperature: $Tmax = 2.957$, and for maximal value of reaction rate $Rmax = 252.20$. For $\rho = 1$ we have $Tmax = 5.969, Rmax = 593.31$.
The results of calculation are represented in folowing Figs. 2.88, 2.89. Using **two simple reaction** we obtain following 1-D reaction-diffusion problem ($\rho = 1$):

$$\begin{cases} \frac{\partial T}{\partial t} + w\frac{\partial T}{\partial x} = P_1 \frac{\partial^2 T}{\partial x^2} + \beta_1 A_1\, C_1 \exp(-\frac{\delta_1}{T}) \\ +\beta_2 A_2\, C_2 \exp(-\frac{\delta_2}{T}), \\ \frac{\partial C_1}{\partial t} + w\frac{\partial C_1}{\partial x} = P_2 \frac{\partial^2 C_1}{\partial x^2} - A_1\, C_1 \exp(-\frac{\delta_1}{T}), \\ \frac{\partial C_2}{\partial t} + w\frac{\partial C_2}{\partial x} = P_2 \frac{\partial^2 C_2}{\partial x^2} - A_2\, C_2 \exp(-\frac{\delta_2}{T}), \\ t \in (0, t_f), x \in (0, L), T(0,t) = 1, C_1(0,t) = 0.8, C_2(0,t) = 0.2, \\ \frac{\partial (s(L,t)}{\partial x} = 0, s = T; C_1; C_2, T(x,0) = 1, \\ C_1(x,0) = 0.8\exp(-\alpha x), C_2(x,0) = 0.2\exp(-\alpha x), \end{cases} \tag{2.59}$$

where $A_1 = A = 5.10^4, A_2 = 5.10^5, \beta_1 = 5, \beta_2 = 1, \delta_1 = 10, \delta_2 = 15, t_f = 1, L = 2; 4,\ \alpha \in [0,6]$ is the parameter for the initial fuel amount in the plate. The results of calculation using **2 reactions** with $Tmax = 5.269$ for $w = 1$ $(T(2,1) = 1.248)$ and $w = 4$ $(T(2,1) = 5.200)$ are represented in the folowing Figs. 2.90, 2.91.

Using **one reaction** $(C_2 = 0, C_1(0,t) = 1)$ for $w = 1$ $(Tmax = 6.0881, T(2,1) = 1.295)$ and $w = 4$ $(Tmax = 6.114, T(2,1) = 1.248)$ we have the following Figs. 2.92, 2.93.

In the following Figs. 2.94, 2.95 for one reaction we can see the surface in (x,t) plane at $w = 4, L = 4, \alpha = 6, P_1 = P_2 = 0.1$, $(Tmax = 6.281, T(4,1) = 4.085)$ and profile of temperature for $w = 3, P_1 = 0.01, P_2 = 0.001$ $(Tmax = 5.690, T(2,1) = 4.000)$.

For stacionary reaction-diffusion equation by $(A_1 = 5.10^4, A_2 = 5.10^8, \delta_1 = 10, \delta_2 = 20,\ \rho = 1, w = 0, T = T(x), C_1 = C_1(x), C_2 = C_2(x), x \in [0,2])$
with BCs $T(0) = 1, C_1(0) = 0.8, C_2(0) = 0.2, T(2)' = C_1'(2) = C_2'(2) = 0$, by multiply second both equations with β_1, β_2 and summed the equations for $x \to 2$ follows that
$Tmax = T(2) = 1 + \frac{\beta_1 0.8 + \beta_2 0.2}{Le}, Le = \frac{P_1}{P_2}$.
Using Matlab solver "bvp4c" it is obtained, that the increasing of the axial velocity w leads to an increasing of Tmax at $Le > 1$ and to decreasing of Tmax at $Le < 1$, but for $Le = 1$ Tmax is not depending on w (see Tab. 2.5). For one simple reactions :$C_2 = 0, C_1 = C, \beta_1 = \beta, \delta_1 = \delta$.

Table 2.5 The values of MaxT , T'(0), C'(0) depends on P_1, P_2, w, β

P_1	P_2	w	β	$MaxT$	$T'(0)$	$C'(0)$
0.01	0.01	0.0	1.0	2.00	68.6	−68.6
0.01	0.01	1.0	1.0	2.00	9.15	−9.15
0.02	0.01	0.0	1.0	**1.50**	17.6	−35.3
0.02	0.01	1.0	1.0	**1.83**	12.9	−9.02
.005	0.01	0.0	1.0	3.00	98.1	−78.3
.005	0.01	0.5	1.0	2.54	84.9	−69.5

2.12 Conclusions

- The FDSES and trigonometric interpolation are equivalent.
- The all eigenvalues and eigenvectors for finite difference operators are obtained.
- The algorithm of discrete Fourier method are formed in different wise.
- The advantages of the FDSES and DM methods for solution the problems with PBCs are demonstrated in comparision with local FDS methods.

The nonlinear heat transfer problem is approximated with the nonlinear initial value problems of a system of ODEs of the first order. Depending on the parameters two type of solutions are obtained:
1) for large value of the time t the solution is stationary or tends to zero,
2) in the fixed time moment the solution have blow up phenomena - the solution tends to infinity in a small interval or in all domain by a fixed time moment.
Two monotonous functions corresponding to the paths with increasing and decreasing source term may be constructed and two different solutions exist in definite ranges of the source term. This leads to hysteresis phenomena in the solutions transformations at change of the source term.
The ill-posed problems for parabolic PDE are regularized and solved with Matlab solver "ode15" by discretizing the spatial derivatives with finite differences.
The 1-D nonstationary diffusion-convection initial-boundary value problem (IBVP) in layered domain is reduced to problem of ODEs using the hyperbolic type splines. The nonlinear system of PDEs (thermal diffusion coefficients depend on temperature) is studied for the heat transfer processes in two gypsum layers. The numerical results are compared with the experimental results and the matching results can be considered sufficiently accurate for engineering-technical calculations.
The nonstationary physical model for simple chemical reaction with temperature and 2 reaction-diffusion equations characterised combustion process with Arrhenius kinetics is considered.

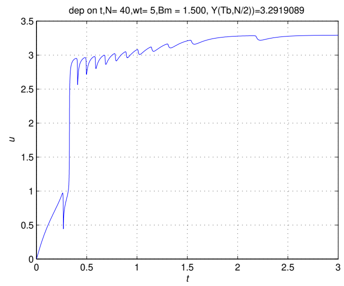


Fig. 2.8 Max β depending on $t$by $Bm = 1.5, \omega\tau = 5$

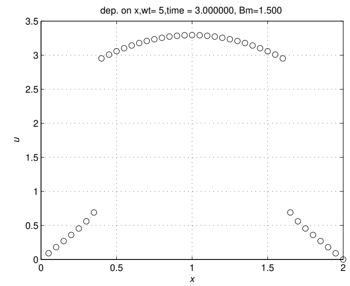


Fig. 2.9 β depending on $x$by $Bm = 1.5, \omega\tau = 5, t = 3$

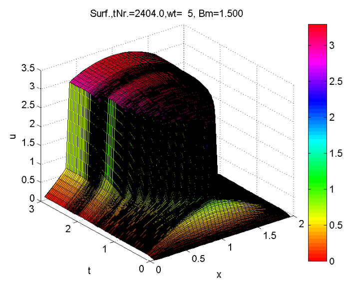


Fig. 2.10 $\beta(x,t)$ surf. by $Bm = 1.5, \omega\tau = 5$

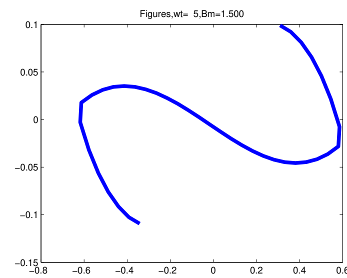


Fig. 2.11 Droplet by $Bm = 1.5, \omega\tau = 5, t = 3$

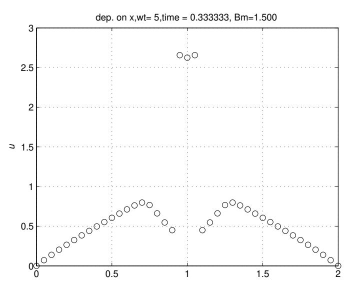


Fig. 2.12 β depending on x by $t = 0.3333$

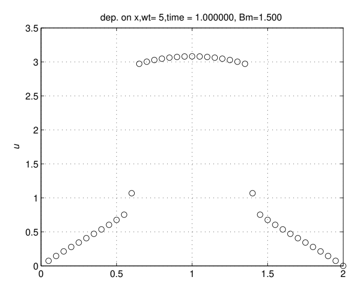


Fig. 2.13 β depending on x by $t = 1.000$

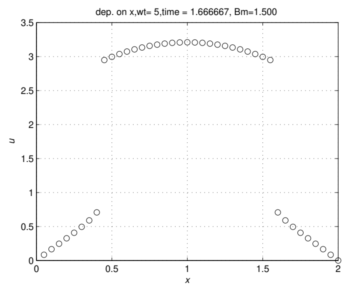


Fig. 2.14 β depending on x by $t = 1.6666$

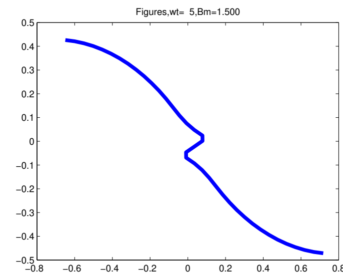


Fig. 2.15 Droplet by $Bm = 1.5, \omega\tau = 5, t = 0.33333$

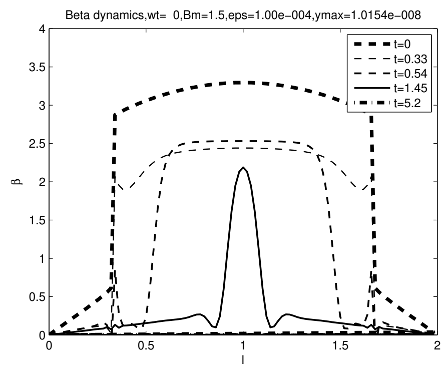


Fig. 2.16 $\beta(l,t_i)$ in dependence on $l$by $Bm = 1.5, \omega\tau = 5 \Rightarrow 0, t_f = 6$ using reverse function

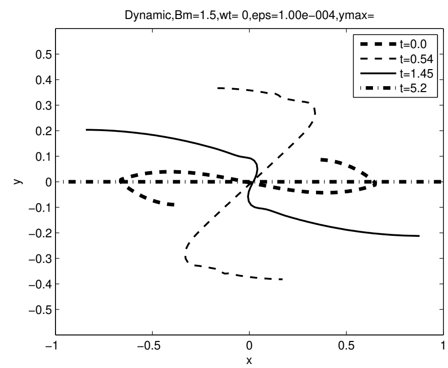


Fig. 2.17 Droplet reverse dynamics at $Bm = 1.5, \omega\tau = 5 \Rightarrow 0, t_f = 6$

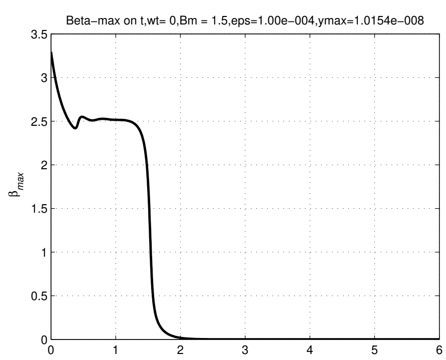


Fig. 2.18 Max β (in dependence on t by $Bm = 1.5, \omega\tau = 5 \Rightarrow 0$ using reverse function

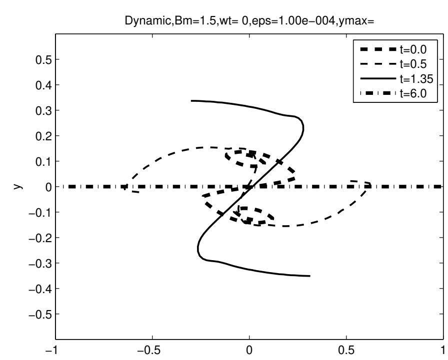


Fig. 2.19 Droplet reverse dynamics at $Bm = 1.5, \omega\tau = 12 \Rightarrow 0, t_f = 6$

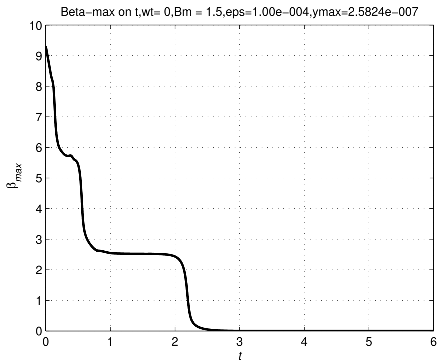


Fig. 2.20 Max β in dependence on t by $Bm = 1.5, \omega\tau = 12 \Rightarrow 0$ using reverse function

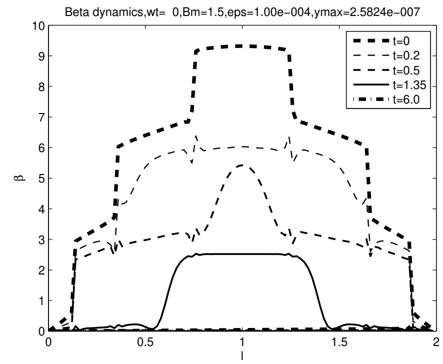


Fig. 2.21 $\beta(l,t_i)$ in dependence on $l$by $Bm = 1.5, \omega\tau = 12 \Rightarrow 0, t_f = 6$ using reverse function

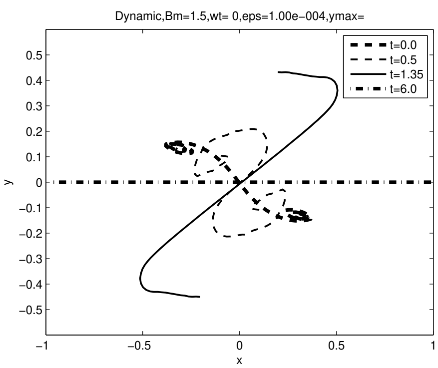


Fig. 2.22 Droplet reverse dynamics at $Bm = 1.5, \omega\tau = 24 \Rightarrow 0, t_f = 6$

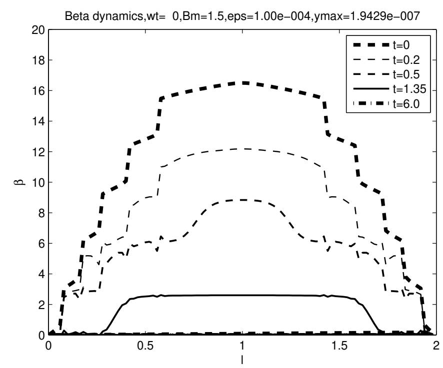


Fig. 2.23 $\beta(l,t_i)$ in dependence on l by $Bm = 1.5, \omega\tau = 24 \Rightarrow 0, t_f = 6$ using reverse function

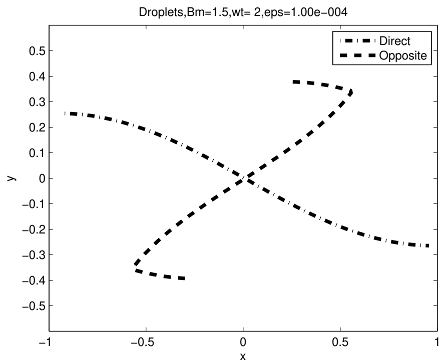


Fig. 2.24 Droplets at $Bm = 1.5, \omega\tau = 0 \Rightarrow 2 \Rightarrow 5 \bigcup 5 \Rightarrow 2, t_f = 6$

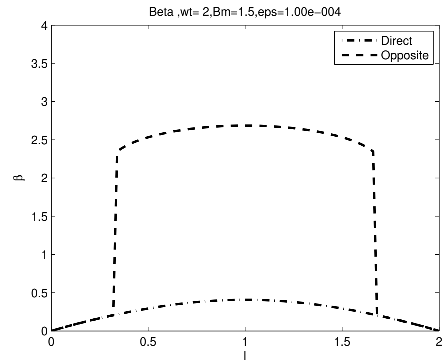


Fig. 2.25 $\beta(l)_{stac}$ by $Bm = 1.5, \omega\tau = 0 \Rightarrow 2 \Rightarrow 5 \bigcup 5 \Rightarrow 2, t_f = 6$

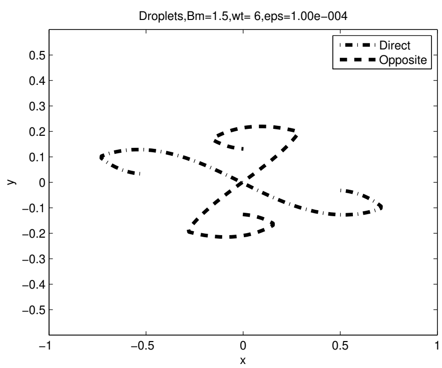


Fig. 2.26 Droplets at $Bm = 1.5, \omega\tau = 0 \Rightarrow 6 \Rightarrow 8 \bigcup 8 \Rightarrow 6, t_f = 6$

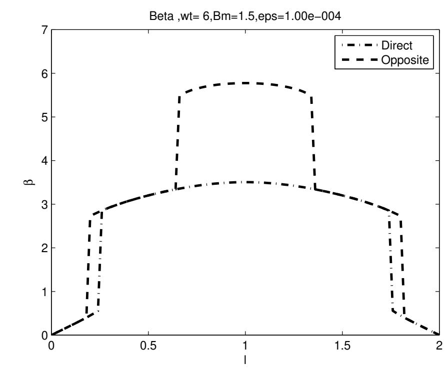


Fig. 2.27 $\beta(l)_{stac}$ by $Bm = 1.5, \omega\tau = 0 \Rightarrow 6 \Rightarrow 8 \bigcup 8 \Rightarrow 6, t_f = 6$

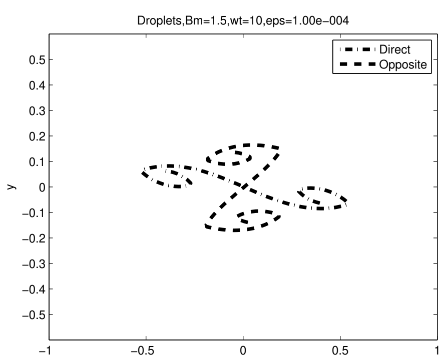


Fig. 2.28 Droplets at $Bm = 1.5, \omega\tau = 0 \Rightarrow 10 \Rightarrow 12 \bigcup 12 \Rightarrow 10, t_f = 6$

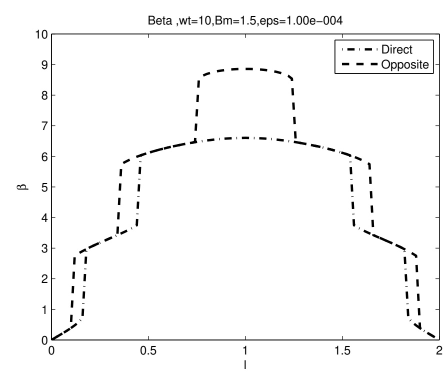


Fig. 2.29 $\beta(l)_{stac}$ by $Bm = 1.5, \omega\tau = 0 \Rightarrow 10 \Rightarrow 12 \bigcup 12 \Rightarrow 10, t_f = 6$

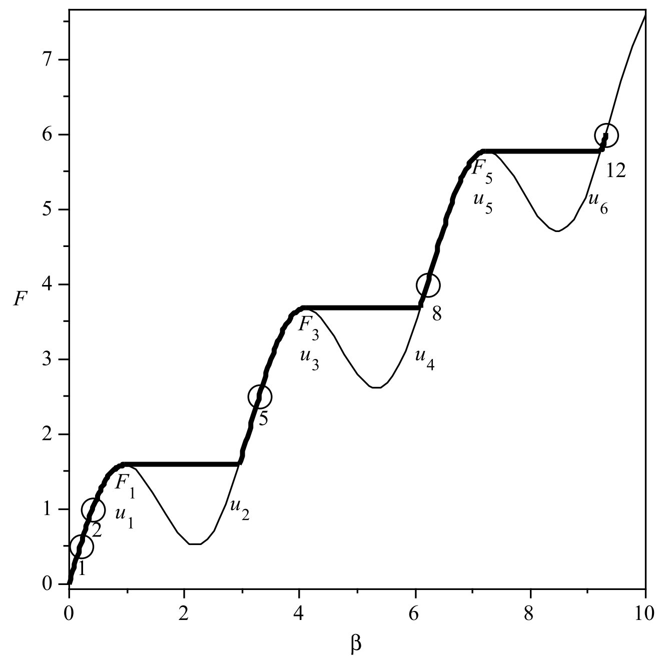


Fig. 2.30 Direct function $F(u)$ at $Bm = 1.5$

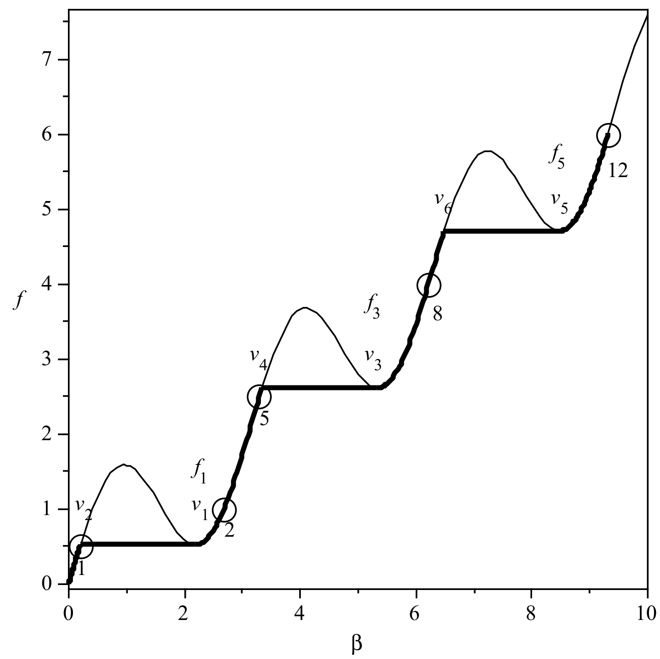


Fig. 2.31 Reverse function $F(u)$ at $Bm = 1.5$

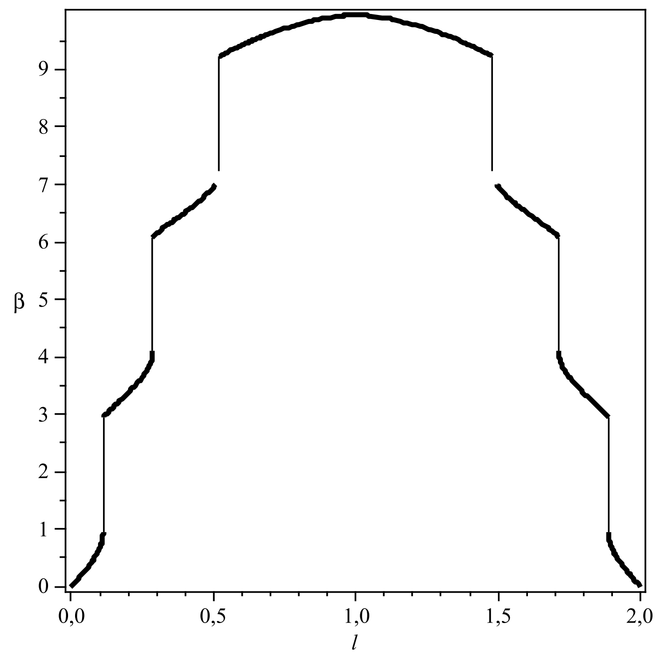


Fig. 2.32 Stationary solution at $Bm = 1.5, \omega\tau = 15$ in the case of direct function

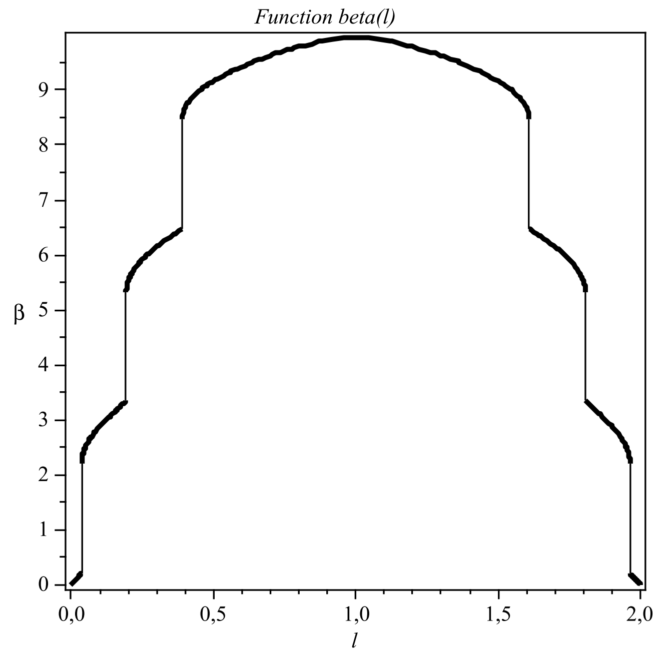


Fig. 2.33 Stationary solution at $Bm = 1.5, \omega\tau = 15$ in the case of reverse function

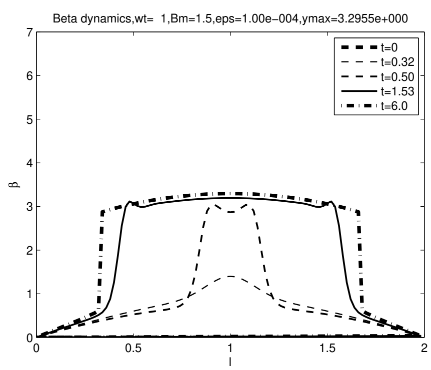


Fig. 2.34 $\beta(l, t_i)$ in dependence on l by $Bm = 1.5, \omega\tau = 0 \Rightarrow 5, t_f = 6$

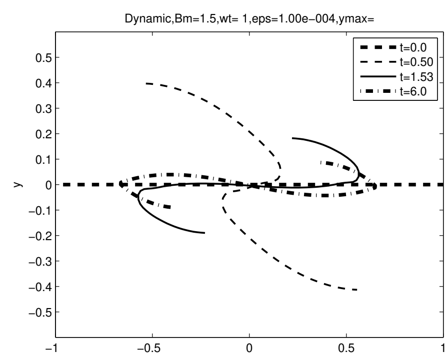


Fig. 2.35 Droplet dynamics at $Bm = 1.5, \omega\tau = 0 \Rightarrow 5, t_f = 6$

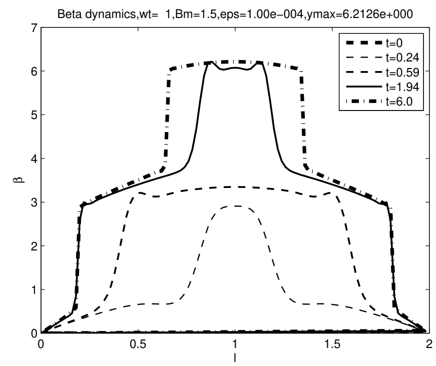


Fig. 2.36 $\beta(l,t_i)$ in dependence on l by $Bm = 1.5, \omega\tau = 0 \Rightarrow 8, t_f = 6$

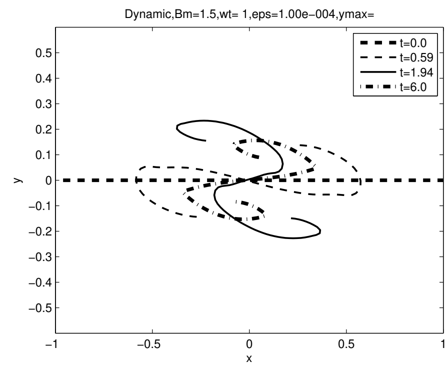


Fig. 2.37 Droplet dynamics at $Bm = 1.5, \omega\tau = 0 \Rightarrow 8, t_f = 6$

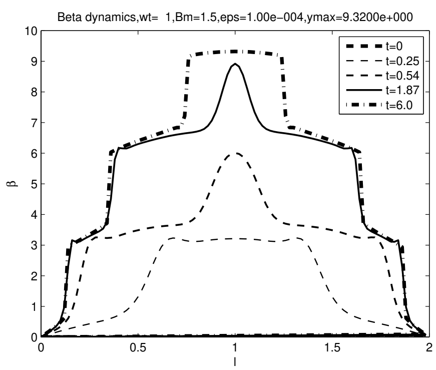


Fig. 2.38 $\beta(l,t_i)$ in dependence on $l$by $Bm = 1.5, \omega\tau = 0 \Rightarrow 12, t_f = 6$

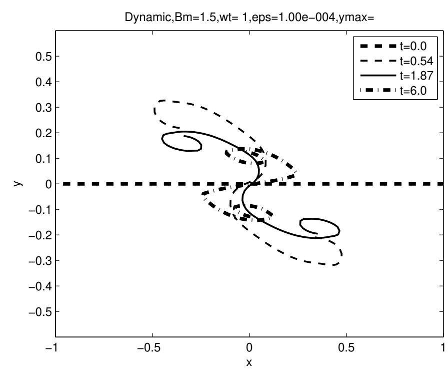


Fig. 2.39 Droplet dynamics at $Bm = 1.5, \omega\tau = 0 \Rightarrow 12, t_f = 6$

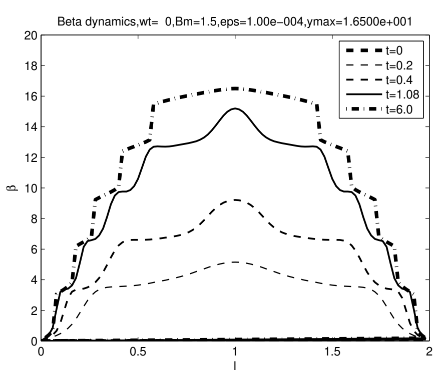


Fig. 2.40 $\beta(l,t_i)$ in dependence on $l$by $Bm = 1.5, \omega\tau = 0 \Rightarrow 24, t_f = 6$

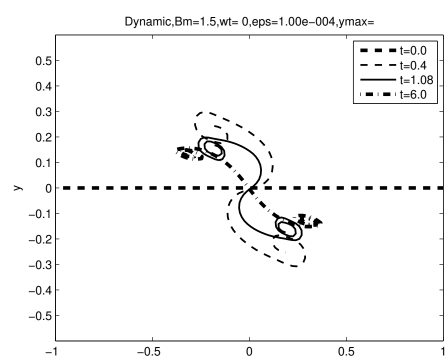


Fig. 2.41 Droplet dynamics at $Bm = 1.5, \omega\tau = 0 \Rightarrow 24, t_f = 6$

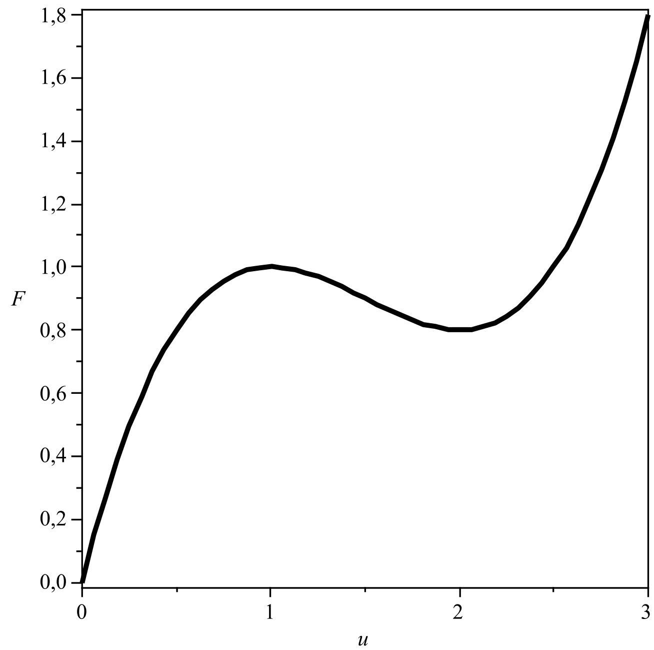


Fig. 2.42 Function $F(u)$

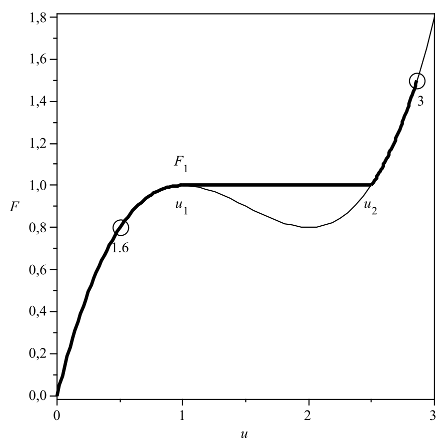


Fig. 2.43 Direct modified function $F(u)$. The following numerical values are shown in figure: $u_1 = 1, u_2 = 2.5, F_1 = 1$; In the fixed points $\odot$ there are values of g with the coordinates $(u_s(1), g_0/2)$

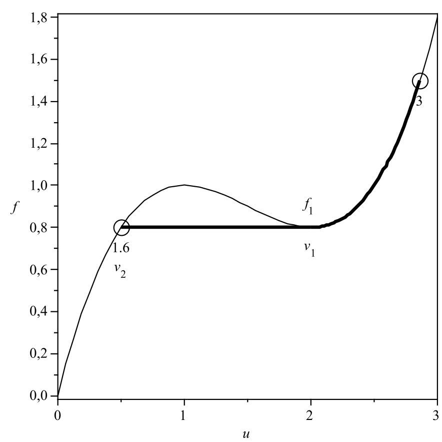


Fig. 2.44 Reverse modified function $f(u)$. The following numerical values are shown in figure:$v_1 = 2, v_2 = 0.5, f_1 = 0.8$; In the fixed points $\odot$ there are values of g_0 with the coordinates $(u_s(1), g_0/2)$

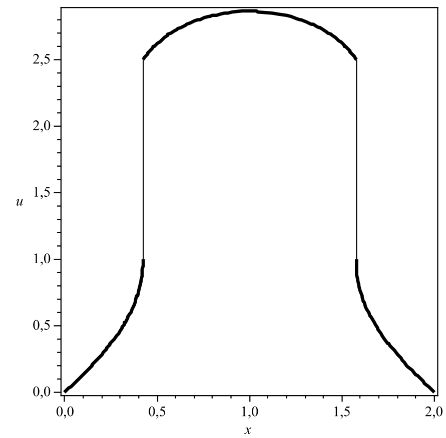


Fig. 2.45 Stationary solution $u_s(x)$ at $g_0 = 3$ in the case of direct function

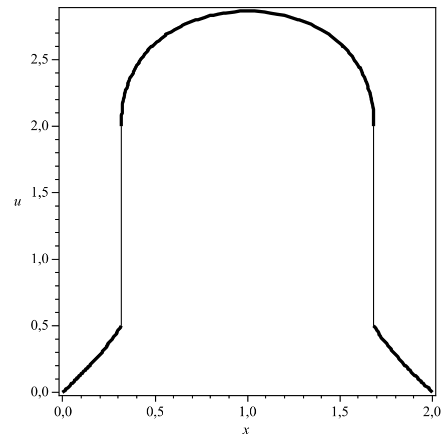


Fig. 2.46 Stationary solution $u_s(x)$ at $g_0 = 3$ in the case of reverse function

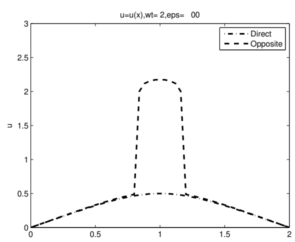


Fig. 2.47 $u(x)_{stac}$ at $g_0 = 0 \Rightarrow 1.6 \Rightarrow 3 \bigcup 3 \Rightarrow 1.6, T = 6$

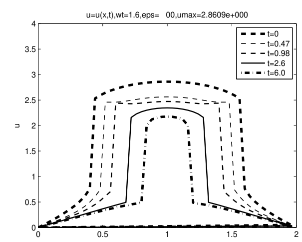


Fig. 2.48 Dynamic of solution $u = u(x,t)$ obtained with reverse function for $g_0 = 3 \Rightarrow 1.6, T = 6$

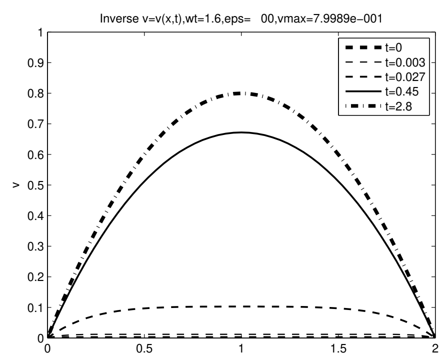


Fig. 2.49 Nonstationary solution $v(x,t)$ for $g_0 = 1.6$

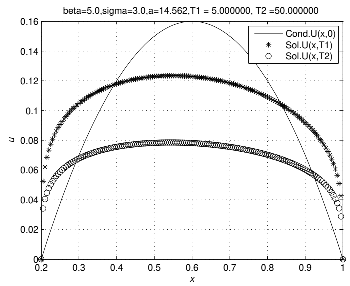


Fig. 2.50 Solution in one layer, $u_{st} = 0$for $r \in [0.2, 1]$, $\beta = 5, \sigma = 3, a = 14.5615$

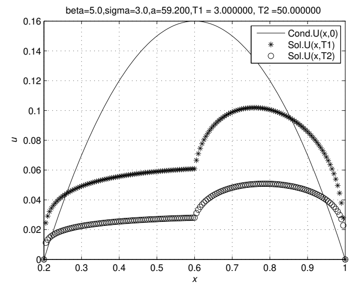


Fig. 2.51 Solution in two layers, $u_{st} = 0$ for $r \in [0.2, 1]$, $\beta = 5, \sigma = 3, a = 59.2001$, $\lambda_1 = 100, \lambda_2 = 1$

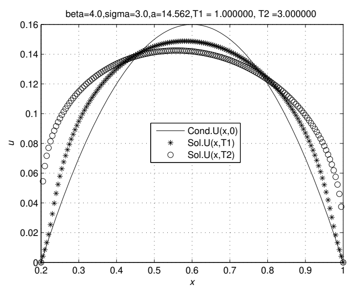


Fig. 2.52 Solution in one layer, $u_{st}$for $r \in [0.2, 1]$, $\beta = 4, \sigma = 3, a = 14.5615$

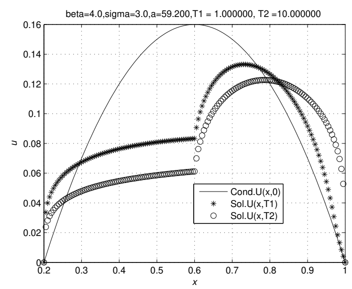


Fig. 2.53 Solution in two layers, u_{st} for $r \in [0.2, 1]$, $\beta = 4, \sigma = 3, a = 59.2001$, $\lambda_1 = 100, \lambda_2 = 1$

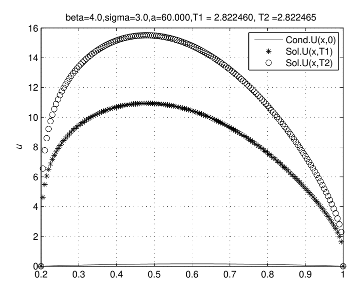


Fig. 2.54 Solution in one layer $u \to \infty$ for $r \in (0.2, 1)$, $\beta = 4, \sigma = 3, a = 60$

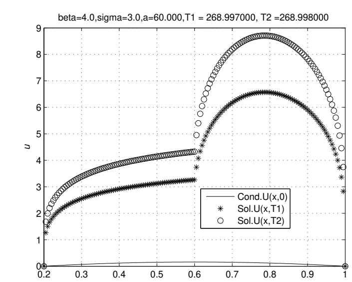


Fig. 2.55 Sol. in two layers $u \to \infty$ for $r \in (0.2, 1)$, $\beta = 4, \sigma = 3, a = 60$, $\lambda_1 = 100, \lambda_2 = 1$

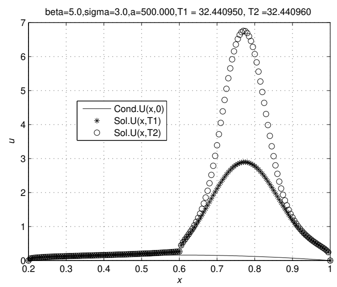


Fig. 2.56 Solution in two layer $u \to \infty$ for $r = 0.75$, $\beta = 5, \sigma = 3, a = 500$, $\lambda_1 = 100, \lambda_2 = 1$

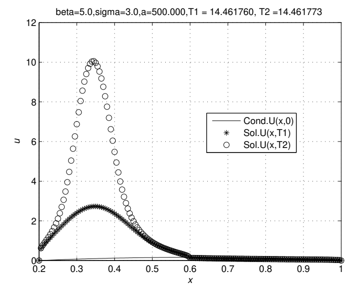


Fig. 2.57 Solution in two layers $u \to \infty$ for $r = 0.25$, $\beta = 5, \sigma = 3, a = 500$, $\lambda_1 = 1, \lambda_2 = 100$

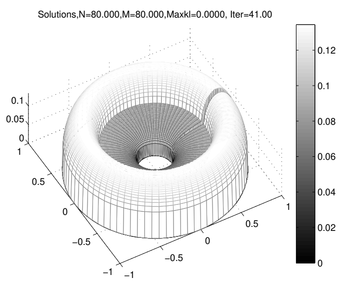


Fig. 2.58 2D solution in two layers $u_{st} \neq 0$, $\beta = 4, \sigma = 3, a = \mu_1 = 59.2001$, $\lambda_1 = 100, \lambda_2 = 1$

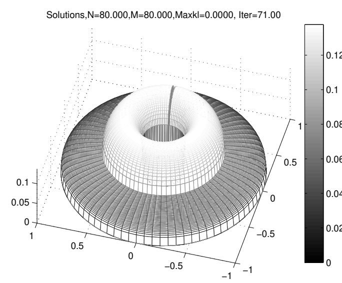


Fig. 2.59 2D solution in two layers $u_{st} \neq 0$, $\beta = 4, \sigma = 3, a = \mu_1 = 58.9950$, $\lambda_1 = 1, \lambda_2 = 100$

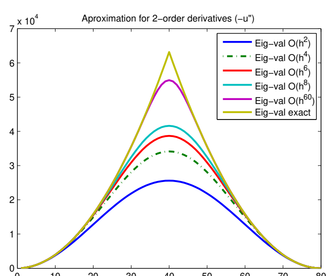


Fig. 2.60 Eigenvalues for $-u''$ by $N = 80, L = 1, n = 1;2;3;4;30$

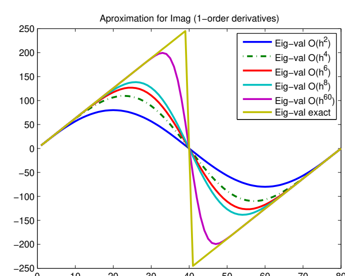


Fig. 2.61 Imaginary part of eigenvalues for u' by $N = 80, L = 1, n = 1;2;3;4;30$

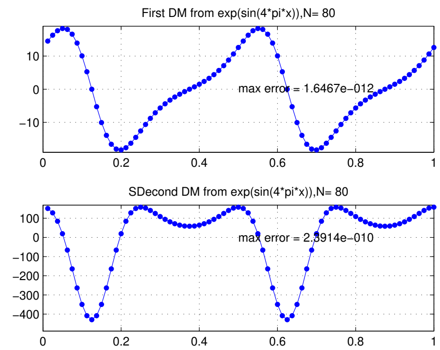


Fig. 2.62 First and second derivatives from $exp(\sin(4\pi x)$, obtained with the differentiation matrix A_1, A_2 of trigonometrical interpolant by$N = 80, L = 1$, (maximal errors: 1.65 e-12, 2.39 e-10)

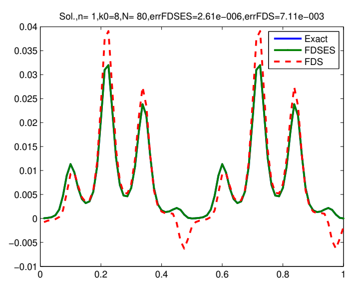


Fig. 2.63 Solutions: exact, FDS (n=1), FDSES by $k_0 = 8, N = 80$

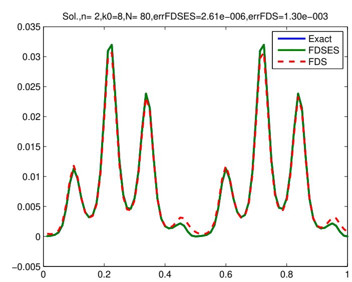


Fig. 2.64 Solutions: exact, FDS (n=2), FDSES by $k_0 = 8, N = 80$

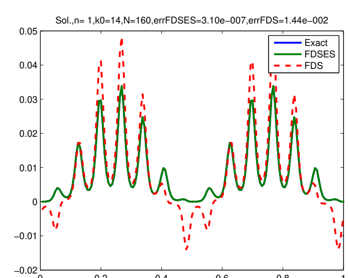


Fig. 2.65 Solutions: exact, FDS (n=1), FDSES by $k_0 = 14, N = 160$

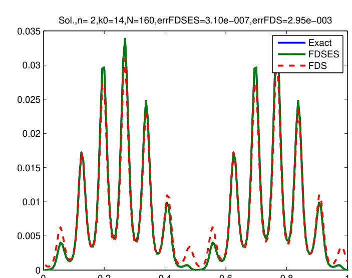


Fig. 2.66 Solutions: exact, FDS (n=2), FDSES by $k_0 = 14, N = 160$

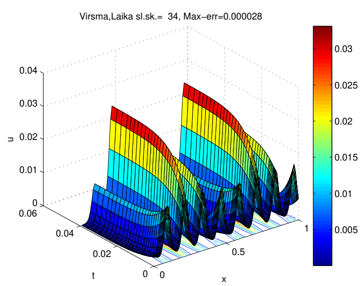


Fig. 2.67 Solutions FDSES, u(x,t) by $k_0 = 8, N = 80$

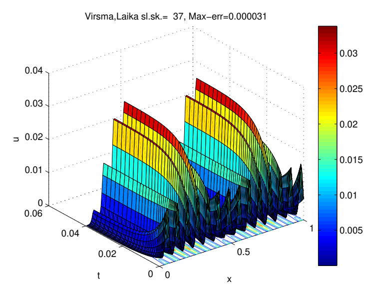


Fig. 2.68 Solutions FDSES, u(x,t) by $k_0 = 14, N = 160$

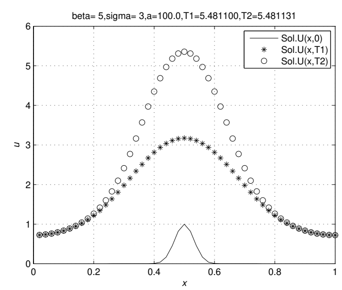


Fig. 2.69 $U \to \infty$ for $x = 0.5$, $\beta = 5, \sigma = 3, a = 100, T_* = 5.481136$

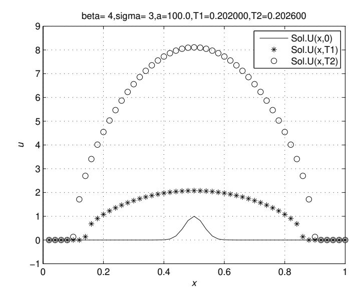


Fig. 2.70 $U \to \infty$ for $x \in (0,1)$, $\beta = 4, \sigma = 3, a = 100, T_* = 0.2020261$

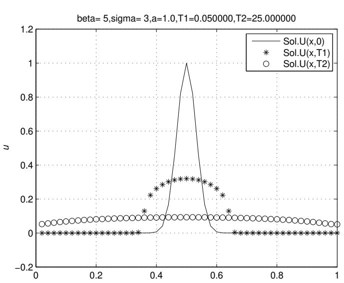


Fig. 2.71 $U \to 0$ if $t \to \infty$, for $\beta = 5, \sigma = 3, a = 1$

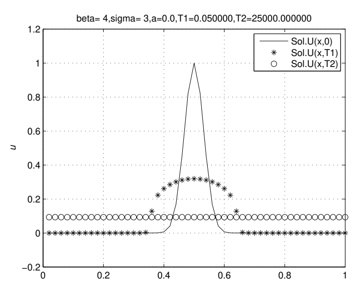


Fig. 2.72 $U \to$ stationary if $t \to \infty$, for $\beta = 4, \sigma = 3, a = 0.01$

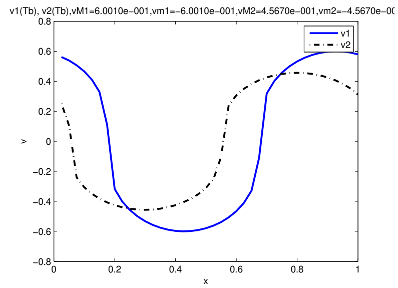


Fig. 2.73 Solutions by $t = 0.1, N = 40$ depending on x

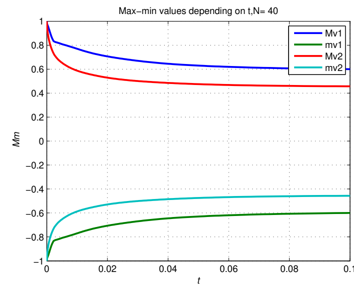


Fig. 2.74 Maximal and minimal values depending on t

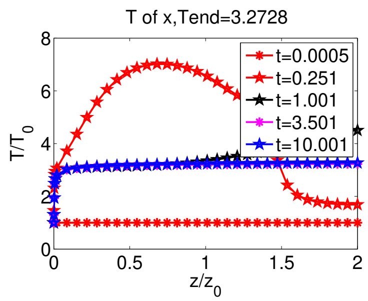


Fig. 2.75 Temperature depending on x for fixed t

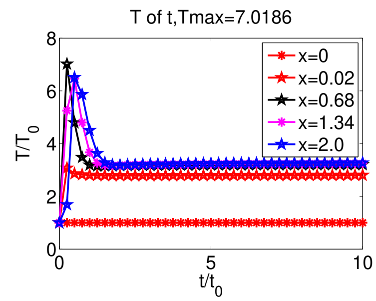


Fig. 2.76 Temperature depending on time t for fixed x

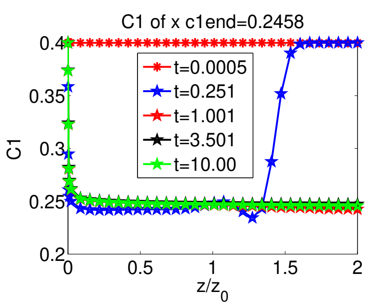


Fig. 2.77 Propane C_3H_8 concentration vs.x for fixed t

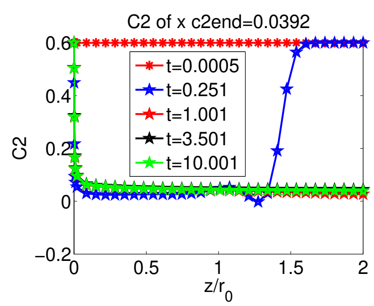


Fig. 2.78 O_2 concentration vs.x for fixed t

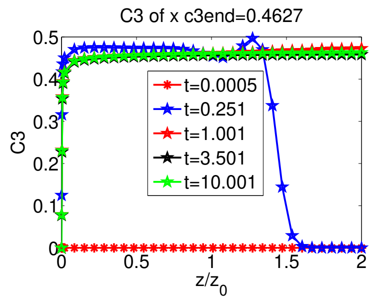


Fig. 2.79 *CO_2* concentration vs.x for fixed t

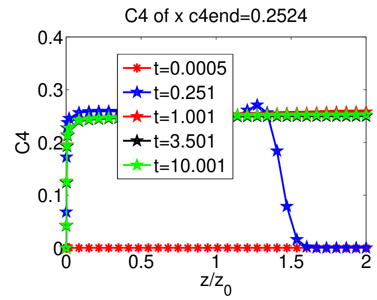


Fig. 2.80 *H_2O* concentration vs.x for fixed t

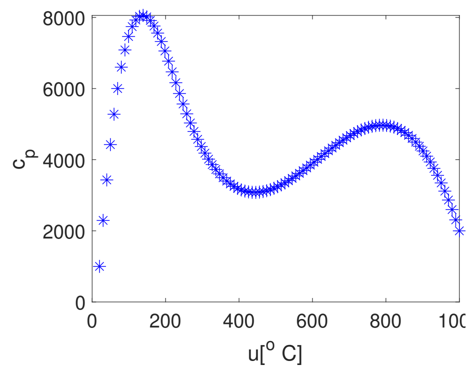


Fig. 2.81 Specific heat c_p dependence on temperature u

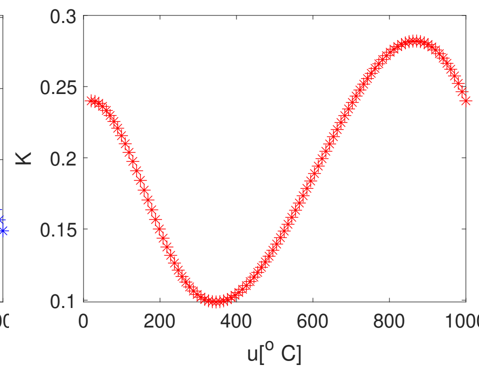


Fig. 2.82 Thermal condutivity K dependence on temperature u

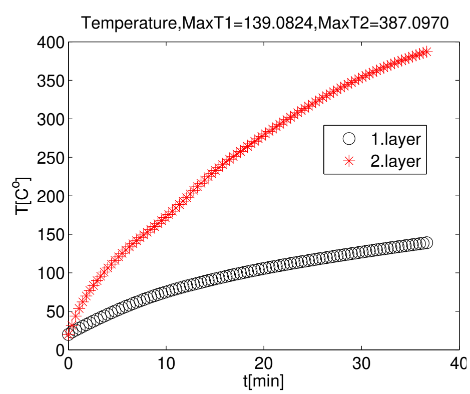


Fig. 2.83 Averaging temperature depends on t for $t_f = 2000[s]$

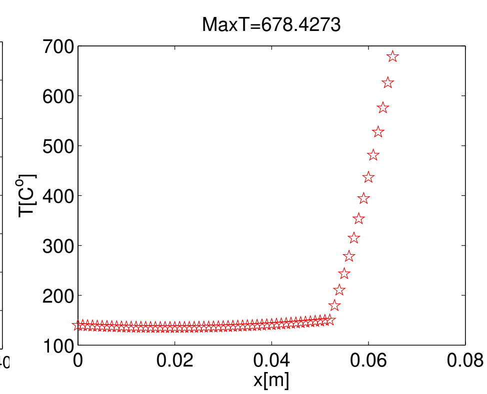


Fig. 2.84 Temperature depends on x for $t_f = 2000[s]$

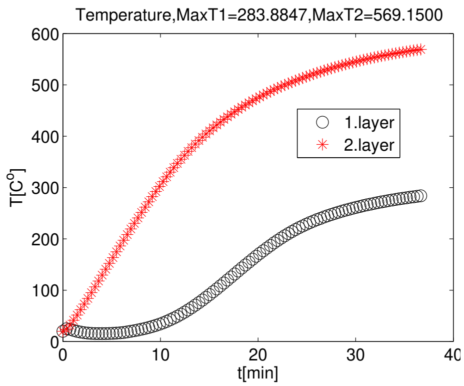


Fig. 2.85 Backward averaging temperature depends on t for $t_f = 2000[s]$

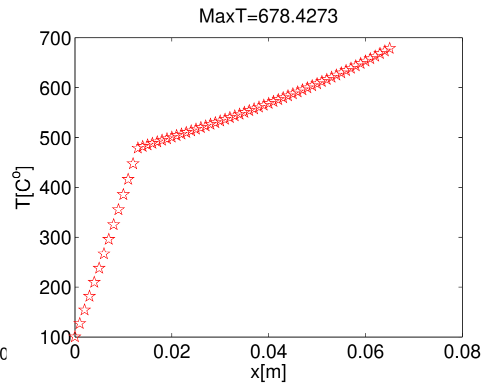


Fig. 2.86 Backward temperature depends on x for $t_f = 2000[s]$

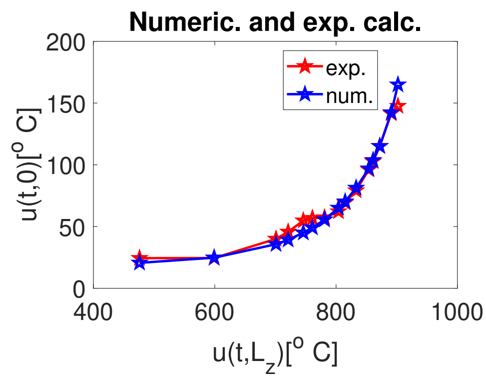


Fig. 2.87 Comparison the temperature u of numerical and experimental results

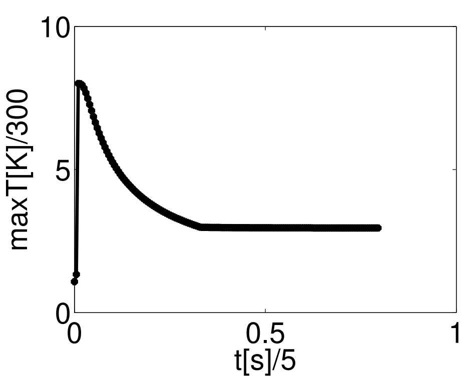


Fig. 2.88 Maximal temperature 2.957 at outlet on time t for $\rho = 1/T$

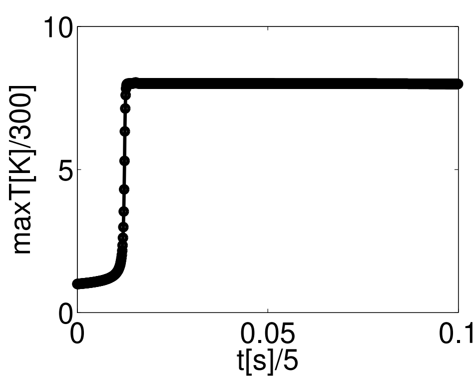


Fig. 2.89 Maximal temperature 5.969 depending on time t for $\rho = 1$

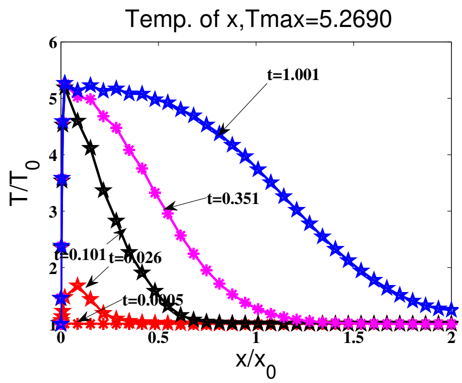


Fig. 2.90 Profile of temperature depending on x in fixed time t for $w = 1, \alpha = 6, P_1 = P_2 = 0.1$

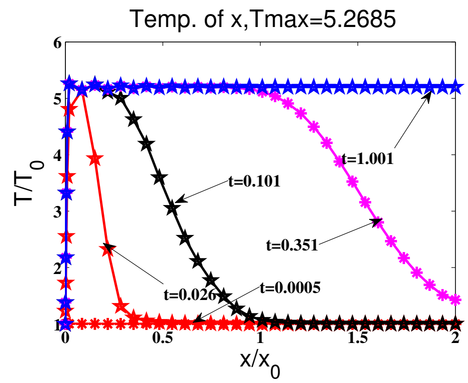


Fig. 2.91 Profile of temperature depending on x in fixed time t for $w = 4, \alpha = 6, P_1 = P_2 = 0.1$

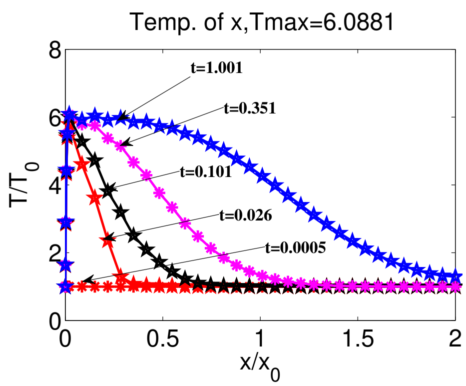


Fig. 2.92 Profile of temperature depending on x in fixed time t for $w = 1, \alpha = 6, P_1 = P_2 = 0.1$

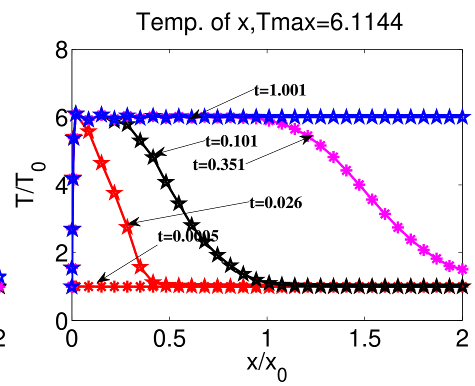


Fig. 2.93 Profile of temperature depending on x in fixed time t for $w = 4, \alpha = 6, P_1 = P_2 = 0.1$

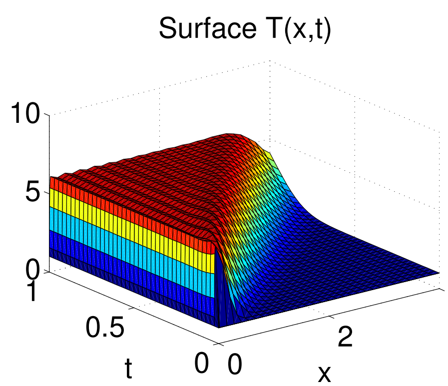


Fig. 2.94 Temperature depending on(x,t) for $w = 4, L = 4, \alpha = 6, P_1 = P_2 = 0.1$

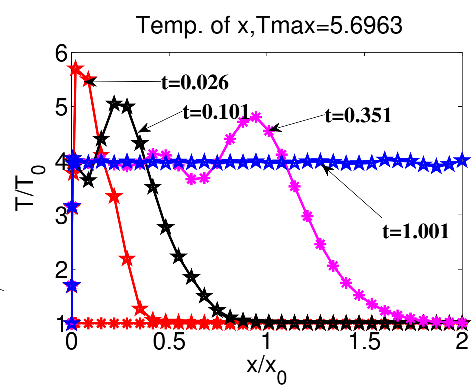


Fig. 2.95 Profile of temperature depending on x in fixed time t for $w = 3, L = 2, \alpha = 6, P_1 = 0.01, P_2 = 0.001$

Chapter 3
The hyperbolic type PDEs: H. Kalis, S. Rogovs, 2011 [74]

For numerical experiments we consider the linear initial-boundary value problem for hyperbolic type equations in following form:

$$\begin{cases} \alpha_2 \frac{\partial^2 T(x,t)}{\partial t^2} + \alpha_1 \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}(\bar{k}\frac{\partial T(x,t)}{\partial x}) + f(x,t), \\ x \in (0,L), t \in (0,t_f), \\ \frac{\partial T(0,t)}{\partial x} - \sigma_1(T(0,t) - T_l(t)) = g_1(t), \\ \frac{\partial T(L,t)}{\partial x} + \sigma_2(T(L,t) - T_r(t)) = g_2(t), t \in (0,t_f), \\ T(x,0) = T_0(x), \frac{\partial T(x,0)}{\partial t} = \bar{T}_0(x), x \in (0,L), \end{cases} \tag{3.1}$$

where $\bar{k} > 0, \alpha_1 \geq 0, \alpha_2 \geq 0, \sigma_1 \geq 0, \sigma_2 \geq 0 (\sigma_1^2 + \sigma_2^2 \neq 0)$, are the constant parameters, t_f is the final time, $T_l(t), T_r(t), T_0(x), \bar{T}_0(x)$ are given functions.

For the 1-D hiperbolic heat conduction equation the parameters $\alpha_1 = 1$ and $\alpha_2 = \tau$ is the relaxation time (small parameter), the function $T(x,t)$ is the distribution of the temperature for modelling an example intensive steel quenching or laser pulse duration.
In this case we have two problems: the direct problem with given function $\bar{T}_0(x)$ from second initial condition and the inverse problem with unknown second initial condition. For the inverse problem the function $\bar{T}_0(x)$ is unknown and then we can used the aditional condition $T(x,t_f) = T_f(x)$, where T_f is given final temperature.

If $T(x,t) = V(x,t) + (C_1(t)x + C_2(t))$ then we have for the function $V(x,t)$ the problem with homogenous BCs

$$\begin{cases} \alpha_2 \frac{\partial^2 V(x,t)}{\partial t^2} + \alpha_1 \frac{\partial V(x,t)}{\partial t} = \frac{\partial}{\partial x}(\bar{k}\frac{\partial V(x,t)}{\partial x}) + \bar{f}(x,t), \\ x \in (0,L), t \in (0,t_f), \\ \frac{\partial V(0,t)}{\partial x} - \sigma_1 V(0,t) = 0, \frac{\partial V(L,t)}{\partial x} + \sigma_2 V(L,t) = 0, t \in (0,t_f), \\ V(x,0) = V_0(x), \frac{\partial V(x,0)}{\partial t} = \bar{V}_0(x), x \in (0,L), \end{cases} \tag{3.2}$$

where $V_0(x) = T_0(x) - C_1(0)x - C_2(0), \bar{V}_0(x) = \bar{T}_0(x) - \dot{C}_1(0)x - \dot{C}_2(0)$,
$\bar{f}(x,t) = f(x,t) - \alpha_2(\ddot{C}_1(t)x + \ddot{C}_2(t)) - \alpha_1(\dot{C}_1(t)x + \dot{C}_2(t))$,
$C_1(t) = (\sigma_2 g_1(t) + \sigma_1 g_2(t) + \sigma_1\sigma_2(T_r(t) - T_l(t)))/\sigma_0$,
$C_2(t) = (\sigma_2 T_l(t) + \sigma_2 T_r(t) + g_2(t) - g_1(t) + \sigma_1\sigma_2 T_l(t)L-$
$\sigma_2 g_1(t)L)/\sigma_0$, $\sigma_0 = \sigma_1 + \sigma_2 + \sigma_1\sigma_2 L$.
For the inverse problem the function $\bar{V}_0(x)$ is unknown.

3.1 The analytical solution of the problem with homogenous BCs

Using the finite differences of second order approximation for partial derivatives of second order respect to x we obtain from (3.2) the initial value problem for system of ODEs of second order in the following matrix form

$$\begin{cases} \alpha_2 \ddot{U}(t) + \alpha_1 \dot{U}(t) + \bar{k}AU(t) = F(t), \\ U(0) = U_0, \dot{U}(0) = \bar{U}_0, \end{cases} \tag{3.3}$$

where A is the 3-diagonal matrix of $N+1$ order, $U(t), \dot{U}(t), \ddot{U}(t), U_0$, $\bar{U}_0, F(t)$ are the column-vectors of $N+1$ order with elements $u_j(t) \approx V(x_j,t))$, $\dot{u}_j(t) \approx \frac{\partial V(x_j,t)}{\partial t}, \ddot{u}_j(t) \approx \frac{\partial^2 V(x_j,t)}{\partial t^2}$,
$u_j(0) = V_0(x_j), \dot{u}_j(0) = \bar{V}_0(x_j), f_j(t) = \bar{f}(x_j,t), j = \overline{0,N}$.

We can consider the **analytical solutions** of (3.3) using the spectral representation of matrix $A = WDW^T$. From transformation $V = W^T U$ follows the seperate system of ODEs

$$\begin{cases} \alpha_2 \ddot{V}(t) + \alpha_1 \dot{V}(t) + \bar{k}DV(t) = G(t), \\ V(0) = W^T U_0, \dot{V}(0) = W^T \bar{U}_0, \end{cases} \tag{3.4}$$

where $V(t), \dot{V}(t), \ddot{V}(t), V(0), \dot{V}(0), G(t) = P^T F(t)$ are the column-vectors of M order with elements
$v_k(t), \dot{v}_k(t), \ddot{v}_k(t), v_k(0), \dot{v}_k(0), g_k(t) k = \overline{1,M}, M = N+1$.
The solution of this system is the function

$$\begin{cases} v_k(t) = exp(-0.5\alpha_1 t/\alpha_2)(C_k \sinh(\kappa_k t) + B_k \cosh(\kappa_k t)) + \\ \frac{1}{\kappa_k \alpha_2} \int_0^t exp(-0.5\frac{\alpha_1}{\alpha_2}(t-\tau)) \sinh(\kappa(t-\tau))G(\tau)d\tau, \end{cases} \tag{3.5}$$

where
$\kappa_k = \sqrt{0.25\alpha_1^2/\alpha_2^2 - \bar{k}\mu_k/\alpha_2}$, $B_k = v_k(0), C_k = \frac{1}{\kappa}(\dot{v}_k(0) + \frac{\alpha_1}{2\alpha_2}v_k(0)$.
If $4\bar{k}\mu_k\alpha_2/\alpha_1^2 > 1$, then the hyperbolic functions to need replaced with the trigonometrical and the parameter κ_k with
$\sqrt{\bar{k}\mu_k/\alpha_2 - 0.25\alpha_1^2/\alpha_2^2}$.
Note: in (3.4) the first and last components of vectors $U_0, \bar{U}_0, F$ are divided with $\sqrt{2}$, but $v_1(t), v_{N+1}(t)$ need to multiply with $\sqrt{2}$.
If $\kappa_k = 0$, then $v_k(t) = \exp(-0.5t/\tau)[t(\dot{v}_k(0) + 0.5v_k(0)/\tau) + v_k(0)]$.

For hyperbolic type equation ($\alpha_2 \neq 0$) by finite σ_1, σ_2 the system of ODEs (3.3) can be rewritten in a normal form

$$\dot{u} = Bu + \bar{F}, u(0) = u_0, \tag{3.6}$$

where $\bar{F}, u, \dot{u}, u_0$ are the column-vectors of $2M = 2N+2$ order in the form
$(0;F)^T, (U;\dot{U})^T, (\dot{U};\ddot{U})^T, (U_0;\bar{U}_0)^T$,
B is the matrix of $2N+2$ order in the following form

$$B = \begin{pmatrix} 0 & E \\ -\alpha_2^{-1}\bar{k}A & -\frac{\alpha_1}{\alpha_2}E \end{pmatrix}$$

E is the unit matrix of $N+1$ order, T is the symbol of transposition.

3.2 The analytical solution of the problem with nonhomogenous BCs

The solution of the problem (3.1) with nonhomogenos boundary conditions we can also obtained by Fourier method in the form [7]
$T(x,t) = \sum_{k=1}^{\infty} v_k(t)y_k(x)$, where $v_k(t) = \int_0^L T(x,t)y_k(x)dx =$
$-\frac{1}{\lambda_k^2}\int_0^L T(x,t)y_k''(x)dx$, $y''(x) = \frac{d^2y}{dx^2}$.
Integrating this integral by parts twice we get
$v_k(t) = -\frac{1}{\lambda_k^2}[\int_0^L \frac{\partial^2 T(x,t)}{\partial x^2}y_k(x)dx +$
$(T(x,t)y_k'(x) - \frac{\partial T(x,t)}{\partial x}y_k(x))|_{x=0}^{x=L}]$.

From $v_k(0)=\int_0^L T_0(x)y_k(x)dx, \dot{v}_k(0)=\int_0^L \bar{T}_0(x)y_k(x)dx$,
$\frac{\partial^2 T(x,t)}{\partial x^2}=\frac{\alpha_2}{\bar{k}}\frac{\partial^2 T(x,t)}{\partial t^2}+$
$\frac{\alpha_1}{\bar{k}}\frac{\partial T(x,t)}{\partial t}-\frac{f(x,t)}{\bar{k}}$ follows the ODEs:

$$\alpha_2\ddot{v}_k(t)+\alpha_1\dot{v}_k(t)+\bar{k}\lambda_k^2 v_k(t)=\bar{G}_k(t), \tag{3.7}$$

where $\bar{G}_k(t)=\bar{k}R_k(t)+\int_0^L f(x,t)y_k(x)dx$,
$R_k(t)=-[T(x,t)y_k'(x)-\frac{\partial T(x,t)}{\partial x}y_k(x))]|_{x=0}^{x=L}=(\sigma_2 T_r(t)+g_2(t))y_k(L)+(\sigma_1 T_l(t)-g_1(t))y_k(0)$.
If $\sigma_1=\infty$ then $R_k(t)=T_l(t)y_k'(0)$.
If $\sigma_1=\sigma_2=\infty$ then $y_k(x)=\sqrt{\frac{2}{L}}sin(k\pi x/L)$,
$R_k(t)=T_l(t)y_k'(0)-T_r(t)y_k'(L)=\frac{k\pi}{2}(T_l(t)-(-1)^k T_r(t))$.
The analytical solution of this problem is also in the form (3.5), where G,μ_k are replaced with $\bar{G},\lambda_k^2$.

Similarly this method can be applied for the method of lines or finite Fourier method ($x=x_j=jh, j=\overline{0,N}$) with the finite differences $u_{\bar{x}x}\approx\frac{\partial^2 u}{\partial x^2}$ of the second order of approximation for the problem (3.1) in the following form

$$\begin{cases} u_{\bar{x}x,j}=M(x_j,t), j=\overline{1,N-1}, t\in(0,t_f),\\ u_{x,0}-\sigma_1 u(0,t)-0.5hM(0,t)=-\sigma_1 T_l(t))+g_1(t),\\ u_{\bar{x},N}+\sigma_2 u(L,t)+0.5hM(L,t)=\sigma_2 T_r(t))+g_2(t), t\in(0,t_f),\\ u(x_j,0)=T_0(x_j), \dot{u}(x_j,0)=\bar{T}_0(x_j), j=\overline{0,N}, \end{cases} \tag{3.8}$$

where
$M(x,t)=\bar{k}^{-1}(\alpha_2\ddot{u}(x,t)+\alpha_1\dot{u}(x,t)-f(x,t)), x=x_j\ u_{x,0}=(u_1-u_0)/h$,
$u_{\bar{x},N}=(x_N-x_{N-1})/h, u_{\bar{x}x,j}=(u_{j+1}-2u_j+u_{j-1})/h^2, u_j=u(x_j,t), j=\overline{0,N}$.
The solution of the problem (3.8) we can obtain in the form
$u(x,t)=\sum_{k=1}^{N+1}v_k(t)y^k(x), (x=x_j)$,
where
$v_k(t)=[u,y^k]=\frac{1}{\mu_k}[u,-y_{\bar{x}x}^k]=\frac{1}{\mu_k}((u,-y_{\bar{x}x}^k)_h+0.5h(u_0y_0^k+u_Ny_N^k))$,
$(u,y)_h=h\sum_{j=1}^{N-1}u_jy_j$.
Using the finite first Green formula [3] follows
$(u,-y_{\bar{x}x}^k)_h=(y^k,-u_{\bar{x}x})_h+(u_0y_{x,0}^k-y_0^ku_{x,0})-(u_Ny_{\bar{x},N}^k-y_0^ku_{\bar{x},N})$
and $v_k(t)=\frac{1}{\mu_k}([y^k,-M]+y_0^k(\sigma_1T_l(t)-g_1(t))+y_N^k(\sigma_2T_r(t)+g_2(t)))$.

Therefore we have the ODEs (3.7), where
$\bar{G}_k(t) = \bar{k}(y_0^k(\sigma_1 T_l(t) - g_1(t)) + y_N^k(\sigma_2 T_r(t) + g_2(t))) + [y^k, f]$,
$v_k(0) = [T_0, y^k], \dot{v}_k(0) = [\bar{T}_0, y^k]$ and λ_k^2 is replaced with μ_k. The analytical solution of this problem can by obtained from (3.5).

3.3 The wave equation with BC of first kind

We consider the second-order hyperbolic equation in one dimension ($\alpha_2 = 1, \alpha_1 = 0, \bar{k} = a^2$)

$$\begin{cases} \frac{\partial^2 T(x,t)}{\partial t^2} = a^2 \frac{\partial^2 T(x,t)}{\partial x^2} + f(x,t), x \in (0,L), t \in (0,t_f), \\ T(0,t) = 0, T(L,t) = 0, t \in (0,t_f), \\ T(x,0) = T_0(x), \frac{\partial T(x,0)}{\partial t} = \bar{T}_0(x), x \in (0,L). \end{cases} \tag{3.9}$$

Notice that replacing
$\frac{\partial^2 T(x,t)}{\partial t^2}$ by t^2, $\frac{\partial^2 T(x,t)}{\partial x^2}$ by x^2 and f by one,
the wave equation becomes $t^2 - a^2 x^2 = 1$ which represents an hyperbola in the (x,t) plane.

The change of variables
$\omega_1(x,t) = \frac{\partial T(x,t)}{\partial x}, \omega_2(x,t) = \frac{\partial T(x,t)}{\partial t}$
transforms (3.9) into the first order system

$$\begin{cases} \frac{\partial \omega(x,t)}{\partial t} + C \frac{\partial \omega(x,t)}{\partial x} = F(x,t), x \in (0,L), t \in (0,t_f), \\ \omega(x,0) = (T_0'(x), \bar{T}_0(x),)^T, x \in (0,L), \end{cases} \tag{3.10}$$

where $\omega = (\omega_1, \omega_2)^T, F = (0, f)^T$ are the column-vectors of the second order,

$$C = \begin{pmatrix} 0 & -1 \\ -a^2 & 0 \end{pmatrix}$$

is a matrix of the second order.
If (3.10) is set on a bounded interval $(0,L)$ then the number of positive eigenvalues of the matrix C determines the number of BC that can be assigned at $x = 0$, whereas at $x = L$ it is admissible to assign a number of conditions which equals the number of negative eigenvalues. The matrix C presents two distinct real eigenvalues $\pm a$ representing the propagation velocities of the wave. Moreover, one BC needs to be prescribed at every end point.

We consider uniform grid in the space $x_j = jh, j = \overline{0,N}, Nh = L$.
Using the FDS we obtain from (3.9) the initial value problem for system of ODEs of the second order in the following matrix form

$$\begin{cases} \ddot{U}(t) + a^2 AU(t) = F(t), \\ U(0) = U_0, \dot{U}(0) = \bar{U}_0 \end{cases} \tag{3.11}$$

where A is the standart 3-diagonal matrix of $N-1$ order in the form

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 & 0 \\ .. & .. & .. & ... & .. & .. & \\ 0 & 0 & 0 & ... & -1 & 2 & -1 \\ 0 & 0 & 0 & ... & 0 & -1 & 2 \end{pmatrix}$$

$U(t), \ddot{U}(t), U_0, \bar{U}_0, F(t)$ are the column-vectors of $N-1$ order
with elements $u_j(t) \approx T(x_j,t))$, $\ddot{u}_j(t) \approx \frac{\partial^2 T(x_j,t)}{\partial t^2}$,
$u_j(0) = T_0(x_j), \dot{u}_j(0) = \bar{T}_0(x_j), f_j(t) = f(x_j,t), j = \overline{1,N-1}$.
The matrix A represented in the form $A = WDW$, $(W = W^T = W^{-1})$ is the symmetrical orthogonal matrix of $N-1$ order with elements $w_{i,j} = \sqrt{\frac{2}{N}} \sin\frac{\pi ij}{N}$, $i, j = \overline{1,N-1}$.
The diagonal matrix D contain the eigenvalues of the matrix A : $\mu_n = \frac{4}{h^2}\sin^2(\frac{n\pi}{2N}), n = \overline{1,N-1}$.
We can consider the **analytical solutions** of (3.11) using the spectral representation of matrix $A = WDW$. From transformation $V = WU(U = WV)$ follows the seperate system of ODEs

$$\begin{cases} \ddot{V}(t) + a^2 DV(t) = G(t), \\ V(0) = WU_0, \dot{V}(0) = W\bar{U}_0 \end{cases} \tag{3.12}$$

where $V(t), \ddot{V}(t), V(0), \dot{V}(0), G(t) = WF(t)$ are the column-vectors of $N-1$ order with elements $v_k(t), \ddot{v}_k(t), v_k(0), \dot{v}_k(0), g_k(t) k = \overline{1,N-1}$.
The solution of this system is

$$v_k(t) = \frac{\dot{v}_k(0)}{\kappa_k}\sin(\kappa_k t) + v_k(0)\cos(\kappa_k t) + \frac{1}{\kappa_k}\int_0^t \sin(\kappa_k(t-\tau))g_k(\tau)d\tau, \tag{3.13}$$

where $\kappa_k = \sqrt{a^2 d_k}, d_k = \mu_k$. For FDSES $d_k = \lambda_k$.
We can used also the **Fourier method** for solving (3.9) in the form

$$T(x,t) = \sum_{k=1}^{\infty} v_k(t) y_k(x),$$

where $y_k(x)$ are the orthonormed eigenvectors
$(y_k, y_n) = \int_0^L y_k(x) y_n(x) dx = \delta_{k,n})$, $v_k(t)$ is the solution (3.13), with $d_k = \lambda_k, v_k(0) = (T_0, y_k), \dot{v}_k(0) = (\bar{T}_0, y_k)$.
The solution we can also obtained in following form:
$T(x,t) = \sum_{k=1}^{\infty} a_{ks}(t) \sin \frac{\pi k x}{L}$, $f(x,t) = \sum_{k=1}^{\infty} b_{ks}(t) \sin \frac{\pi k x}{L}$,
$b_{ks}(t) = \frac{2}{L} \int_0^L f(\xi,t) \sin \frac{\pi k \xi}{L} d\xi$,
where $a_{ks}(t)$ are the corresponding solutions of (3.13) by
$a_{ks}(0) = \frac{2}{L} \int_0^L T_0(\xi) \sin \frac{\pi k \xi}{L} d\xi, \dot{a}_{ks}(0) = \frac{2}{L} \int_0^L \bar{T}_0(\xi) \sin \frac{\pi k \xi}{L} d\xi$, $g_k(t) = b_{ks}(t)$.

For he discrete problem (3.3) we can similarly obtain
$f_j(t) = \sum_{k=1}^{N-1} b_{ks}(t) \sin \frac{\pi k j}{N})$,
$b_{ks}(t) = \frac{2}{N} \sum_{j=1}^{N-1} f_j(t) \sin \frac{\pi k j}{N}, k = \overline{1, N-1}$,
and for the solution $u_j(t) = \sum_{k=1}^{N-1} a_{ks}(t) \sin \frac{\pi k j}{N}$,
$u_j(0) = \sum_{k=1}^{N-1} a_{ks}(0) \sin \frac{\pi k j}{N}$, $\dot{u}_j(0) = \sum_{k=1}^{N-1} \dot{a}_{ks}(0) \sin \frac{\pi k j}{N}$,
with $a_{ks}(0) = \frac{2}{N} \sum_{j=1}^{N-1} u_j(0) \sin \frac{\pi k j}{N}$, $\dot{a}_{ks}(0) = \frac{2}{N} \sum_{j=1}^{N-1} \dot{u}_j(0) \sin \frac{\pi k j}{N}$
we need determine the unknown function $a_{ks}(t)$ of the equation (3.13). For the FDSES the discreate eigenvalues μ_k are replaced with the eigenvalues $\lambda_k, k = \overline{1, N-1}$.

Example 3.1. For numerical calculation we consider two examples:
1) the initial boundary value problem (3.9) with
$f = 0, T_0 = \sin(\pi x), \bar{T}_0 = 0, T(x,t) = \sin(\pi x) cos(\pi t)$,
2) the problem with discontinues condition $T(0,0) \neq T_0(0)$

$$\begin{cases} \frac{\partial^2 V(x,t)}{\partial t^2} = a^2 \frac{\partial^2 V(x,t)}{\partial x^2}, x \in (0,1), t \in (0,t_f), \\ V(0,t) = 1, V(1,t) = 0, t \in (0,t_f), \\ V(x,0) = 0, \frac{\partial V(x,0)}{\partial t} = 0, x \in (0,1). \end{cases} \quad (3.14)$$

Using tranformation $T(xt) = V(x,t) - 1 + x$ we obtain the problem (3.9) with the homogenous BC, where $f = 0, T_0 = x - 1, \bar{T}_0 = 0$.
We have following MATLAB m.file **Wave1:**

```
%system ODE U_tt+a^2 AU=f with BC first kind
%t=Tb,u_(t=0)=x-1,f=0,a^2=1 ,L=1
function Wave1(N)
N1=N+1;MK=20; Tb=0.2;L=1;x=linspace(0,L,N1)';
t=linspace(0,Tb,MK);
h=L/N;N2=N-1;NN=2*(N-1);a=1;a2=a^2;
lk=4/h^2* (sin(pi*h/L*(1:N2)'/2)).^2;% FDS,eig-val.
lk0=(pi/L*(1:N2)').^2; % ODE , eig-val.
A2=zeros(N2,N2);x=x(2:N);
%A2=A2-diag(ones(N2-1,1),1)-. . .
diag(ones(N2-1,1),-1)+2*diag(ones(N2,1));
%A2=A2/h^2;%matrix A control
W= sqrt(2*h/L)*sin(pi*h/L*[1:N2]'*[1:N2]);
A2=W*diag(lk0)*W; %FDS or FDSES
y2=zeros(N2,1);
y1=ones(N2,1).*(x-1);% 1. init-cond
%y1=sin(pi*x); % 2. init-cond exact
P=W*y1;P1=zeros(MK,N2);P0=W*y2;
for k=1:N2
    b=sqrt(a2*lk0(k)); %FDS or FDSES
    P1(:,k)=P(k)*cos(b*t')+P0(k)/b*sin(b*t');
end
P2=(W*P1')';
prec=sin(pi*x)*cos(pi*t);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(MK,1)*x';Y1=t'*ones(1,N2);
figure, surfc(X1,Y1,P2(:,1:N2)+1-X1)% analyt.real
%figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('Anal.,tNr.=%4.1f,. . .
max=%9.7f',MK,Ma1))%analytic sol.
Z1=zeros(N2,N2);E1=eye(N2,N2);
y0=[y1;y2];AT=[Z1,E1;-A2,Z1];
options=odeset('RelTol', 1.0e-7);
[T,Y]=ode15s(@SIST,[0,Tb],y0,options,AT);
K=length(T);
prec1=sin(pi*x)*cos(pi*T');% exact
MA1 = abs(Y(end,1:N2 )-prec1(1:N2,end)');
MA=max(MA1);% max error Matlab
%figure,plot(T,max(Y(:,1:N2)'-prec1),'k*')% max error on t
figure,plot(T,max(Y(:,1:N2)'+1-. . .
(ones(K,1)*x')'),'k*')% max error on t
title(sprintf('Max-sol.on t,time=...
%8.6f,Max=%9.7f',T(end),MA))
xlabel('\itt'), ylabel('\itu')
figure,plot(x,Y(end,1:N2)'+1-x,'ko') % real func.
grid on
title(sprintf('Sol.on x by Tb.,time =...
 %8.6f,Max=%9.7f',T(end),MA))
xlabel('\itx'), ylabel('\itu')
Ma2=max(max(abs(Y(:,1:N2)-prec1'))); %error Matlab
 X2=ones(K,1)*x';Y2=T*ones(1,N2);
```

```
figure, surfc(X2,Y2,Y(:,1:N2)+1-X2)%real Matlab
%figure, surfc(X2,Y2,abs(Y(:,1:N2)-prec1'))%err
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('Matlab.,time=...
%3.06f, Max=%8.6f',K,Ma2))% Matlab
function F=SIST(t,y,AT)
F=AT*y;
```

For the **first example** by operator **Wave1(10)**we obtain following maximal errors $(t_f = 1)$:
0.00748 (FDS), 5.10^{-6} (FDSES by MATLAB solver), 10^{-16} (FDSES analytical) (see Figs. 3.1, 3.2).

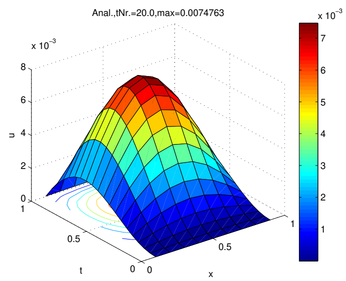


Fig. 3.1 Error with FDS by $N = 10$

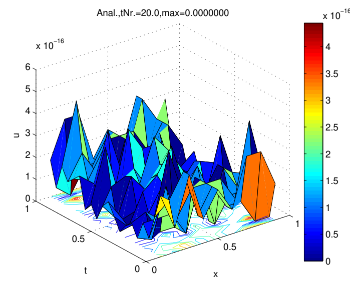


Fig. 3.2 Error with FDSES by $N = 10$

For the **second example** by operator **Wave1(40)** obtained results $(t = 0.2)$ are represented in the Figs. 3.3, 3.4.
The coresponding 3-D graphics are in the Figs. 3.5, 3.6.

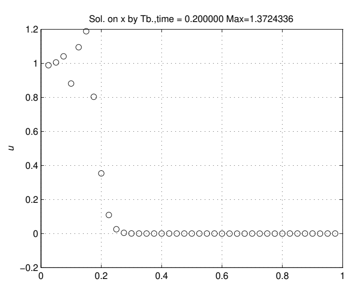


Fig. 3.3 FDS solution depending on x by $N = 40, t = 0.2$

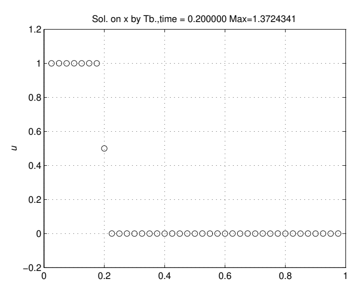


Fig. 3.4 FDSES solution depending on x by $N = 40, t = 0.2$

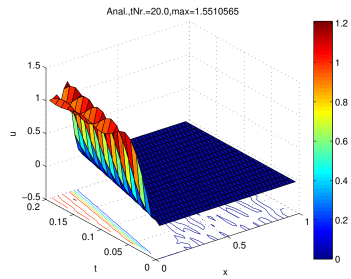


Fig. 3.5 FDS solution by $N = 40, t = 0.2$

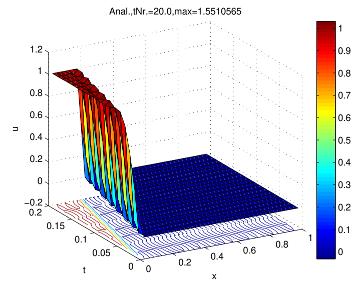


Fig. 3.6 FDSES solution by $N = 40, t = 0.2$

In the Figs. 3.7, 3.8 we can see the maximum of the solution depending on t.

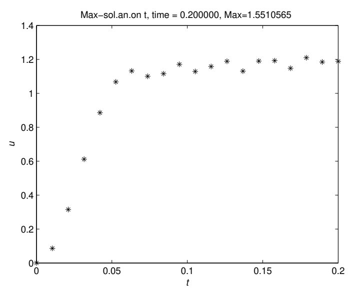


Fig. 3.7 FDS max. solution depending on t by $N = 40$

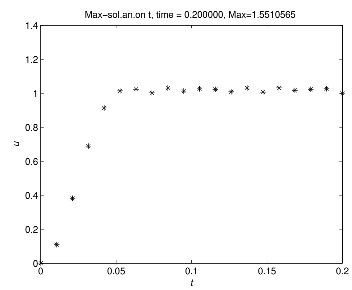


Fig. 3.8 FDSES max. solution depending on t by $N = 40$

3.4 The wave equation with periodical BCs

We consider the initial boundary value problem with periodical BCs

$$\begin{cases} \frac{\partial^2 T(x,t)}{\partial t^2} = a^2 \frac{\partial^2 T(x,t)}{\partial x^2} + f(x,t), x \in (0,L), t \in (0,t_f), \\ T(0,t) = T(L,t), \frac{\partial T(0,t)}{\partial x} = \frac{\partial T(L,t)}{\partial x}, t \in (0,t_f), \\ T(x,0) = T_0(x), \frac{\partial T(x,0)}{\partial t} = \bar{T}_0(x), x \in (0,L). \end{cases} \tag{3.15}$$

Using uniform grid $x_j = jh, j = \overline{0,N}, Nh = L,$ (N is even number) we obtain the system of ODEs (3.11), where A is the 3-diagonal circulant matrix of N order in the form

$$A = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ .. & .. & .. & \dots & .. & .. & \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

or $A = \frac{1}{h^2}[2,-1,0,\cdots,0,-1]$.
From the corresponding spectral problem of the matrix A follows that (see chapter1)
$\mu_k = \frac{4}{h^2}(\sin(k\pi/N)^2$, are the eigenvalues and $w_j^k = \sqrt{\frac{1}{N}}\exp(2\pi ikj/N)$,
$w_{*,j}^k = \sqrt{\frac{1}{N}}\exp(-2\pi ikj/N), k, j = \overline{1,N}$ are the elements of the biorthonormed eigenvectors w^k, w_*^k, where $(w^k, w_*^m) = \sum_{j=1}^N w_j^k w_{*,j}^m = \delta_{k,m}$,
The eigenvalues μ_k are symmetrical as regards $k = N/2$ (with the maximal value $\frac{4}{h^2}$) or $\mu_{N/2+m} = \mu_{N/2-m}, m = \overline{1,N/2}$.
Using the matrices W, W_* with the eigenvectors w^k, w_*^k in the matrices columns we get $AW = WD, WW_* = E, W^{-1} = W_*, A = WDW_*$, where the elements of the diagonal matrix D is $d_k = \mu_k, k = \overline{1,N}$.
For the differential spectral problem follows
$\lambda_k = (2\pi k/L)^2, w^k(x) = \sqrt{\frac{1}{L}}\exp(2\pi ikx/L)$,
$w_*^k(x) = \sqrt{\frac{1}{L}}\exp(-2\pi ikx/L)$, $(w^k, w_*^m)_* = \int_0^L w^k(x)w_*^m(x)dx = \delta_{k,m}$,
$k, m = \overline{-\infty, +\infty}$.
The solution of (3.15) with the Fourier method we have obtained:
$f(x,t) = \sum_{k=-\infty}^{\infty} g_k(t)w^k(x)$, $g_k(t) = (w_*^k, f)$, $T(x,t) = \sum_{k=-\infty}^{\infty} v_k(t)w^k(x)$,
where $v_k(t)$ is the solution of 3.13) by $k \neq 0$. For $k = 0$ we obtain
$v_0(t) = \dot{v}_0(0)t + v_0(0) + \int_0^t (t-\tau)g_0(\tau)d\tau$.
The solution we can also obtained in real form:
$T(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
$f(x,t) = \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{L}\int_0^L f(\xi,t)\cos\frac{2\pi k\xi}{L}d\xi, b_{ks}(t) = \frac{2}{L}\int_0^L f(\xi,t)\sin\frac{2\pi k\xi}{L}d\xi$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (3.13) by
$a_{kc}(0) = \frac{2}{L}\int_0^L T_0(\xi)\cos\frac{2\pi k\xi}{L}d\xi$, $\dot{a}_{kc}(0) = \frac{2}{L}\int_0^L \bar{T}_0(\xi)\cos\frac{2\pi k\xi}{L}d\xi$,

$a_{ks}(0) = \frac{2}{L}\int_0^L T_0(\xi)\sin\frac{2\pi k\xi}{L}d\xi$, $\dot{a}_{ks}(0) = \frac{2}{L}\int_0^L \bar{T}_0(\xi)\sin\frac{2\pi k\xi}{L}d\xi$,
$g_k(t) = b_{kc}(t)$ or $b_{ks}(t)$.

We can consider the **analytical solutions** of (3.11) using the spectral representation of matrix $A = WDW_*$. From transformation $V = W_*U(U = WV)$ follows the seperate system of ODEs (3.12), where the column-vectors are of N order.

The solution of the system (3.12) is in the form (3.13), where $k = \overline{1,N-1}, d_k = \mu_k$. For $k = N$ the solution is $v_N(t) = \dot{v}_N(0)t + v_N(0) + \int_0^t (t-\tau)g_k(\tau)d\tau$. The solution of (3.11) is in the form $U = WV$.

If $d_k = \lambda_k$ then we can obtain the solution of FDSES in following way:
1)$d_k = \lambda_k$ for $k = \overline{1,N_2}$, where $N_2 = N/2$.
2)$d_k = \lambda_{N-k}$ for $k = \overline{N_2,N-1}, d_N = 0$.

We can obtain also the solution of the discrete problem in real form
$u_j(t) = \sum_{k=1}^{*N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}}{2}$,
$f_j(t) = \sum_{k=1}^{*N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\cos\frac{2\pi kj}{N}, b_{ks}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\sin\frac{2\pi kj}{N}$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (3.13) by
$a_{kc}(0) = \frac{2}{N}\sum_1^N T_0(x_j)\cos\frac{2\pi kj}{N}, \dot{a}_{kc}(0) = \frac{2}{N}\sum_1^N \bar{T}_0(x_j)\cos\frac{2\pi kj}{N}$,
$a_{ks}(0) = \frac{2}{N}\sum_1^N T_0(x_j)\sin\frac{2\pi kj}{N}, \dot{a}_{ks}(0) = \frac{2}{N}\sum_1^N \bar{T}_0(x_j)\sin\frac{2\pi kj}{N}$,
$g_k(t) = b_{kc}(t)$ or $b_{ks}(t)$.

The equation $\frac{\partial^2 V(x,t)}{\partial t^2} = \frac{\partial^2 V(x,t)}{\partial x^2} + a^2 V(x,t) + g(x,t)$,
we can reduce to the equation (3.2) using transformation $V(x,t) = \exp(at)T(x,t)$, where
$a = const, \bar{k} = \alpha_1 = 1, \alpha_2 = 2a, f(x,t) = g(x,t)\exp(-at)$. In this case the homogenous BCs of first kind and periodical BCs remained.

For the equations $\frac{\partial^2 T(x,t)}{\partial t^2} = \frac{\partial^2 T(x,t)}{\partial x^2} \pm a^2 T(x,t) + f(x,t)$,
we can use preliminary analytical solutions considering only the new values of the eigenvalues
1) for BCs with the first kind:
$\lambda_k = (\frac{k\pi}{L})^2 - \pm a^2$, $\mu_k = \frac{4}{h^2}\sin(k\pi/2N)^2\gamma - \pm a^2$,
2)for periodical BCs:
$\mu_k = \frac{4}{h^2}\sin(k\pi/N)^2\gamma - \pm a^2$, $\lambda_k = (\frac{k\pi}{L})^2 - \pm a^2$,
where γ is the coefficient in the Bahvalov FDS.

3.5 Example of wave equation with the periodical BC for one wave number

For numerical calculation we consider the initial boundary value problem (3.15) with $f=0, T_0=\sin(2\pi x), \bar{T}_0=0, T(x,t)=\sin(2\pi x)\cos(2\pi t)$. Using the Fourier method we obtain $v_k(0)=0, v_k(t)=0$ for $k\neq\pm 1$, $v_{\pm 1}(0)=\frac{1}{\pm i},\ v_{\pm 1}(t)=\pm\frac{\cos(2\pi t)}{2i},\ T(x,t)=\cos(2\pi t)\sin(2\pi x)$. We have following MATLAB m.file **Wave2:**

```
%system ODE U_tt+a^2 AU=f with periodical BC
%t=Tb,u(x,t)=sin(2 pi x)cos(2 pi t),a=1,f=0,N-even
function Wave2(N)
N1=N+1;MK=20; Tb=1;L=1;x=linspace(0,L,N1)';t=linspace(0,Tb,MK);
h=L/N;N2=N-1;a=1;a2=a^2;x=x(2:N1);
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1));
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
NH=N/2; d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L);
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L);
A2=W*diag(d)*W1; %FDS or FDSES
y2=zeros(N,1);
y1=sin(2*pi*x); % init-cond
P=W1*y1;P1=zeros(MK,N);P0=W1*y2;
for k=1:N2
    b=sqrt(a2*d(k)); %FDS or FDSES
    P1(:,k)=P(k)*cos(b*t')+P0(k)/b*sin(b*t');
end
P1(:,N)=P(N)+P0(N)*t';
P2=(W*P1.').';% operator of transponation
prec=sin(2*pi*x)*cos(2*pi*t);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(MK,1)*x';Y1=t'*ones(1,N);
figure,plot(t',max(abs(P2(:,1:N).'-prec)),'k*')% max error on t
title(sprintf('err. Max-sol.an.on t, Max=%9.7f ',Ma1))
xlabel('\itt'), ylabel('\itu')
figure,plot(x,P2(end,1:N).','ko')
grid on
```

```
title(sprintf('Sol.an.on x by Tb.,Max=%9.7f ',Ma1))
xlabel('\itx'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('err. anal.,tNr.=%4.1f,max=%9.7f',MK,Ma1))
```

Using the operator **Wave2(10)**we obtan following maximal errors $(t_f = 1)$:
0.0755 (FDS-$O(h^2)$), 0.0038 (FDS- $O(h^4)$), 0.00024 (FDS -$O(h^6)$), 0.00002 (FDS-$O(h^8)$), !0^{-12} (FDSES).
By $N = 40$ the results are:
0.0049 (FDS-$O(h^2)$), 0.000016 (FDS- $O(h^4)$), 10^{-7} (FDS -$O(h^6)$),2.10^{-10} (FDS-$O(h^8)$), 10^{-14} (FDSES), (see Figs.3.9, 3.10)

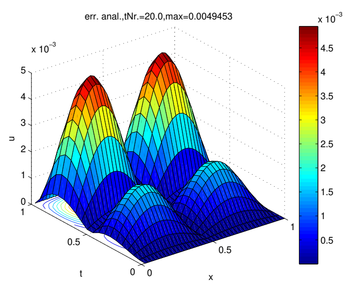


Fig. 3.9 Error with FDS by $N = 40, O(h^2)$

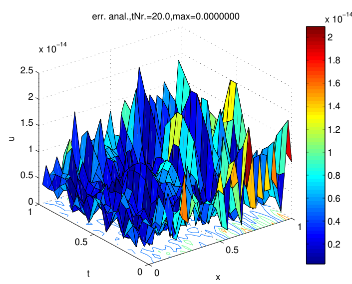


Fig. 3.10 Error with FDSES by $N = 40$

3.6 Example of wave equation with the periodical BC for dfferent wave numbers

We consider the initial boundary value problem (3.15) by $L = 1, a = 1, f = 0, T_0(x) = \sin(2\pi mx), \bar{T}_0 = 0$, where m is integer in $(1,N)$ with $m \leq N/2$. Then the exact solution is $T(x,t) = \cos(2\pi mt)\sin(2\pi mx)$. The solution of (3.15) with the **Fourier method** can be obtained in following form:
$T(x,t) = \sum_{k=-\infty}^{\infty} v_k(t) w^k(x)$, where
$v_k(t)$ is the solution of 3.13) in the form $v_k(t) = \cos(\kappa_k t) v_k(0, \kappa_k = \sqrt{d_k} = 2\pi k$,

$v_k(0) = \int_0^1 T_0(x) w_*^k(x) dx = 0$, $v_k(t) = 0$ for $k \neq \pm m$,
$v_{\pm m}(0) = \frac{0.5}{\pm i}$, $v_{\pm m}(t) = \pm \frac{\cos(2\pi m t)}{2i}$, $T(x,t) = \cos(2\pi m t)\sin(2\pi m x)$.
Therefore we have using the Fourier method the exact solution.
We can consider the **analytical solutions** for FDS of (3.11) using the spectral representation of matrix $A = WDW_*$. From transformation $V = W_* U (U = WV)$ follows the seperate system of ODEs (3.12).
The solution of the system (3.12) is
$V(t) = \cos(\sqrt{D}t)V_0, D = diag(\mu_k)$ or in the form (3.13), where
$v_k(t) = \cos(\kappa_k t) v_k(0, \kappa_k = \sqrt{\mu_k}$,
$v_k(0) = (W^* U_0)_k = 0$, $v_k(t) = 0$ for $k \neq m, k \neq N - m$.
From $\mu_{N-m} = \mu_k, w^{N-k} = w_*^k$ follows $v_m(0) = \frac{\sqrt{N}}{2i}, v_{N-m}(0) = -\frac{\sqrt{N}}{2i}$.
Therefore $U(t) = \cos(\sqrt{\mu_m}t)U_0$,
where $U_0 = (\sin(2\pi m x_1), \cdots, \sin(2\pi m x_N)^T$ is the column-vector of the N order, $x_j = jh, j = \overline{1.N}, Nh = 1$.
The solution can be obtained in the matrix form $U(t) = W\cos(\sqrt{D}t)W^* U_0$.
For the FDSES $\sqrt{\mu_m} = \sqrt{d_m} = 2\pi m$ and we have also the exact solution.
Using the **discrete Fourier** transformation
$U(t) = \sum_{k=1}^N a_k(t) w^k (Aw^k = \mu_k w^k)$, we get $a_k(t) = \cos(\sqrt{\mu_k}t) a_k(0)$,
where $a_k(0) = U_0.w_*^k = 0$ for $k \neq m, k \neq N - m$,
$a_m(0) = \frac{\sqrt{N}}{2i}, a_{N-m}(0) = -\frac{\sqrt{N}}{2i}$.
We have
$U(t) = a_m(t) w^m + a_{N-m}(t) w_*^m = \frac{\sqrt{N}}{2i}\cos(\sqrt{\mu_m}t)(w^m - w_*^m) = \cos(\sqrt{\mu_m}t)U_0$.
For numerical calculation we consider the initial boundary value problem (3.15) with
$t_f = L = 1, f = 0, T_0 = \sin(2\pi m x), \bar{T}_0 = 0, T(x,t) = \sin(2\pi m x)\cos(2\pi m t)$,
for $m = 1;2;3;4, N = 10$.
We have following MATLAB m.file **Wave2 m:**

```
%system ODE U_tt+ AU=0 with periodical BC
%t=Tb,u(x,t)=sin(2 pi m x)cos(2 pi m t),m<N-even
function Wave2m(N)
N1=N+1;MK=3;m=2; Tb=1;L=1;x=linspace(0,L,N1)';
t=linspace(0,Tb,MK);
h=L/N;N2=N-1;a=1;a2=a^2;x=x(2:N1);
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1));
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
```

```
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
%8/45*(sin(pi*h*NT)).^6);%O(h^6)
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
%8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
%NH=N/2; d(1:NH)=lk0(1:NH);
%d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L);
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L);
A2=W*diag(d)*W1; %FDS or FDSES
y1=sin(2*pi*m*x); % init-cond
P=zeros(N,1);P=W1*y1;P1=zeros(MK,N);
for k=1:N
    b=sqrt(a2*d(k)); %FDS or FDSES
    P1(:,k)=cos(b*t')*P(k);
    end
P2=(W*P1.').';% this is transponation operator
P21=W*diag(cos(sqrt(d)*t(end)))*W1*y1;%okei !
prec=sin(2*pi*m*x)*cos(2*pi*m*t);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(MK,1)*x';Y1=t'*ones(1,N);
figure,plot(t',max(abs(P2(:,1:N).'-prec)),'k*')% max error on t
title(sprintf('err. Max-sol.an.on t, Max=%9.7f ',Ma1))
xlabel('\itt'), ylabel('\itu')
figure,plot(x,P21,'ko',x,prec(1:N,end),'*',x,P2(end,1:N),'-')
%figure,plot(x,P2(end,1:N)','ko')
grid on
title(sprintf('Sol.an.on x by Tb.,Max=%9.7f ',Ma1))
xlabel('\itx'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('err. anal.,tNr.=%4.1f,max=%9.7f',MK,Ma1))
```

Using the operator **Wave2m(10)**we obtan following maximal errors $(t_f = 1)$ (see Table 3.1):

Table 3.1 The FDS maximal error depending on order of approximation and m by $N = 10$

Method	m=1	m=2	m=3	m=4
$O(h^2)$	0.0050	0.2958	1.797	—–
$O(h^4)$	1.210^{-7}	0.0110	0.4301	—-
$O(h^6)$	3.10^{-8}	6.10^{-4}	0.0911	1.550
$O(h^8))$	2.10^{-10}	3.10^{-5}	0.0219	0.9724
FDSES	2.10^{-15}	1.10^{-15}	3.10^{-15}	1.10^{-15}

In the Figs. 3.11, 3.12 we can see the FDSES exact solutions by $m = 4$ and $N = 10, N = 20$.

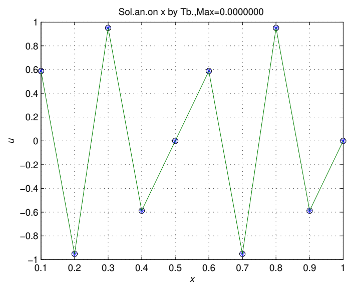


Fig. 3.11 FDSES solutions by $N = 10, m = 4, t_f = 1$

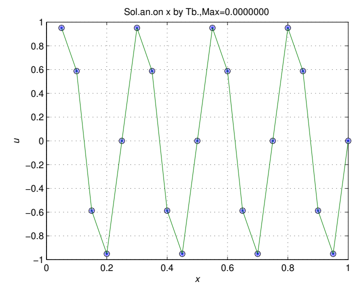


Fig. 3.12 FDSES solutions by $N = 20, m = 4, t_f = 1$

3.7 Example of nonlinear wave equation with the periodical BC

We shall consider the initial - boundary value problem for solving the following nonlinear wave equation:

$$\begin{cases} \frac{\partial^2 T(x,t)}{\partial t^2} = \frac{\partial^2 (g(T))}{\partial x^2} + f(T), x \in (0,L), t \in (0,t_f), \\ T(0,t) = T(L,t), \frac{\partial T(0,t)}{\partial x} = \frac{\partial T(L,t)}{\partial x}, t \in (0,t_f), \\ T(x,0) = T_0(x), \frac{\partial T(x,0)}{\partial t} = \bar{T}_0(x), x \in (0,L). \end{cases} \tag{3.16}$$

where $g(T), f(T)$ is nonlinear given functions. Using the FDS we obtain from (3.16) the initial value problem for system of nonlinear ODEs of the second order in the following matrix form

$$\begin{cases} \ddot{U}(t) = -AG(U) + F(U), \\ U(0) = U_0, \dot{U}(0) = \bar{U}_0, \end{cases} \tag{3.17}$$

where G, F are the vectors-column of N order with elements $g_k = g(u(x_k,t)), f_k = f(u(x_k,t)), k = \overline{1,N}$.
For using the Matlab solvers we need write the system of ODEs in the normal form

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = -AG(y_1) + F(y_1), \\ y_1(0) = U_0, y_2(0) = \bar{U}_0, \end{cases} \tag{3.18}$$

$(U(t) = y_1(t)$ or

$$\dot{y}(t) = A_1 G(y) + BF(y),$$

where y is the the column-vectors of $2N$ order with elements (y_1, y_2), A_1, B are the matrices of $2N$ order in following form:

$$A_1 = \begin{pmatrix} 0 & E \\ -A & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}$$

The numerical experiment with $L = 1, t_f = 0.1$ and $F = aT^{\beta}, g(T) = T^{\sigma+1}, \sigma = 2, T_0 = \sin(2\pi x), \bar{T}_0 = 0, \beta = a = 0$ is produced by MATLAB 7.4 solver "ode15s".

We have following MATLAB m.file **svarst3per:**

```
%PDE U_tt=AG +F with periodic BC
%t=Tb, A =WDW* with different aproksimation
function svarst3per(N)
sigma=2;sigma1=sigma+1;beta=0;a=0;
N1=N+1; Tb=0.1;L=1;x=linspace(0,L,N1)';
h=L/N;N2=N-1;NT=[1:N]/L;NN=2*N;NH=N/2;
lk0=(2*pi/L*(1:N)').^2; % precizas ipasv.
lk2=4/h^2*(sin(pi*h*NT)).^2; %O^2
lk4=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O^4
lk6=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
8/45*(sin(pi*h*NT)).^6);%O^6
lk8=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
8/45*(sin(pi*h*NT)).^6+ 4/35*(sin(pi*h*NT)).^8);%O^8
W=exp(2*pi*h*i*[1:N]'*[1:N]/L);x=x(2:N1);lk=zeros(N,1);
%a=lk2(1)
%A2=zeros(N,N);
%A2=A2+...
%diag(ones(N2,1),1)+diag(ones(N2,1),-1)-2*diag(ones(N,1));
%A2(1,N)=1; A2(N,1)=1;A2=A2/h^2;
%D=-h*(1./W)*B2*W;
lk(1:NH)=lk0(1:NH);
lk(NH:N2)=lk0(NH:-1:1);%FDSES
A2=-h*W*diag(lk8)*conj(W);
A=zeros(NN,NN);y1=(sin(2*pi*x));y2=zeros(N,1);
y0=[y1;y2];Z1=zeros(N,N);A=[Z1,eye(N,N);A2,Z1];
B=[Z1,Z1; eye(N,N),Z1];
options=odeset('RelTol', 1.0e-7);
```

```
[T,Y]=ode15s(@SIST,[0 Tb],y0,options,A,sigma1,beta,a,B);
im=max(abs(imag(Y(end,:))));
MA=max(abs(real(Y(end,1:N))));
figure,plot(x,real(Y(end,1:N)'),'ko')
grid on
title(sprintf('End time., maxim=%8.6f,time = . . .
%8.6f Max=%9.7f ',im,T(end),MA))
xlabel('\itx'), ylabel('\itu')
figure
plot(T(:),max(real(Y(:,1:N)')))
grid on
title(sprintf('FDS in time,N=%3.0f, time = %8.6f ',N,T(end)))
xlabel('\itt'), ylabel('\itu')
K=length(T);X1=ones(K,1)*x';Y1=T*ones(1,N);
figure, surfc(X1,Y1,real(Y(:,1:N)))
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('Surface,imag=%8.6f,Laika sl.sk.=%3.06f,. . .
 Max=%8.6f',im,K,MA))
function F=SIST(t,y,A,sigma1,beta,a,B)
F=A*y.^sigma1+a*B*(y).^beta;
```

In Table 3.2 are shown maximal values of solution max $|y_1(t_f)|$ depending on N and order of approximation.

Table 3.2 The maximal values max $|y_1(t_f)|$ depending on order of approximation and N

Method	N=10	N=20	N=40
$O(h^2)$	0.6622	0.5515	0.5735
$O(h^4)$	0.6078	0.5883	0.5928
$O(h^6)$	0.5579	0.5947	0.5957
$O(h^8)$)	0.5322	0.5961	0.5964
FDSES	0.5058	0.5950	0.5928

In the Figs. 3.13, 3.14 we can see the FDSES solutions by $N = 20; 40$.

The numerical experiment with $L = 1, t_f = 0.8$ and $F = a(sin(T))^{\beta}$, $g(T) = T^{\sigma+1}, \sigma = 0, T_0 = \sin^{100}(\pi x), \bar{T}_0 = 0, \beta = a = 1$ is produced also by MATLAB 7.4 solver "ode15s" with operator $F = A * y.^{\sigma+1} + a * B * sin(y).^{\beta}$;
In Table 3.3 are shown maximal values of solution max $|y_1(t_f)|$ depending on N and order of approximation.

In the Figs. 3.15, 3.16 we can see the FDSES solutions by $N = 80$.

In the Figs. 3.17, 3.18 we can see the FDSES and FDS $O(h^2)$ solutions by $N = 80, t_f = 0.8$.

Table 3.3 The maximal values $\max|y_1(t_f)|$ depending on order of approximation and N

Method	N=10	N=20	N=40	N=80
$O(h^2)$	0.4789	0.3076	0.4325	0.5038
$O(h^4)$	0.4097	0.3890	0.4806	0.5227
$O(h^6)$	0.4936	0.4124	0.5073	0.5242
$O(h^8))$	0.4593	0.3859	0.5163	0.5243
FDSES	0.5306	0.5243	0.5243	0.5243

In the Figs. 3.19, 3.20 we can see the FDSES and FDS $O(h^8)$ solutions by $N = 80, t_f = 0.8$.

We can see, that FDS methods give the solutions with oscillations. FDSES method is without oscilations and the solution is positive even if $N = 10$.

3.8 The mathematical model for wave equations with convection

We consider the linear wave equation in the following form:

$$\frac{\partial^2 T(x,t)}{\partial t^2} = \frac{\partial^2 T(x,t)}{\partial x^2} + a\frac{\partial T(x,t)}{\partial x} + f(x,t) \tag{3.19}$$

with the periodical boundary conditions (3.15) (a=const).
We can used the **Fourier method** for solving the initial-boundary value problem in the form
$T(x,t) = \sum_{k\in Z} a_k(t)w^k(x), f(x,t) = \sum_{k\in Z} b_k(t)w^k(x)$,
where $w^k(x)$ are the orthonormed eigenvectors, $b_k(t) = (f, w^k_*(x))$(see chapter 1).
Then for the unknown functions $a_k(t)$ get the complex initial value problem for ODEs of second order:

$$\begin{cases} \ddot{a}_k(t) + a_k(t)\lambda_k = b_k(t), \\ a_k(0) = \frac{1}{L}\int_0^L T_0(s)\exp\frac{-2i\pi ks}{L}ds, \\ \dot{a}_k(0) = \frac{1}{L}\int_0^L \bar{T}_0(s)\exp\frac{-2i\pi ks}{L}ds, \\ b_k(t) = \frac{1}{L}\int_0^L f(s,t)\exp\frac{-2i\pi ks}{L}ds. \end{cases} \tag{3.20}$$

The solution of (3.20) is

$$a_{kc}(t) = \cos(\sqrt{\lambda_k} t) a_k(0) + \frac{\sin(\sqrt{\lambda_k} t))}{\sqrt{\lambda_k}} \dot{a}_k(0) + \int_0^t \frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}} b_k(s) ds.$$

The solution with the Fourier method can also obtain in real form:
$f(x,t) = \sum_{k=1}^{\infty}(b_k(t)w^k(x) + b_{-k}(t)w^{-k}(x)) + \frac{b_0(t)}{\sqrt{L}} =$
$\frac{1}{2}\sum_{k=1}^{\infty}((b_k(t)+b_k(t))(w^k(x)+w_*^k(x)) + (b_k(t)-b_{-k}(t))(w^k(x)-w_*^k(x))) + \frac{b_0(t)}{\sqrt{L}} =$
$\sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2}$,
where $\lambda_k = (2\pi k/L)^2 - 2\pi kai/L, w^k(x) = \sqrt{\frac{1}{L}}\exp(2\pi ikx/L)$,
$w_*^k(x) = \sqrt{\frac{1}{L}}\exp(-2\pi ikx/L) = w^{-k}(x)$,
$b_{kc}(t) = \frac{1}{\sqrt{L}}(b_k(t) + b_{-k}(t)) =$
$\frac{1}{\sqrt{L}}\int_0^L f(x,t)(w_*^k(x) + w^k(x))dx = \frac{2}{L}\int_0^L f(s,t)\cos\frac{2\pi ks}{L}ds$,
$b_{ks}(t) = \frac{i}{\sqrt{L}}(b_k(t) - b_{-k}(t)) = \frac{i}{\sqrt{L}}\int_0^L f(x,t)(w_*^k(x) - w^k(x))dx =$
$\frac{2}{L}\int_0^L f(s,t)\sin\frac{2\pi ks}{L}ds$.
Then,
$T(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
where $a_{kc}(t), a_{ks}(t)$ are unknown functions.
From $f(x,t) = \frac{\partial^2 T(x,t)}{\partial t^2} - (\frac{\partial^2 T(x,t)}{\partial x^2} + a\frac{\partial T(x,t)}{\partial x})$
follows $f(x,t) = \sum_{k=1}^{\infty}(\ddot{a}_{kc}(t)\cos\frac{2\pi kx}{L} +$
$\ddot{a}_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{\ddot{a}_{0c}(t)}{2} +$
$\sum_{k=1}^{\infty}((a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k))\cos\frac{2\pi kx}{L} +$
$(a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k))\sin\frac{2\pi kx}{L})$,
because $(a_k(t)\lambda_k + a_{-k}(t)\lambda_{-k})/sqrtN = a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k)$,
$i(a_k(t)\lambda_k - a_{-k}(t)\lambda_{-k})/\sqrt{L} = a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k)$,
where $a_{kc}(t) = \frac{a_k(t)+a_{-k}(t)}{\sqrt{L}}, a_{ks}(t) = \frac{i(a_k(t)-a_{-k}(t)}{\sqrt{N}}$ are the coefficients in the expression from the solution $T(x,t)$.
Therefore we obtain the initial boundary value problem for the system of two ODEs:

$$\begin{cases} \ddot{a}_{kc}(t) + a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k) = b_{kc}(t), \\ \ddot{a}_{ks}(t) + a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k) = b_{ks}(t), \\ a_{kc}(0) = \frac{2}{L}\int_0^L T_0(s)\cos\frac{2\pi ks}{L}ds, a_{ks}(0) = \frac{2}{L}\int_0^L T_0(s)\sin\frac{2\pi ks}{L}ds, \\ \dot{a}_{kc}(0) = \frac{2}{L}\int_0^L \bar{T}_0(s)\cos\frac{2\pi ks}{L}ds, \dot{a}_{ks}(0) = \frac{2}{L}\int_0^L \bar{T}_0(s)\sin\frac{2\pi ks}{L}ds. \end{cases} \tag{3.21}$$

In the matrix form we have

$$\ddot{A}_k(t) + \Lambda_k A_k(t) = B_k(t), A_k(0) = A_{k0}, \dot{A}_k(0) = \dot{A}_{k0} \tag{3.22}$$

where
$\Lambda_k = \begin{pmatrix} Re(\lambda_k) & Im(\lambda_k) \\ -Im(\lambda_k) & Re(\lambda_k) \end{pmatrix}$ is the matrix of second order,
$A_k(t), B_k(t), A_{k0}, \dot{A}_{k0}$ are the column-vectors with elements
$(a_{kc}(t), a_{ks}(t)), (b_{kc}(t), b_{ks}(t)), (a_{kc}(0), a_{ks}(0)), (\dot{a}_{kc}(0), \dot{a}_{ks}(0))$.
We can represented the matrix Λ_k in the form $\Lambda_k = PDP^{-1}$,
where $P = \begin{pmatrix} 0.5 & -i \\ -0.5i & 1) \end{pmatrix}, P^{-1} = \begin{pmatrix} 1 & i \\ 0.5i & 0.5) \end{pmatrix}$,
$D = \begin{pmatrix} \lambda_k^* & 0 \\ 0 & \lambda_k) \end{pmatrix}$,
where $\lambda_k^* = Re(\lambda_k) - iIm(\lambda_k), Re(\lambda_k) = \frac{4\pi^2 k^2}{L^2}, Im(\lambda_k) = -\frac{2\pi ka}{L}$.
Then the matrix solution of (3.22)
$A_k(t) = \cos(\sqrt{\Lambda_k}t)A_{k0} + \Lambda_k^{-0.5}\sin(\sqrt{\Lambda_k}t)\dot{A}_{k0} +$
$\Lambda_k^{-0.5}\int_0^t \sin(\sqrt{\Lambda_k}(t-s)B_k(s)ds$
with the transformations $\tilde{A}_k(t) = P^{-1}A_k(t), A_k(t) = P\tilde{A}_k(t)$ can obtain in the form

$$A_k(t) = \cos(\sqrt{D}t)\tilde{A}_{k0} + D^{-0.5}\sin(\sqrt{D}t)\tilde{\dot{A}}_{k0} + D^{-0.5}\int_0^t \sin(\sqrt{D}(t-s)\tilde{B}_k(s)ds$$

where $\tilde{A}_{k0} = P^{-1}A_{k0}, \tilde{\dot{A}}_{k0} = P^{-1}\dot{A}_{k0}, \tilde{B}_k(t) = P^{-1}B_k(t)$.
For this seperable we can determine the elements $\tilde{a}_{kc}(t), \tilde{a}_{ks}(t)$ of the column-vector $\tilde{A}_k$
depending on the elements $\tilde{a}_{kc}(0) = a_{kc}(0) + ia_{ks}(0), \tilde{a}_{ks}(0) = 0.5ia_{kc}(0) + 0.5a_{ks}(0)$,
$\tilde{\dot{a}}_{kc}(0) = \dot{a}_{kc}(0) + i\dot{a}_{ks}(0), \tilde{\dot{a}}_{ks}(0) = 0.5i\dot{a}_{kc}(0) + 0.5\dot{a}_{ks}(0)$,
$\tilde{b}_{kc}(t) = b_{kc}(t) + ib_{ks}(t), \tilde{b}_{ks}(t) = 0.5ib_{kc}(t) + 0.5b_{ks}(t)$
of the column-vectors $\tilde{A}_{k0}, \tilde{\dot{A}}_{k0}, \tilde{B}_k(t)$ we obtain the solution of the ODEs system (6.8) in following form

$$\begin{cases} a_{kc}(t) = Re(\cos(\sqrt{\lambda_k}t))a_{kc}(0) + a_{ks}(0)Im(\cos(\sqrt{\lambda_k}t)) + \\ Re(\frac{\sin(\sqrt{\lambda_k}t))}{\sqrt{\lambda_k}})\dot{a}_{kc}(0) + \dot{a}_{ks}(0)Im(\frac{\sin(\sqrt{\lambda_k}t))}{\sqrt{\lambda_k}}) + \\ \int_0^t(Re(\frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}})b_{kc}(s) + b_{ks}(s)Im(\frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}}))ds, \\ a_{ks}(t) = -Im(\cos(\sqrt{\lambda_k}t))a_{kc}(0) + a_{ks}(0)Re(\cos(\sqrt{\lambda_k}t)) - \\ Im(\frac{\sin(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}})\dot{a}_{kc}(0) + \dot{a}_{ks}(0)Re(\frac{\sin(\sqrt{\lambda_k}t))}{\sqrt{\lambda_k}}) - \\ \int_0^t(Im(\frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}})b_{kc}(s) + b_{ks}(s)Re(\frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}}))ds, \end{cases} \tag{3.23}$$

For $k = 0$ we obtain $a_{0c}(t) = \int_0^t b_{0c}(s)ds + a_{0c}(0) + t\dot{a}_{0c}(0)$.

For numerical calculation we consider the initial boundary value problem with $f(x,t) = -2\pi a\cos(2\pi x)\cos(2\pi t), T_0 = \sin(\pi x), \bar{T}_0 = 0, T(x,t) = \sin(\pi x)cos(\pi t)$.

From (3.20) follows:

$T(x,t) = a_1(t)\exp(2\pi ix) + a_{-1}(t)\exp(-2\pi ix), a_1(0) = \frac{1}{2i}, a_{-1}(0) = -\frac{1}{2i}, b_1(t) = b_{-1}(t) = -\pi a\cos(2\pi t)$,

$\dot{a}_1(0) = \dot{a}_{-1}(0) = 0, \lambda_1 = 4\pi^2 - 2\pi ai$,

$\lambda_{-1} = 4\pi^2 + 2\pi ai, a_1(t) = \cos(\sqrt{\lambda_1}t)/(2i) + (\cos(2\pi t) - cos(\sqrt{\lambda_1}t))/(2i)$,

$a_{-1}(t) = -\cos(\sqrt{\lambda_{-1}}t)/(2i) - (\cos(2\pi t) - cos(\sqrt{\lambda_{-1}}t))/(2i)$.

Therefore $T(x,t) = (\cos(\sqrt{\lambda_1}t)\exp(2\pi ix) - \cos(\sqrt{\lambda_{-1}}t)\exp(-2\pi ix))/(2i) + ((\cos(2\pi t) - \cos(\sqrt{\lambda_1}t)\exp(2\pi ix))\exp(2\pi ix) - (\cos(2\pi t) - \cos(\sqrt{\lambda_{-1}}t))\exp(-2\pi ix))/(2i) = \cos(2\pi t)sin(2\pi x)$.

From (3.23) we have:

$T(x,t) = a_{1c}\cos(2\pi x) + a_{1s}\sin(2\pi x), a_{1c}(0) = 0, a_{1s}(0) = 1, \dot{a}_{1c}(0) = \dot{a}_{1s}(0) = 0$,

$b_{1s}(0) = 0, b_{1c}(0) = -2\pi a\cos(2\pi t), a_{1c}(t) = Im(\cos(\sqrt{\lambda_1}t)) + Re((\cos(2\pi t) - cos(\sqrt{\lambda_1}t))/i$,

$a_{1s}(t) = Re(\cos(\sqrt{\lambda_1}t)), T(x,t) = Re(\cos(\sqrt{\lambda_1}t))\sin(2\pi x)$.

We have for $a = L = 1$ following MATLAB operators:

$\mathbf{t = 0:0.01:1; l = 4*\pi^2 - 2*\pi*i; v = real}(\cos(\sqrt{(}\mathbf{l})*\mathbf{t})); \mathbf{v1} = \cos(\mathbf{2}*\pi*\mathbf{t})$;
$\mathbf{plot}(\mathbf{t}, \mathbf{v}, '*', \mathbf{t}, \mathbf{v1}, '\mathbf{o}')$.

For the **discrete problem** we have the system of N ODEs in the form of (3.11), where $a^2 = 1$ and the circulant matrix
$A = \frac{1}{h^2}[2\gamma, -(\gamma+\alpha), 0, 0, ...0, -(\gamma-\alpha)]$,

with the eigenvalues $\mu_k = \frac{4}{h^2}(\sin(k\pi/N)^2(\gamma - i\alpha\cot\frac{k\pi}{N})$,
and with the elements of the orthonormed eigenvectors
$w_j^k = \sqrt{\frac{1}{N}}\exp(2\pi ikj/N), w_{*j}^k = \sqrt{\frac{1}{N}}\exp(-2\pi ikj/N)k, j = \overline{1,N}$.
Here $\gamma = \alpha\coth(\alpha), \alpha = ah/2$.
For the column-vector $F(t)$ elements $f_j(t)$ we similarly from the chapter 1 obtain
$f_j(t) = \sum_{k=1}^{*N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
where
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^{N} f_j(t)\cos\frac{2\pi kj}{N}$,
$b_{ks}(t) = \frac{2}{N}\sum_{j=1}^{N} f_j(t)\sin\frac{2\pi kj}{N}, k = \overline{1,N_2}$,
$b_0(t) = b_N(t) = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} f_j(t)$,
$b_{0c}(t) = b_{Nc}(t) = \frac{2}{\sqrt{N}}b_0(t), b_{N_2,c}(t) = \frac{2}{\sqrt{N}}$
$b_{N_2}(t) = \frac{2}{N}\sum_{j=1}^{N}\cos(j\pi), N_2 = \frac{N}{2}, b_{N_2s}(t) = b_{Ns}(t) = 0$,
$\sum_{k=1}^{*N_2}\beta_k = \sum_{k=1}^{N_2-1}\beta_k + \frac{\beta_{N/2}}{2}$.
For the solution
$u_j(t) = \sum_{k=1}^{*N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$,
$u_j(0) = \sum_{k=1}^{*N_2}(a_{kc}(0)\cos\frac{2\pi kj}{N} + a_{ks}(0)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(0)}{2}$
$\dot{u}_j(0) = \sum_{k=1}^{*N_2}(\dot{a}_{kc}(0)\cos\frac{2\pi kj}{N} +$
$\dot{a}_{ks}(0)\sin\frac{2\pi kj}{N}) + \frac{\dot{a}_{0c}(0)}{2}$
with
$a_{kc}(0) = \frac{2}{N}\sum_{j=1}^{N} u_j(0)\cos\frac{2\pi kj}{N}$, $a_{ks}(0) = \frac{2}{N}\sum_{j=1}^{N} u_j(0)\sin\frac{2\pi kj}{N}$,
$\dot{a}_{kc}(0) = \frac{2}{N}\sum_{j=1}^{N}\dot{u}_j(0)\cos\frac{2\pi kj}{N}$, $\dot{a}_{ks}(0) = \frac{2}{N}\sum_{j=1}^{N}\dot{u}_j(0)\sin\frac{2\pi kj}{N}$
we need determine the unknown functions
$a_{kc}(t), a_{ks}(t)$ of the following expressions
$f_j(t) = \ddot{u}_j + (Au)_j = \sum_{k=1}^{*N_2}(\ddot{a}_{kc}(t)\cos\frac{2\pi kj}{N} + \ddot{a}_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{\ddot{a}_{Nc}(t)}{2} +$
$\sum_{k=1}^{*N_2}(a_{kc}(t)Re(\mu_k) + a_{ks}(t)Im(\mu_k))\cos\frac{2\pi kj}{N} +$
$(a_{ks}(t)Re(\mu_k) - a_{kc}(t)Im(\mu_k))\sin\frac{2\pi kj}{N})$.

Therefore, for the determine the functions $a_{kc}(t), a_{ks}(t)$ we obtain the systems of ODEs (3.21,3.22) and the solution (3.23), where the eigenvalues λ_k are replaced with the discreate eigenvalues $\mu_k, k = \overline{1,N}$.

3.9 The system of hyperbolic type equations with periodical BCs

We consider the initial-boundary problem of linear M-order system in following form :

$$\begin{cases} \frac{\partial^2 T_m}{\partial t^2} = \sum_{s=1}^{M}(\frac{\partial}{\partial x}(k_{m,s}\frac{\partial T_m}{\partial x}) + p_{m,s}\frac{\partial T_m}{\partial x}) + f_m, \\ T_m(0,t) = T_m(L,t), \frac{\partial T_m(0,t)}{\partial x} = \frac{\partial T_m(L,t)}{\partial x}, \\ T_m(x,0) = T_{m,0}(x), \frac{\partial T_m(x,0)}{\partial t} = \tilde{T}_m(x), x \in (0,L), m = \overline{1,M}, \end{cases} \tag{3.24}$$

where $\tilde{K}$ is the positive definite M-order matrix with different positive eigenvalues $\mu_K > 0$ and the elements $k_{m,s}$, P is the real M-order matrix with different real eigenvalues μ_P and the elements $p_{m,s}, m,s = \overline{1,M}$..
This system we can rewriten in the matrix form

$$\begin{cases} \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}(\tilde{K}\frac{\partial u(x,t)}{\partial x}) + P\frac{\partial u(x,t)}{\partial x} + f(x,t), \\ u(0,t) = u(L,t), \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x}, t \in (0,t_f), \\ u(x,0) = u_0(x), \frac{\partial u(x,0)}{\partial t} = \tilde{u}_0(x), x \in (0,L), \end{cases} \tag{3.25}$$

where u, f are column-vectors with elements$T_m, f_m, m = \overline{1,M}$.
Using the Fourier series the solution we can obtained in the following form:
$u(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
$f(x,t) = \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{L}\int_0^L f(\xi,t)\cos\frac{2\pi k\xi}{L}d\xi, b_{ks}(t) = \frac{2}{L}\int_0^L f(\xi,t)\sin\frac{2\pi k\xi}{L}d\xi$,
where the column-vectors $a_{kc}(t), a_{ks}(t)$ of the M order
are the corresponding solutions of the following differential equations

$$\begin{cases} \ddot{a}_{kc}(t) + \lambda_k\tilde{K}a_{kc}(t) - \frac{2\pi k}{L}Pa_{ks}(t) = b_{kc}(t), \\ \ddot{a}_{ks}(t) + \lambda_k\tilde{K}a_{ks}(t) + \frac{2\pi k}{L}Pa_{kc}(t) = b_{ks}(t), \end{cases} \tag{3.26}$$

where
$b_{kc}(t), b_{ks}(t)$ are the column-vectors of M order,
$a_{kc}(0) = \frac{2}{L}\int_0^L u_0\xi)\cos\frac{2\pi k\xi}{L}d\xi,, a_{ks}(0) = \frac{2}{L}\int_0^L u_0\xi)\sin\frac{2\pi k\xi}{L}d\xi,,$
$\dot{a}_{kc}(0) = \frac{2}{L}\int_0^L \tilde{u}_0\xi)\cos\frac{2\pi k\xi}{L}d\xi,, \dot{a}_{ks}(0) = \frac{2}{L}\int_0^L \tilde{u}_0\xi)\sin\frac{2\pi k\xi}{L}d\xi.$
For the **discrete problem** ($O(h^{2n})$ order of approximation) we have the system of ODEs in the following form:

$$\begin{cases} \ddot{v}_j(t) = \tilde{K}\Lambda v_j(t) + P\Lambda^0 v_j(t) + f_j(t), t \in [0,t_f], \\ v_j(0) = u_0(x_j), \dot{v}_j(0) = \tilde{u}_0(x_j), x_j = jh, Nh = L, j = \overline{1,N}, \end{cases} \quad (3.27)$$

where the column-vectors of the M order $v_j(t) \approx u(x_j,t), f_j(t) = f(x_j,t)$,
the expressions of the finite difference operators in multi-points stencil with 2n+1-points $(N \geq 2n+1)$
$\Lambda v_j = \frac{1}{h^2}(C_n(v_{j-n}+v_{j+n}) + ... + C_1(v_{j-1}+v_{j+1}) + C_0 v_j)$,
$C_0 = -\sum_{p=1}^n C_p$, $C_p = \frac{2(n!)^2(-1)^{p-1}}{p^2(n-p)!(n+p)!}, p = \overline{1,n}$
$\Lambda^0 v_j = \frac{1}{h}(c_n(v_{j+n}-v_{j-n}) + ... + c_1(v_{j+1}-v_{j-1}))$,
$c_p = \frac{(n!)^2(-1)^{p-1}}{p(n-p)!(n+p)!}, p = \overline{1,n}$ (see section 1).
We have following matrix representation for circulant matrices $A = -\Lambda = -\frac{1}{h^2}[C_0, C_1, ..., C_n, 0, ..., 0, C_n, ..., C_1]$,
with the eigenvalues $\mu_k = \frac{4}{h^2}\sum_{p=1}^n Q_p \sin^{2p}(\pi k/N), Q_p = \frac{2((p-1!))^2 4^{p-1}}{(2p)!}$
and $A^0 = \Lambda^0 = \frac{1}{h}[0, c_1, ..., c_n, 0, ..., 0, -c_n, ..., -c_1]$,
with the eigenvalues $\mu_k^0 = \frac{2i}{h}\sum_{p=1}^n c_p \sin\frac{2\pi pk}{N} =$
$\frac{2i}{h}\sin\frac{2\pi k}{N}\sum_{p=1}^n q_p \sin^{2p-2}\frac{\pi k}{N}$,
where $q_p = \frac{(p!)^2 4^{p-1}}{p(2p)!}$.
Using the discrete Fourier method the solution we can obtained in the following form:
$v_j(t) = \sum_{k=1}^{N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$,
$f_j(t) = \sum_{k=1}^{N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\cos\frac{2\pi kj}{N}, b_{ks}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\sin\frac{2\pi kj}{N}$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (3.26) and $\lambda_k, \frac{2\pi k}{L}$ are replaced with $\mu_k, Im(\mu_k^0)$,
$a_{kc}(0) = \frac{2}{N}\sum_{j=1}^N u_0(x_j)\cos\frac{2\pi kj}{N}$, $a_{ks}(0) = \frac{2}{N}\sum_{j=1}^N u_0(x_j)\sin\frac{2\pi kj}{N}$.
$\dot{a}_{kc}(0) = \frac{2}{N}\sum_{j=1}^N \tilde{u}_0(x_j)\cos\frac{2\pi kj}{N}$, $\dot{a}_{ks}(0) = \frac{2}{N}\sum_{j=1}^N \tilde{u}_0(x_j)\sin\frac{2\pi kj}{N}$.

This real form we can obtain also from following expressions
$A\cos_k = \mu_k \cos_k, A^0\cos_k = -|\mu_k^0|\sin_k, A\sin_k = \mu_k\sin_k$,
$A^0\sin_k = |\mu_k^0|\cos_k$, where $\sin_k, \cos_k$
are N-order column-vectors with the elements
$\sin\frac{2\pi kj}{N}, \cos\frac{2\pi kj}{N}$ and using the orthonormed conditions
$\sum_{j=1}^N \sin_k\cos_s = \sum_{j=1}^N \sin_k\sin_s = \sum_{j=1}^N \cos_k\cos_s = \frac{N}{2}\delta_{k,s}$.

Then for fixed frequency k in the initial data the solution can be writen in the form $u(t) = d_s(t)\sin_k + d_c(t)\cos_k$,
where $d_s(t), d_c(t)$ are unknown the time-depending vector-functions.
Then $\ddot{u}(t) = \ddot{d_s}(t)\sin_k + \ddot{d_c}(t)\cos_k$,
$Au(t) = \mu_k(d_s(t)\sin_k + d_c(t)\cos_k), A^0u(t) = |\mu_k^0|(d_s(t)\cos_k - d_c(t)\sin_k)$.
For FDSES, replaced the discrete eigenvalue $\mu_k, Im(\mu_k^0)$ with $\lambda_k, \frac{2\pi k}{L}$ we obtain the exact solutions for initial data with the frequency $\leq N/2$.

The time-dependent difference equation (3.27) using the MN-order column-vector $v(t)$ with the elements $v_j^m(t), j = \overline{1,N}, m = \overline{1,M}$ in the form

$$\ddot{v}(t) = (-\tilde{K}\bigotimes A + P\bigotimes A^0)v(t) + f(t) \tag{3.28}$$

serves as an approximation to the differential problem, in the sence that any smooth solution $u(t)$ satisfies the approximation (3.27) modulo a small local truncation error $\Psi(h,t) = O(h^{2n})$:

$$\frac{\partial^2 u(t)}{\partial t^2} = (-\tilde{K}\bigotimes A)u(t) + (P\bigotimes B)u(t) + f(t) + \Psi(h,t), \tag{3.29}$$

where MN- order matrices

$$\tilde{K}\bigotimes A = \begin{pmatrix} k_{1,1}A & \cdots & k_{1,M}A \\ \cdots & \cdots & \cdots \\ k_{M,1}A & \cdots & k_{M,M}A \end{pmatrix}, P\bigotimes A^0 = \begin{pmatrix} p_{1,1}A^0 & \cdots & p_{1,M}A^0 \\ \cdots & \cdots & \cdots \\ p_{M,1}A^0 & \cdots & p_{M,M}A^0 \end{pmatrix},$$

are Kronecker tensor products, $u(t), u(0), \tilde{u}(0), f(t)$ are MN column-vectors with the elements
$u_j^m(t), u_j^m(0), \tilde{u}_j^m(0), f_j^m, m = \overline{1,M}, j = \overline{1,N}$.
Matrices also can be defined with the representation $A = WDW^*$, $A^0 = WD^0W^*$ and solved numerically with the Matlab using the operator "kron",
If eigenvalues of matrices $\tilde{K}, P$ are $\lambda_s(K) > 0, \lambda_s(P), s = \overline{1,M}$ then exist the transformation of M-order matrices W_K, W_P and the representation
$K = W_K D_K W_K^{-1}, P = W_P D_P W_P^{-1}$,
where $D_K = diag(\lambda_s(K)), D_P = diag(\lambda_s(P))$
are the diagonal matrices. From properties of Kroneker tensor product follows $(W^* = W^{-1})$:
$B = -K\bigotimes A + P\bigotimes A^0 =$
$-(W_K D_K W_K^{-1}\bigotimes WDW^*) + (W_P D_P W_P^{-1}\bigotimes WD^0W^*) =$

$(W_K \bigotimes W)(-D_K \bigotimes D)(W_K^{-1} \bigotimes W^*)+$
$(W_P \bigotimes W)(-D_P \bigotimes D^0)(W_P^{-1} \bigotimes W^*)=$
$(W_K \bigotimes W)(-D_K \bigotimes D)(W_K \bigotimes W)^{-1} + (W_P \bigotimes W)(-D_P \bigotimes D^0)(W_P \bigotimes W)^{-1}.$
The eigenvalues
$\lambda(B)$ of matrix B are $-\mu_k \lambda_s(K) + \mu_k^0 \lambda_s(P), k = \overline{1,N}, s = \overline{1,M}$ with the $Re(\lambda(B)) \leq 0$ and the system of ODEs is stable.
For the approximation, if $v_j^m(t) = T_m(x_j,t), m = \overline{1,M}$ and for every time moment t
$u_j'' = \Lambda u_j + E_{2n} \frac{h^{2n} u^{(2n+2)}(\xi_j)}{2n+2)!}, E_{2n} = -2\sum_{k=1}^n C_k k^{2n+2}$
$u_j' = \Lambda^0 u_j + e_{2n} \frac{h^{2n} u^{(2n+1)}(\xi_j)}{2n+1)!}, e_{2n} = -2\sum_{k=1}^n c_k k^{2n+1}, x_{n-j} < \xi_j < x_{n+j},$
then
$\ddot{u}(t) = Bv(t) + f(t) + \Psi(h,t),$
where v, f, Ψ are MN-order column- vectors and $\Psi(h,t) = O(h^{2n})$,or
$||\Psi(h,t)|| \leq h^{2n}(\frac{|E_{2n}|}{(2n+2)!}||K||M_{2n+2}(t)+$
$\frac{|e_{2n}|}{(2n+1)!}||P||M_{2n+1}(t)).$
$M_s = \max|\frac{\partial^s T(x,t)}{\partial x^s}$- is the maksimal estimate for corresponding derivatives.
Given stability we can now estimate the global error $e(t) = v(t) - u(t)$ and to find, that the error $e(t)$ is governed by the eror equation

$$\frac{\partial^2 e(t)}{\partial t^2} = Be(t) + \Psi(h,t).$$

The solution of this equation is given by $e(t) = \cos(\sqrt{B}t)e(0) + \sin(\sqrt{B}t)(\sqrt{B})^{-1}\dot{e}(0)+$
$(\sqrt{B})^{-1}\int_0^t \sin(\sqrt{B}(t-\xi))\Psi(h,\xi)d\xi.$
From $Re(\lambda(B)) \leq 0$ and that $||\Psi||$ and$||e(0)||, ||\dot{e}(0)||$ are of order $O(h^{2n})$ follows the 2n- order of convergence rate later on $||e(t)|| = O(h^{2n}).$

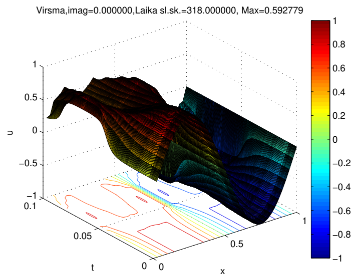


Fig. 3.13 FDSES solutions-surface by $N = 40$

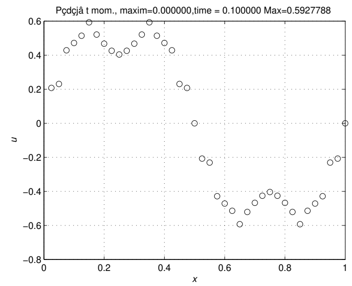


Fig. 3.14 FDSES solutions by $N = 20, t = t_f = 0.1$

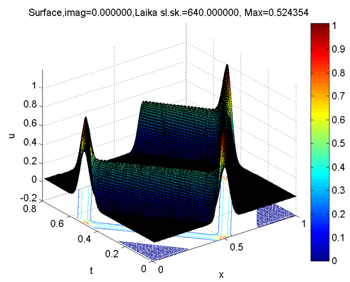


Fig. 3.15 FDSES solutions-surface by $N = 80$

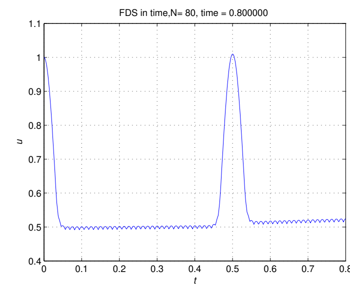


Fig. 3.16 FDSES max solutions depending on t by $N = 80$

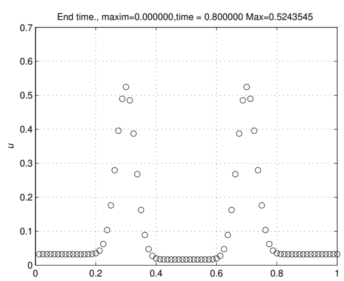


Fig. 3.17 FDSES solutions by $N = 80, t_f = 0.8$

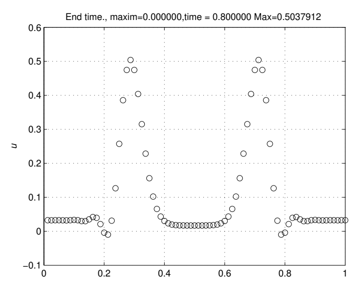


Fig. 3.18 FDS $-O(h^2)$ solutions by $N = 80, t_f = 0.8$

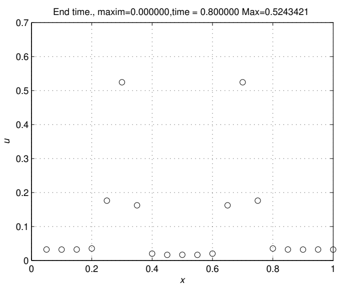


Fig. 3.19 FDSES solutions by $N = 20, t_f = 0.8$

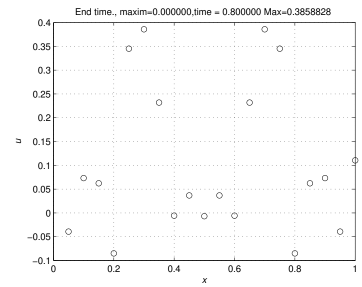


Fig. 3.20 FDS $-O(h^8)$ solutions by $N = 20, t_f = 0.8$

Chapter 4
1-D HHC equation: H. Kalis, A. Buikis, 2011 [39]

The hyperbolic heat conduction (HHC) problems are used for modelling of intensive steel quenching [6].

We consider the following homogenous 1-D hyperbolic heat conduction problem in plate :

$$\begin{cases} \tau\frac{\partial^2 T(x,t)}{\partial t^2}+\frac{\partial T(x,t)}{\partial t}=\frac{\partial}{\partial x}(\bar{k}\frac{\partial T(x,t)}{\partial x}), x\in(0,L), t\in(0,t_f), \\ \frac{\partial T(0,t)}{\partial x}-\alpha_0(T(0,t)-T_l)=0, \frac{\partial T(L,t)}{\partial x}+\alpha_1(T(L,t)-T_r)=0, \\ t\in(0,t_f), \\ T(x,0)=T_0(x), \frac{\partial T(x,0)}{\partial t}=V_0(x), x\in(0,L), \end{cases} \tag{4.1}$$

where $\bar{k}$ is the heat conductivity, t_f is the final time, τ is the relaxation time ($\tau < 1$), $T_l, T_r, T_0(x)$ are given temperatures, α_0, α_1 are the heat transfer coefficients (for boundary conditions of first kind $\alpha_0 = \alpha_1 = \infty$)

For the inverse problem the function $V_0(x)$ is unknown and then we can used the aditional condition $T(x,t_f) = T_f(x)$, where T_f is given final temperature. If the temperature T_l, T_r are constant values then using the transformation $V(x,t) = T(x,t) - T_h(x)$ we have the problem ($\alpha_0 = \alpha_1 = \infty$) with homogenous boundary conditions (BC) of first kind [6]:

$$\begin{cases} \tau\frac{\partial^2 V(x,t)}{\partial t^2}+\frac{\partial V(x,t)}{\partial t}=\frac{\partial}{\partial x}(\bar{k}\frac{\partial V(x,t)}{\partial x}), x\in(0,L), t\in(0,t_f), \\ V(0,t)=0, V(L,t)=0, t\in(0,t_f), \\ V(x,0)=v_0(x), \frac{\partial V(x,0)}{\partial t}=V_0(x), x\in(0,L), \end{cases} \tag{4.2}$$

where $v_0(x) = T_0(x) - T_h(x), T_h(x) = (xT_r + (L-x)T_l)/L$.

If $\alpha_0 = 0, \alpha_1 = \alpha, T_r = 0$ then we have from (4.1) the following problem with BC of the third kind

$$\begin{cases} \tau \frac{\partial^2 V(x,t)}{\partial t^2} + \frac{\partial V(x,t)}{\partial t} = \frac{\partial}{\partial x}(\bar{k}\frac{\partial V(x,t)}{\partial x}), x \in (0,L), t \in (0,t_f), \\ \frac{\partial V(0,t)}{\partial x} = 0, \frac{\partial V(L,t)}{\partial x} + \alpha V(L,t) = 0, t \in (0,t_f), \\ V(x,0) = T_0(x), \frac{\partial V(x,0)}{\partial t} = V_0(x), x \in (0,L), \end{cases} \tag{4.3}$$

The following 1-D hyperbolic heat conduction problem in the sphere with holes $(0 < r_0 < r < R)$ by radial symmetry is:

$$\begin{cases} \tau \frac{\partial^2 T(r,t)}{\partial t^2} + \frac{\partial T(r,t)}{\partial t} = \frac{\bar{k}}{r}\frac{\partial^2 (rT(r,t))}{\partial r^2}, r \in (r_0,R), t \in (0,t_f), \\ \frac{\partial T(r_0,t)}{\partial r} - \alpha_0 T(r_0,t) = 0, \frac{\partial T(R,t)}{\partial r} + \alpha_1 T(R,t) = 0, t \in (0,t_f), \\ T(r,0) = T_0(r), \frac{\partial T(r,0)}{\partial t} = V_0(r), r \in (r_0,R), \end{cases} \tag{4.4}$$

where r is the polar coordinate, R, r_0 are the radius of the sphere and hole. For the inverse problem the function $V_0(r)$ is unknown and then we can used the aditional condition $T(r,t_f) = T_f(r)$, where T_f is given temperature.
Using the tranformation $V = Tr, x = r - r_0$ we can the problem (4.4) reduced to the following hyperbolic heat conduction problem in the plate:

$$\begin{cases} \tau \frac{\partial^2 V(x,t)}{\partial t^2} + \frac{\partial V(x,t)}{\partial t} = \frac{\partial}{\partial x}(\bar{k}\frac{\partial V(x,t)}{\partial x}) + f(x+r_0,t), x \in (0,L), \\ \frac{\partial V(0,t)}{\partial x} - \sigma_1 V(0,t) = 0, \frac{\partial V(L,t)}{\partial x} + \sigma_2 V(L,t) = 0, t \in (0,t_f), \\ V(x,0) = T_0(x+r_0)(x+r_0), \frac{\partial V(x,0)}{\partial t} = V_0(x+r_0)(x+r_0), x \in (0,L) \end{cases} \tag{4.5}$$

where $\sigma_1 = \alpha_0 + \frac{1}{r_0}, \sigma_2 = \alpha_1 - \frac{1}{R} \geq 0, L = R - r_0, T(r,t) = \frac{V(x+r_0,t)}{x+r_0}$.

In the case of full sphere $(r_0 \to 0)$ the first BC condition is $V(0,t) = 0$ or $\sigma_1 = \infty$ and $T(0,t) = \frac{\partial V(0,t)}{\partial x}$.

4.1 The methods for solving the direct problem with the BC of first kind

We consider uniform grid in the space $x_k = kh, k = \overline{0,N}, Nh = L$.
Using the finite differences of second order approximation for partial derivatives of second order respect to x we obtain from (4.2) the initial

value problem for system of ordinary differential equations (ODEs) of second order in the following matrix form

$$\begin{cases} \tau\ddot{U}(t)+\dot{U}(t)+\bar{k}AU(t)=0, \\ U(0)=U_0, \dot{U}(0)=V_0, \end{cases} \tag{4.6}$$

where A is the standard 3-diagonal matrix of $N-1$ order with elements $\frac{4}{h^2}\{1;-2;1\}$. $U(t),\dot{U}(t),\ddot{U}(t),U_0,V_0$ are the column-vectors of $N-1$ order with elements $u_k(t)\approx V(x_k,t))$, $\dot{u}_k(t)\approx\frac{\partial V(x_k,t)}{\partial t}, \ddot{u}_k(t)\approx\frac{\partial^2 V(x_k,t)}{\partial t^2}, u_k(0)=v_0(x_k), v_k(0)=V_0(x_k), k=\overline{1,N-1}$.
The matrix A have the eigenvalues $\mu_k{=}\frac{4}{h^2}\sin^2(\frac{\pi k}{2N})$.

The system of ODEs (4.6) can be rewritten in a normal form

$$\dot{u}=Bu, u(0)=u_0, \tag{4.7}$$

where $u,\dot{u},u_0$ are the column-vectors of $2N-2$ order in the form $(U;\dot{U})^T,(\dot{U};\ddot{U})^T,(U_0;V_0)^T$,
B is the matrix of $2N-2$ order in the following form

$$B=\begin{pmatrix} 0 & E \\ -\tau^{-1}\bar{k}A & -\tau^{-1}E \end{pmatrix}$$

E is the unit matrix of $N-1$ order, T is the symbol of transposition.

For the difference sheme with **exact spectrum** (FDSES) the matrix A is represented in the form $A=WDW$, where $W=W^{-1}$ is the symmetrical orthogonal matrix with elements $w_{i,j}{=}\sqrt{\frac{2}{N}}\sin\frac{\pi ij}{N}$, $i,j{=}\overline{1,N-1}$. The column of the matrix W and the diagonal matrix D contains the first $N-1$ orthonormed eigenvectors $w_k(x_j){=}\sqrt{\frac{2h}{L}}\sin\frac{\pi k x_j}{L}, (x_j=jh,\ j{=}\overline{1,N-1})$ and eigenvalues $d_k=(k\pi/L)^2, k{=}\overline{1,N-1}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$ $(AW=WD)$.

We can consider the **analytical solutions** of (4.6) using the spectral representation of matrix A in form $A=WDW$.
From transformation $P=WU$ follows the seperate system of ODEs

$$\begin{cases} \tau\ddot{P}(t)+\dot{P}(t)+\bar{k}DP(t)=0, \\ P(0)=WU_0, \dot{P}(0)=WV_0, \end{cases} \tag{4.8}$$

where $P(t), \dot{P}(t), \ddot{P}(t), P_0, \dot{P}(0)$ are the column-vectors of $N-1$ order with elements $p_k(t), \dot{p}_k(t), \ddot{p}_k(t), p_k(0), \dot{p}_k(0), k = \overline{1, N-1}$.
The solution of this system is the function

$$\begin{cases} p_k(t) = exp(-0.5t/\tau)(\frac{\sinh(\kappa_k t)}{\kappa_k}(\dot{p}_k(0) + 0.5p_k(0)/\tau) + \\ \cosh(\kappa_k t)p_k(0)), \end{cases} \tag{4.9}$$

where $\kappa_k = \sqrt{0.25/\tau^2 - \bar{k}d_k/\tau}$. If $4\bar{k}d_k\tau > 1$, then the hyperbolic functions are replaced with the trigonometrical and the parameter κ_k with $\sqrt{\bar{k}d_k/\tau - 0.25/\tau^2}$.
If $\kappa_k = 0$, then $p_k(t) = \exp(-0.5t/\tau)[t(\dot{p}_k(0) + 0.5p_k(0)/\tau) + p_k(0)]$.

We can consider also the analytical solutions of (4.2) using the **Fourier method** in following form

$$V(x,t) = \sum_{k=1}^{\infty} p_k(t) w_k(x),$$

where $w_k(x) = \sqrt{2/L}\sin(\pi kx/L)$ are the orthonormed eigenvectors $((w_k, w_m) = \int_0^L w_k(x)w_m(x)dx = \delta_{k,m})$ of the differential operator $(\frac{\partial^2}{\partial x^2})$ with homogenous boundary conditions ($\delta_{k,m}$ – the symbol of Kronecker), $p_k(t)$ is the solution (4.9),with $p_k(0) = (v_0, w_k), \dot{p}_k(0) = (V_0, w_k)$.

For the approximations the derivatives with **matrix of derivatives** we can consider nonuniform grid with the grid points of the roots of the Chebyshev polynomials of the second kind

$$x_k = 0.5L(1 - \cos(\pi(k-1)/N)), \quad k = \overline{1, N+1}. \tag{4.10}$$

Using this grid points we can approximate the derivative $\frac{\partial^2}{\partial x^2}$ in the equations (4.2) with matrix D_e^2 of derivatives of $N+1$ order in the form [10]

$$V_h'' = D_e^2 V_h,$$

where $V_h = (V(x_1,t), V(x_2,t), \ldots, V(x_{N+1},t))^T$,
$V_h'' = (V''(x_1,t), V''(x_2,t), \ldots, V''(x_{N+1},t))^T$ are the colomns-vectors of the corresponding values $V''(x_k,t) \approx \partial^2 V(x_k,t)/\partial x^2$.
From the Lagrange interpolation follows, that the elements of matrix D_e are in the form

$$d_{j,k} = \frac{dl_k(x_j)}{dx}, j,k = \overline{1,N+1}, \tag{4.11}$$

where $l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)}$ are the elementary Lagrange multipliers, $\omega = \prod_{k=1}^{N+1}(x-x_k)$.
For this nonuniform grid the interpolation error is small [10].
The determinant of derivatives matrix D^2 are equal to zero (this matrix are singular). Therefore to need decrease the orders of this matrix to $N-1$ order using the homogenous boundary conditions and deleting the first and last columns and rows.In this case we need in (4.6, 4.7) replaced the matrix A with $-D_e^2$.

4.2 The methods for solving the direct problem with the BC of the 3-th kind

Similarly we obtain from (4.3) the initial value problem for the system of ODEs (4.6,4.7), where the column-vectors and unit matrix are of the $N+1$ order , B is the matrix of $2N+2$ order.
The 3-diagonal matrix A of $N+1$ order (similarly from Chapter 2, $\sigma_1 = 0, \sigma_2 = \alpha$) can be represented with difference
operator of second order approximation and solved the corresponding spectral problem.
The **spectral problem** $Ay^n = \mu_n y^n, n = \overline{1,N+1}$ have following solution

$$\begin{cases} y^n = C_n(\frac{1}{\sqrt{2}}\cos(p_n x_0), \cos(p_n x_1), ..., \cos(p_n x_{N-1}), \\ \frac{1}{\sqrt{2}} cos(p_n x_N)^T, \mu_n = \frac{4}{h^2}\sin^2(p_n h/2), \end{cases} \tag{4.12}$$

where p_n are the positive roots of the following transcendental equation

$$\tan(p_n L) sin(p_n h)/h = \alpha, n = \overline{1,N}$$

and the constants $C_n = \sqrt{\frac{2}{L+0.5h\sin(2p_nL)/\tan(p_nh)}}$ give the orthonormal eigenvectors y^n, y^m with the scalar product $[y^n, y^m] = \delta_{n,m}$ for $n,m = \overline{1,N}$.

For p_{N+1} from (4.12) follows that $p_{N+1} = 2\pi N/L - p_N$ and $\mu_{N+1} = \mu_N$. Therefore for $n = N+1$ we have following special solution of the spectral problem:

$$\begin{cases} y^n = C_n(\frac{1}{\sqrt{2}}\cosh(p_n x_0), \cosh(p_n x_1), ..., \cosh(p_n x_{N-1}), \\ \frac{1}{\sqrt{2}} cosh(p_n x_N)^T, \mu_n = -\frac{4}{h^2}\cosh^2(p_n h/2),\ n = N+1 \end{cases} \tag{4.13}$$

where p_{N+1} is the positive root of the new following transcendental equation

$$\tanh(p_{N+1}L) sinh(p_{N+1}h)/h = \alpha$$

and the constants $C_{N+1} = \sqrt{\frac{2}{L+0.5h\sinh(2p_{N+1}L)/\tanh(p_{N+1}h)}}$ give the orthonormed eigenvectors y^n, y^m with the scalar product $[y^n, y^m] = \delta_{n,m}$ for all $n, m = \overline{1, N+1}$.

The matrix A can be represented in form $A = WDW^T$, where the column of the matrix W and the diagonal matrix D contains $N+1$ orthonormed eigenvectors y^n and eigenvalues $\mu_n, n=\overline{1,N+1}$. From $W^T W = E$ follows that $W^{-1} = W^T$.

The solution of the **spectral problem** for differential equations

$$-y''(x) = \lambda^2 y(x), x \in (0, L), y'(0) = 0, y'(L) + \alpha y(L) = 0,$$

is in following form $y_n(x) = C_n \cos(\lambda_n x)$,
$C_n = \sqrt{\frac{2}{L+0.5\sin(2\lambda_n)/\lambda_n}} = \sqrt{\frac{2}{L+\alpha/(\alpha^2+\lambda_n^2)}}$, where
$(y_n, y_m) = \int_0^L y_n(x) y_m(x) dx = \delta_{n,m}$ and λ_n are positive roots of the following transcendental equation:

$$\tan(\lambda_n L)\lambda_n = \alpha, n = 1, 2, 3, ...$$

We can used also the **Fourier method** for solving (4.3) in the form

$$V(x,t) = \sum_{k=1}^{\infty} p_k(t) y_k(x),$$

where $y_k(x)$ are the orthonormed eigenvectors, $p_k(t)$ is the solution (4.9),with $p_k(0) = (T_0, y_k), \dot{p}_k(0) = (V_0, y_k)$.

For the **FDSES** the matrix A is represented in form $A = WDW^T$ and the diagonal matrix D contains the first $N+1$ eigenvalues $d_n = \lambda_n^2$, $n=\overline{1,N+1}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$ correspondly.

Similarly we can consider the **analytical solutions** of (4.6) using the spectral representation of matrix A in form $A = WDW^T$.

From transformation $P = W^T U$ follows the seperate system of ODEs (4.8), where $P(0) = W^T U_0, \dot{P}(0) = W^T V_0$ are the column-vectors of $N+1$ order. The solution of this system is in the form (4.9).

Note: In this case to consider the scalar product the first and last components of vectors U_0, V_0 are divided with $\sqrt{2}$, but $p_1(t), p_{N+1}(t)$ need to multiply with $\sqrt{2}$.

Using **averaged value with constant function** [9]
$\tilde{v}(t) = \frac{1}{L}\int_0^L V(x,t)dx, V(L,t) = \tilde{v}(t)$ from equation (4.3) follows the initial value problem for ODEs

$$\tau\tilde{v}'' + \tilde{v}' + \gamma\tilde{v} = 0, \tilde{v}(0) = \int_0^L T_0(x)dx, \tilde{v}'(0) = \int_0^L V_0(x)dx, \quad (4.14)$$

where $\gamma = \bar{k}\alpha/L, v' = \frac{dv}{dt}, v'' = \frac{d^2v}{dt}$.
The analytical solution of this problem is

$$\tilde{v}(t) = \exp(-0.5t/\tau)[\tilde{v}(0)\cosh(t\beta) + (\frac{\tilde{v}'(0)}{\beta} + \frac{\tilde{v}(0)}{2\tau\beta})\sinh(t\beta)],$$

where $\beta = \frac{1}{2\tau}\sqrt{1-4\tau\gamma}$. If $\gamma > \frac{1}{4\tau}$, then the hyperbolic function is replaced with the trigonometrical and the parameter β with $\frac{1}{2\tau}\sqrt{4\tau\gamma - 1}$.
If $\beta = 0$, then $\tilde{v}(t) = \exp(-0.5t/\tau)[\tilde{v}(0)(1+0.5t/\tau) + t\tilde{v}'(0)]$.
Using **averaged value with quadratic polynomial**[9]

$$V(x,t) \approx \tilde{v}(t) + m(t)(x-0.5L) + e(t)((x-0.5L)^2 - L^2/12)$$

with the unknown functions $m(t), e(t)$.
We can determined this functions from BC (4.3) in following form: $m(t) = e(t)L = -\frac{0.5\alpha}{1+L\alpha/3}\tilde{v}(t)$. Then

$$V(x,t) \approx 0.5\tilde{v}(t)\frac{2 + L\alpha - \alpha x^2/L}{1 + L\alpha/3}.$$

If $\alpha = 0$ then $V(x,t) \approx \tilde{v}(t)$.
The corresponding solution $\tilde{v}(t)$ of the problem (4.14) remains one's own form, where $\gamma = \bar{k}\alpha/(L + L^2\alpha/3)$.

4.3 The methods for solving the inverse problem

For the inverse problem the vector V_0 in (4.1,4.6) is unknown and we have additional condition $U(t_f) = u_f$, where u_f is the vectors-column with elements $u_{f,k} = T_f(x_k), k = \overline{1,M}$, ($M = N-1$ (4.2) or $M = N+1$ (4.3).
The **analytical solutions** of this problem can be obtain from (4.8) replaced the second initial condition $\dot{P}(0) = WV_0$ with $P(t_f) = Wu_f$ for the problem (4.2) or $\dot{P}(0) = W^T V_0$ with $P(t_f) = W^T u_f$ for the problem (4.3).
Then the solution is following

$$\begin{cases} p_k(t) = exp(-0.5t/\tau)[\frac{\sinh(\kappa_k t)}{\sinh(\kappa_k t_f)}(\exp(0.5t_f/\tau)p_k(t_f)- \\ w_k(0)\cosh(\kappa_k t_f)) + cosh(\kappa_k t)p_k(0)], \end{cases} \tag{4.15}$$

where $p_k(t_f)$ are the components of vector $P(t_f)$ (the **note** for the vectors V_0, u_f in (4.3) is valid).
From (4.15) follows that the second conditions is in following form

$$\dot{p}_k(0) = \frac{\kappa_k}{\sinh(\kappa_k t_f)}[exp(0.5t_f/\tau)p_k(t_f) - p_k(0)\cosh(\kappa_k t_f)] - \frac{p_k(0)}{2\tau},$$

$V_0 = W\dot{P}(0)$.
Replaced this expression in (4.9) we obtain the solution (4.15).

For **Fourier method** we obtain the Fourier coefficients $p_k(t), \dot{p}_k(0)$ from (4.15), where $p_k(t_f) = (u_f, Q_k), V_0(x) = \sum_{k=1}^{\infty} \dot{p}_k(0)Q_k(x)$, $Q_k(x) = w_k(x)$ for the problem (4.2) or $Q_k(x) = y_k(x)$ for the problem (4.3).

Using **averaged method** (4.14) for given value $\tilde{v}(t_f) = \frac{1}{L}\int_0^L V(x,t_f)dx$ we can obtained the value $\tilde{v}'(0)$ in the form

$$\tilde{v}'(0) = \tilde{v}(t_f)\frac{\beta\exp(0.5t_f/\tau)}{\sinh(t_f\beta)} - \tilde{v}(0)(\beta\coth(t_f\beta) + 0.5/\tau). \tag{4.16}$$

If $\beta = 0$ then $\tilde{v}'(0) = \tilde{v}(t_f)\frac{\exp(0.5t_f/\tau)}{t_f} - \tilde{v}(0)(1/t_f + 0.5/\tau)$.
From **averaged method with quadratic polynomial** follows that $V_0(x) \approx 0.5\tilde{v}'(0)\frac{2+L\alpha-\alpha x^2/L}{1+L\alpha/3}$.

4.4 The methods for solving the problem in the sphere with holes

We obtain from (4.5) the initial value problem for system of ODEs of second order in the matrix form (4.6), where A is the 3-diagonal matrix of $N+1$ order in the form

$$A = \frac{1}{h^2}\begin{pmatrix} 2+h\sigma_1 & -2 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ .. & .. & .. & \dots & .. & .. & \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -2 & 2+h\sigma_2 \end{pmatrix}$$

$u_k(0) = U_0(x_k+r_0)(x_k+r_k), v_k(0) = V_0(x_k+r_0)(x_k+r_0), k = \overline{0,N}$.

The matrix A can be represented in form $A = PDP^T, (AP = PD)$ where the column of the matrix P and the diagonal matrix D contain M orthonormed eigenvectors y^n and eigenvalues $\mu_n, n = \overline{1,M}$, ($M = N+1$ for finite value of σ_1 and σ_2, $M = N$ for infinite σ_1), T is the symbol of transposition. From $P^T P = E$ follows that $P^{-1} = P^T$.

We can consider the **analytical solutions** of (4.6) using the spectral representation of matrix A $(A = PDP^T)$
and obtain the seperate system of ODEs (4.8) with the solution (4.9). Similarly we can obtain the solution of the inverse problem.

4.5 Some examples and numerical results

4.5.1 The initial-boundary problem with the BC of first kind

The numerical experiment for the direct problem (4.2, 4.7) with $L = \bar{k} = 1, \tau = 0.1, t_f = 0.2, T_l = 1, T_r = 0, T_0 = 0, V_0 = 0$ (the initial and boundary conditions are discontinuous) is produced by MATLAB solver ”ode15s” or with help of calculation the matrix-function $exp(Bt)$, using Matlab operator $expm(B*t)$. The solutions of the problem (4.7) is $u(t) = expm(B*t)*u_0$.
For the solution of the inverse problem with the conjugate operator (4.7) we have $w(t) = expm(B^T*(t_f-t))*w(t_f)$ but for the method of superposition $U_i(t) = expm(B*t)*U_{i,0}$, where $U_{i,0}, i = 1;2$ are special initial conductions.
For hyperbolic equations (4.2) (without the term $\frac{\partial V(x,t)}{\partial t}$) follows that

at time t_f a front of the right-moving wave will reach point $x_f(t_f) = \frac{t_f}{\sqrt{\tau/\bar{k}}} = 0.63245$ [6].
The grafics of numerical results for direct problem obtained from the method of FDSES by $N = 30$ are in the Fig. 4.1 and Fig. 4.2.
For testing the inverse problem for $V_0 = 1$ is obtained the vector u_f by solved the direct problem. Then by solving the inverse problem for given u_f is obtained the vector V_0. The maximal relative error for values V_0 is 10^{-7} for $N > 20$.
In the Figs. 4.3, 4.4 we can see the results obtained with finite differences approximation and FDSES method by $N = 200$.

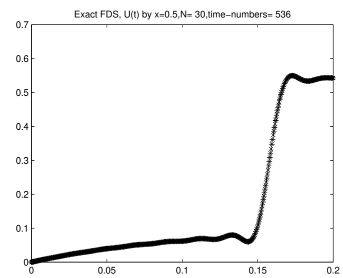


Fig. 4.1 Solution depending on t by $x = 0.5$

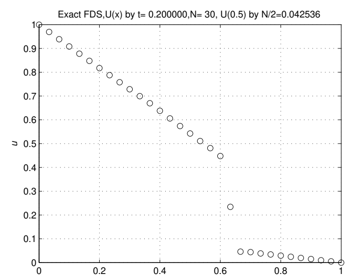


Fig. 4.2 Solution depending on x by $t = 0.2$

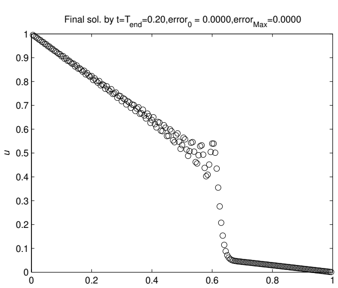


Fig. 4.3 Solution of FDS by $N = 200, t = 0.2$

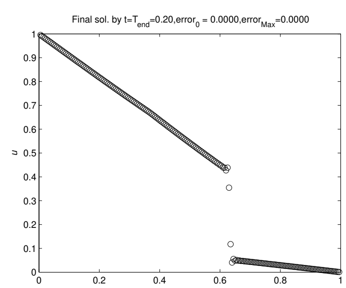


Fig. 4.4 Solution of FDSES by $N = 200, t = 0.2$

In the Figs. 4.5, 4.6 are represented the results of Fourier obtained from the series

$$V(x,t) = -2\exp(-0.5t/\tau)\sum_{k=1}^{\infty}\frac{\sin(k\pi x)}{k\pi}(\cosh(\kappa_k t) + \frac{\sinh(\kappa_k t)}{2\kappa_k\tau}).$$

with $w_k(0) = -\frac{\sqrt{2}}{\pi k}, \dot{w}_k(0) = 0, \infty \approx 200, N = 80$ (the Gibbs efect is observed).
Using the matrix of derivative D_e^2 in (2.2) the coresponding results with oscilations by $N = 100$ can be seen in Figs. 4.7, 4.8.

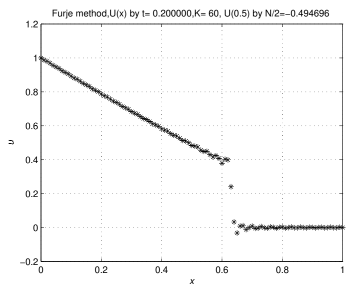


Fig. 4.5 Solution $T(x,0.2)$ with Fourier method by $N = 80$

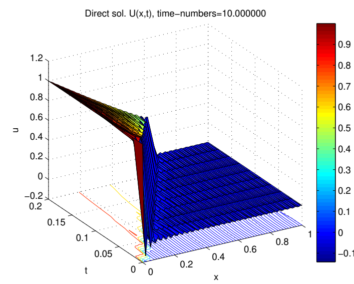


Fig. 4.6 Solution $T(x,t)$ with Furier method by $N = 80$

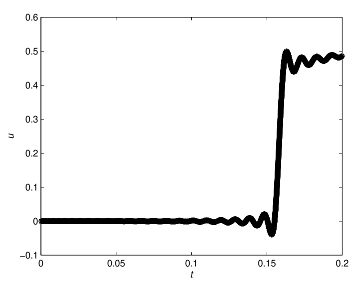


Fig. 4.7 Solution $(U(0.5,t))$ with matrix-derivatives method by $N = 100$

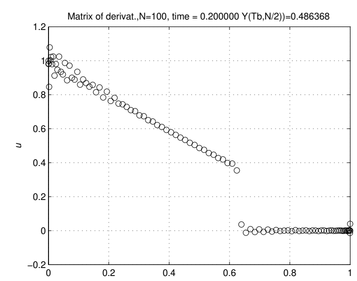


Fig. 4.8 Solution $U(x,t)$ with matrix-derivatives method by $N = 100$

4.5.2 The initial-boundary problem with the BC of 3-th kind

The numerical results for the problem (4.3, 4.7) are obtained for intensive steel quenching models with $\bar{k} = 0.0001738602$ ($\bar{k} = \frac{k}{c\rho}, k =$

$429\frac{W}{mC}$ - the heat conductivity, $\rho = 1.05.10^4\frac{kg}{mC}$ - the density of the steel, $c = 235\frac{J}{kgC}$ - the heat capacity),$\alpha = \frac{30}{429}$ ($\alpha = \frac{h}{k}, h$ - the heat transfer coefficient) and $L = 1, \tau = 0.1; 0.5, T_0(x) = 600, t_f = 5, T_f(x) = 0, N = 10; 20$. We have following inequalities for eigenvalues $\lambda_n : \pi(n-1.5) < \lambda_n < \pi(n-0.5), n = \overline{1, N+1}$.
The first 6 eigenvalues λ_n are: 0.2614; 3.1637; 6.2943; 9.4322; 12.5719; 15.7124.
For $\tau = 0.5$ we have $V_0 = -1200, 107; -1200, 161; -1200, 047$ correspondly for superposition, conjugate operators and averaged methods (see the Figs. 4.9, 4.10).
For $\tau = 0.1$ we have correspondly $V_0 = -6000, 0527; -6000, 0532; -5999, 9927$ (see the Figs. 4.11, 4.12).

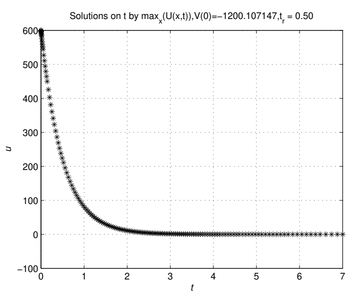


Fig. 4.9 Solution $max(U(x,t))$ depending on t by $\tau = 0.5, t_f = 5$

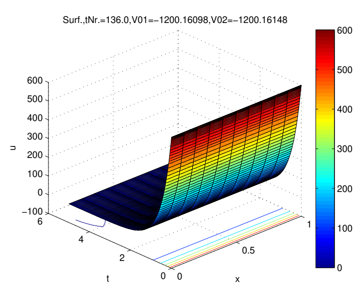


Fig. 4.10 Solution $U(x,t)$ by $\tau = 0.5, t_f = 5$

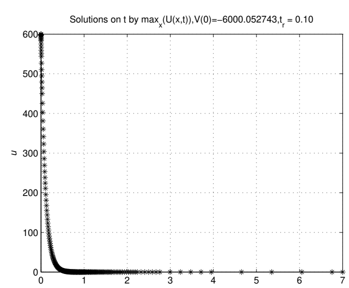


Fig. 4.11 Solution $max(U(x,t))$ depending on t by $\tau = 0.1$

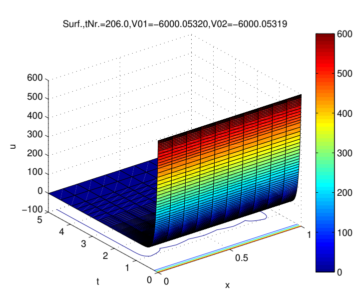


Fig. 4.12 Solution $U(x,t)$ by $\tau = 0.1$

4.5.3 The initial-boundary problem for holow sphere

The numerical results for the problem (4.4, 4.6) are obtained by the above-mentioned parameters with
$R = 8, r_0 = 1, T_0(r) = 600; 600\frac{r-r_0}{R-r_0}, t_f = 0.5; 5, T_f(r) = 0, N = 10; 20.$
In the Figs. 4.13, 4.14, 4.15 are presented the results of the calculation by $\tau = 0.5, T_0(r) = 600, r_0 = 1, t_f = 5$, where $V_0 = -1200, 1$.
In the Figs. 4.16, 4.17, 4.18 are presented the results of the calculation by $\tau = 0.1, T_0(r) = 600, r_0 = 1, t_f = 0.5$, where $V_0 = -6000, 4$.

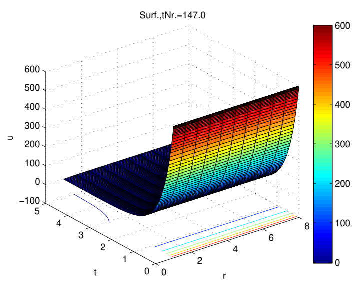


Fig. 4.13 Solution depending on t, r by $\tau = 0.5, T_0(r) = 600$

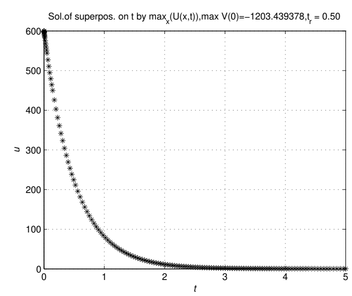


Fig. 4.14 Solution $max|u|$ depending on t by $\tau = 0.5, T_0(r) = 600$

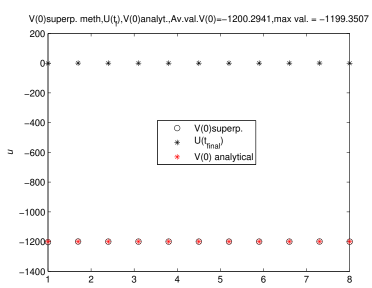


Fig. 4.15 Functions V_0, U_f depending on r by $\tau = 0.5, T_0(r) = 600$

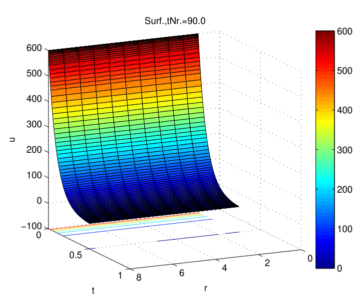


Fig. 4.16 Solution depending on t, r by $\tau = 0.1, T_0(r) = 600$

In the Figs. 4.19, 4.20 are presented the results of the calculation by $\tau = 0.5, T_0(r) = 600\frac{r-r_0}{R-r_0}, r_0 = 1, t_f = 5$,
where the averaged value $V_0 = -600, 1$.
In the Figs. 4.21, 4.22 are presented the results of the calculation by

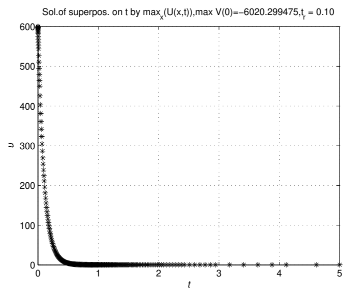


Fig. 4.17 Solution $max|u|$ depending on t by $\tau = 0.1, T_0(r) = 600$

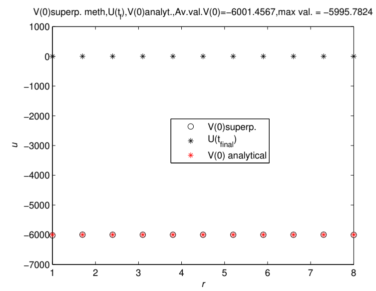


Fig. 4.18 Functions V_0, U_f depending on r by $\tau = 0.1, T_0(r) = 600$

$\tau = 0.1, T_0(r) = 600\frac{r-r_0}{R-r_0}, r_0 = 1, t_f = 0.5,$
where the averaged value $V_0 = -3000, 5$.

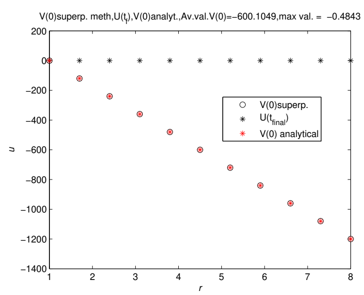


Fig. 4.19 Functions V_0, U_f depending on r by $\tau = 0.5, T_0(r) = 600\frac{r-r_0}{R-r_0}$

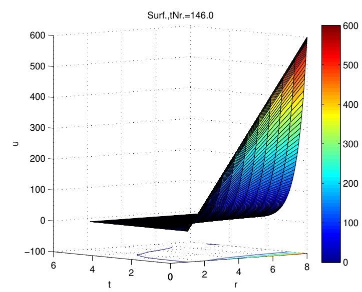


Fig. 4.20 Solution depending on t, r by $\tau = 0.5, T_0(r) = 600\frac{r-r_0}{R-r_0}$

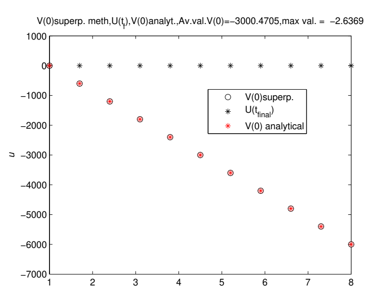


Fig. 4.21 Functions V_0, U_f depending on r by $\tau = 0.1, T_0(r) = 600\frac{r-r_0}{R-r_0}$

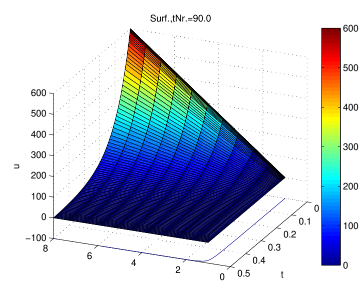


Fig. 4.22 Solution depending on t, r by $\tau = 0.5, T_0(r) = 600\frac{r-r_0}{R-r_0}$

4.5.4 MATLAB programm for solvig PDE in the holow sphere

We consider the m.file **hipSiltLL.**

```
%$ODE  tau*U_{tt}+U_t+a2 AU =f with solvers ,3.kind boun-cond.
%Operator method and superposition,anal. sol.
% $x=r-r_0,r_0;R-radii of sphere,L=R-r_0
%$t=Tb,U_{(t=0)}=600,U_{t(t=0)} =v_0=?,U_t=T =0,
%$U_{x(x=0)}-1/r0 U{(x=0)}=0,U_{x(x=L)}+ (alfa-1/R)U_{(x=L)}=0,
%FDSES,method of lines, use m.file"ipas3v(N,sig1,sig2,L)"
function hipSiltLL(N)
MK=50;r0=1;R=8;N1=N+1; Tb=0.5;L=R-r0;x=linspace(0,L,N1)';
t=linspace(0,Tb,MK);h=L/N;NN=2*N1; tau=0.1;k1=429;
c1=235;r1=10500;a2=k1/(c1*r1);
alfa=60/429;sig1=1/r0;sig2=alfa-1/R;
[lk0,lk,W]=ipas3v(N,sig1,sig2,L); % solve the spectral problem
Z1=zeros(N1,N1);E1=eye(N1,N1);A=zeros(NN,NN);
 y1=600.*(x+r0);y1(1)=y1(1)/sqrt(2);y1(N1)=y1(N1)/sqrt(2);
P=W'*y1;P1=zeros(MK,N1);P0=zeros(N1,1);
for k=1:N1
    b=sqrt(0.25/tau^2 -a2*lk(k)/tau);
P1(:,k)=P(k)*exp(-0.5*t'/tau).*(cosh(b*t')-...
sinh(b*t')*coth(b*Tb));
    P0(k)=-P(k)*(0.5/tau +b*coth(b*Tb));
end
P2=(W*P1')';V0=W*P0;V0(1)=V0(1)*sqrt(2);
V0(N1)=V0(N1)*sqrt(2);V0=V0./(x+r0);
P2(:,1)=P2(:,1)*sqrt(2);P2(:,N1)=P2(:,N1)*sqrt(2);
P2=P2./(ones(MK,1)*(x+r0)');
X4=ones(MK,1)*(x+r0)';Y4=t'*ones(1,N1);
figure, surfc(X4,Y4,P2)
colorbar
xlabel('r'), ylabel('t'), zlabel('u')
title(sprintf('Surf.,tNr.=%4.1f',MK))
A2=a2*W*diag(lk0).^2*W';
A=[Z1,E1;-A2/tau,-E1/tau];MI2=zeros(N1,1);
y01=[y1;zeros(N1,1)];uT1=expm(A*Tb)*y01;
y02=[zeros(N1,1);ones(N1,1)];uT2=expm(A*Tb)*y02;
MI2(1:N1,1)=(zeros(1:N1,1)-uT1(1:N1,1))./uT2(1:N1,1);
y0=[y1;MI2];y0T=expm(A*Tb)*y0;y0T(1)=y0T(1)*sqrt(2);
y0T(N1)=y0T(N1)*sqrt(2);
MI2(1)=MI2(1)*sqrt(2);MI2(N1)=MI2(N1)*sqrt(2);
SM=sum(MI2./(x+r0))/N1;MSM=max(MI2./(x+r0));
figure
plot(x+r0,MI2./(x+r0),...
'ko',x+r0,y0T(1:N1)./(x+r0),'k*',x+r0,V0,'r*')
title(sprintf('V(0)superp. meth,U(t_f),V(0)analyt.,Av.V(0)=. . .
%8.4f,max val. = %8.4f ',SM,MSM))
legend('V(0)superp.','U(t_{final})', 'V(0) analytical')
xlabel('r'), ylabel('u')
options=odeset('RelTol', 1.0e-7);
[T1,Y1]=ode15s(@SIST,[0 Tb],y0,options,A);
```

```
Y1(:,1)=Y1(:,1)*sqrt(2); Y1(:,N1)=Y1(:,N1)*sqrt(2);
Y1(:,N1+1)=Y1(:,N1+1)*sqrt(2);Y1(:,NN)=Y1(:,NN)*sqrt(2);
Mv=[(x+r0);(x+r0)]';K=length(T1);Y1=Y1./(ones(K,1)*Mv);
MSM1=min(Y1(1,N1+1:NN));
figure,plot(x+r0,Y1(end,1:N1)','ko')
grid on
title(sprintf('Sol.of superpos. on r by t=Tb=...
%4.2f,t_r =%4.2f ',Tb,tau))
xlabel('r'), ylabel('u')
figure,plot(T1(:),max(Y1(:,1:N1)'),'k*')
title(sprintf('Sol.of superpos. on t by max(U(x,t)),max V(0)=. . .
%8.6f,$t_r$ = %4.2f ',MSM1,tau))
xlabel('t'), ylabel('u')
X11=ones(K,1)*(x+r0)';Y11=T1*ones(1,N1);
figure, surfc(X11,Y11,Y1(:,1:N1))
colorbar
xlabel('r'), ylabel('t'), zlabel('u')
title(sprintf('Surf.,tNr.=%4.1f',K))
function y=f1(x,sig1,sig2,h,L)
y=$tanh(x*L)*(sig1*sig2*h^2+(sinh(x*h))^2)$/h/sinh(x*h) -. . .
(sig2+sig1);
function y=msak(x,sig1,sig2,L)
y= cot(x*L)-(x- sig1*sig2/x)/(sig1+sig2);
function y=msak1(x,sig1,sig2,h,L)
y= cot(x*L)-(sin(x*h)/h -h*sig1*sig2/sin(x*h))/(sig2+sig1);
function F=SIST(t,y,A)
F=A*y;
```

4.5.5 MATLAB programm for matrix of derivatives

Here is the m.file **HipParCh** and **cheb** for matrix of derivatives

```
%$ODE rU_tt +U_t=A U,U(0,t)=1,U(1,t)=U(x,0)=U_t(x,0)=0
%$transf.U=U1 +1-x,U1(0,t)=U1(1,t)=0,U1(x,0)=x-1,U1_t(x,0)=1
function HipParCh(N)
Tb=0.2;r=0.1;
L=1;N1=N+1;N2=N-1;NN=N2*N2;A=zeros(NN,NN);NP=fix(N/2);
[D1,x1]=cheb(N); % Chebihev grid points x1 and matrix D1
D=D1*2/L;
x=0.5*L*(x1+1);
x=x(2:N);
y1=-(1-x);y2=zeros(N2,1);y0=[y1;y2];
B=D^2;A2=eye(N2,N2);A1=zeros(N2,N2);
A3=B(2:N,2:N)/r;A4=-1/r*eye(N2,N2);
A=[A1 A2;A3 A4];
options=odeset('RelTol', 1.0e-7);
```

```
[T,Y]=ode15s(@SIST,[0 Tb],y0,options,A);
Y1=Y(end,1:N2)'-y1;
figure
plot([0;x;L],[1;Y1;0],'ko')
grid on
title(sprintf('Matrix of derivat.,N=%3.0f, time = . . .
%8.6f Y(Tb,N/2))=%8.6f ',N,T(end),Y1(NP)))
xlabel('x'), ylabel('u')
figure
plot(T,Y(:,NP)-y1(NP),'k*')
xlabel('t'), ylabel('u')
hold off
function F=SIST(t,y,A)\\
F=A*y;
function [D,x] = cheb(N) % in segment [-1,1]
if N==0, D=0; x=1; return, end
x=-cos(pi*(0:N)/N)';
c=[2; ones(N-1,1);2].*(-1).$^{(0:N)}$';
X = repmat(x,1,N+1);
dX = X-X';
D =-(c*(1./c)')./(dX+(eye(N+1)));
D = -D+ diag(sum(D'));
```

4.6 The solving the direct problem with the periodical BC

We consider the initial boundary value problem with periodical BCs

$$\begin{cases} \tau\frac{\partial^2 T(x,t)}{\partial t^2} + \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}(\bar{k}\frac{\partial T(x,t)}{\partial x}), x \in (0,L), t \in (0,t_f), \\ T(0,t) = T(L,t), \frac{\partial T(0,t)}{\partial x} = \frac{\partial T(L,t)}{\partial x}, t \in (0,t_f), \\ T(x,0) = T_0(x), \frac{\partial T(x,0)}{\partial t} = V_0(x), x \in (0,L). \end{cases} \tag{4.17}$$

Using uniform grid $x_j = jh, j = \overline{0,N}, Nh = L,$ (N is even number) we obtain the system of ODEs (4.6) where A is the 3-diagonal circulant matrix of N order in the form $A = \frac{1}{h^2}[2,-1,0,\cdots,0,-1]$.
From the corresponding spectral problem of the matrix A follows that (see chapter1)
$\mu_k = \frac{4}{h^2}(\sin(k\pi/N)^2$, are the eigenvalues and $w_j^k = \sqrt{\frac{1}{N}}\exp(2\pi ikj/N)$, $w_{*,j}^k = \sqrt{\frac{1}{N}}\exp(-2\pi ikj/N), k, j = \overline{1,N}$
are the elements of the biorthonormed complex eigenvectors
w^k, w_*^k, where $(w^k, w_*^m) = \sum_{j=1}^N w_j^k w_{*,j}^m = \delta_{k,m}$,
The eigenvalues μ_k are symmetrical as regards $k = N/2$ (with the max-

imal value $\frac{4}{h^2}$) or $\mu_{N/2+m} = \mu_{N/2-m}, m = \overline{1, N/2}$.
Using the matrixes W, W_* with the eigenvectors w^k, w_*^k in the matrixes columns we get $AW = WD, WW_* = E, W^{-1} = W_*, A = WDW_*$, where the elements of the diagonal matrix D is $d_k = \mu_k, k = \overline{1, N}$.
For the differential spectral problem follows
$\lambda_k = (2\pi k/L)^2, w^k(x) = \sqrt{\frac{1}{L}}\exp(2\pi ikx/L)$,
$w_*^k(x) = \sqrt{\frac{1}{L}}\exp(-2\pi ikx/L)$, $(w^k, w_*^m)_* = \int_0^L w^k(x)w_*^m(x)dx = \delta_{k,m}$,
$k, m = \overline{-\infty, +\infty}$.

The solution of (4.17) with the Fourier method can be obtained in following form:
$T_0(x) = \sum_{k=-\infty}^{\infty} p_k(0)w^k(x)$, $p_k(0) = (w_*^k, T_0)$,
$V_0(x) = \sum_{k=-\infty}^{\infty} \dot{p}_k(0)w^k(x)$, $\dot{p}_k(0) = (w_*^k, V_0)$, $T(x,t) = \sum_{k=-\infty}^{\infty} p_k(t)w^k(x)$,
where $p_k(t)$ is the solution of 4.9) by $k \neq 0$.
For $k = 0$ we obtain $p_k(t) = \exp(-0.5t/\tau)[t(\dot{p}_k(0) + 0.5p_k(0)/\tau) + p_k(0)]$.

The solution we can also obtained in real form:
$T(x,t) = \sum_{k=1}^{\infty}(b_k(t)\cos\frac{2\pi kx}{L} + c_k(t)\sin\frac{2\pi kx}{L}) + \frac{b_0(t)}{2}$,
$T_0(x) = \sum_{k=1}^{\infty}(b_k(0)\cos\frac{2\pi kx}{L} + c_k(0)\sin\frac{2\pi kx}{L}) + \frac{b_0(0)}{2}$,
$b_k(0) = \frac{2}{L}\int_0^L T_0(\xi)\cos\frac{2\pi k\xi}{L}d\xi, c_k(0) = \frac{2}{L}\int_0^L T_0(\xi)\sin\frac{2\pi k\xi}{L}d\xi$,
$V_0(x) = \sum_{k=1}^{\infty}(\dot{b}_k(0)\cos\frac{2\pi kx}{L} + \dot{c}_k(0)\sin\frac{2\pi kx}{L}) + \frac{\dot{b}_0(0)}{2}$,
$\dot{b}_k(0) = \frac{2}{L}\int_0^L V_0(\xi)\cos\frac{2\pi k\xi}{L}d\xi, \dot{c}_k(0) = \frac{2}{L}\int_0^L V_0(\xi)\sin\frac{2\pi k\xi}{L}d\xi$,
where $b_k(t), c_k(t)$ are the corresponding solutions of (4.9) with initial conditions $b_k(0), \dot{b}_k(0)$.

We can consider the **analytical solutions** of (4.6) using the spectral representation of matrix $A = WDW_*$. From transformation $P = W_*U(U = WP)$ follows the seperate system of ODEs (4.8), where the column-vectors are of N order.
The solution of the system (4.8) is in the form (4.9), where
$k = \overline{1, N}, d_k = \mu_k$. For $k = N$ the solution is
$p_N(t) = \exp(-0.5t/\tau)[t(\dot{p}_N(0) + 0.5p_N(0)/\tau) + p_N(0)]$.
If $d_k = \lambda_k$ then we can obtain the solution of FDSES in following way:
1)$d_k = \lambda_k$ for $k = \overline{1, N_2}$, where $N_2 = N/2$.
2)$d_k = \lambda_{N-k}$ for $k = \overline{N_2, N-1}, d_N = 0$.

We can obtain the solution of the discrete problem in real form $u_j(t) = \sum_{k=1}^{N_2}(b_k(t)\cos\frac{2\pi kj}{N} + c_k(t)\sin\frac{2\pi kj}{N}) + \frac{b_0}{2}$,
where $b_k(t), c_k(t)$ are the corresponding solutions of (4.9) by
$b_k(0) = \frac{2}{N}\sum_{j=1}^{N} T_0(x_j)\cos\frac{2\pi kj}{N}, \dot{b}_k(0) = \frac{2}{N}\sum_{j=1}^{N} V_0(x_j)\cos\frac{2\pi kj}{N}$,
$c_k(0) = \frac{2}{N}\sum_{j=1}^{N} T_0(x_j)\sin\frac{2\pi kj}{N}, \dot{c}_k(0) = \frac{2}{N}\sum_{j=1}^{N} V_0(x_j)\sin\frac{2\pi kj}{N}$.

The numerical experiment with $L = 1, t_f = 0.4, \tau = 0.5; 0.1$ and $, T_0 = \sin^{100}(\pi x), V_0 = 0$ is produced by MATLAB 7.4 solver ”ode15s” .

We have following MATLAB m.file **hipSiltPer :**

```
%ODEs tau*U_tt+U_t=AU with periodic cond.
%t=Tb,u_t=0=sin(p x)^100
function hipSiltPer(N)
N1=N+1; Tb=0.4;L=1;x=linspace(0,L,N1)';h=L/N;
NT=[1:N]/L;N2=N-1;NN=2*N;NH=N/2;tau=0.1;
lk0=(2*pi/L*(1:N)/L').^2; % dif. eig.
lk2=4/h^2*(sin(pi*h*NT)).^2; %FDS with O(h^2)
lk4=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk6=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk8=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
W=exp(2*pi*h*i*[1:N]'*[1:N]/L);x=x(2:N1);y1=sin(pi*x).^100;
y2=zeros(N,1);Z1=zeros(N,N);E1=eye(N,N);
lk=zeros(N,1);
lk(1:NH)=lk0(1:NH);
lk(NH:N2)=lk0(NH:-1:1);%FDSES
A2=-h*W*diag(lk)*conj(W);
AT=zeros(NN,NN);y11=zeros(N2,1);
y0=[y1;y2];AT=[Z1,E1;A2/tau,-E1/tau];
options=odeset('RelTol', 1.0e-7);
[T,Y]=ode15s(@SIST,[0,Tb],y0,options,AT);
im=max(imag(Y(end,:)));
figure,plot(x,real(Y(end,1:N)'),'ko'), MA=max(abs(Y(end,1:N)));
grid on
title(sprintf('Sol. by Tb, maxim=%8.6f,. . .
time = %8.6f Max=%9.7f ',im,T(end),MA))
xlabel('\itx'), ylabel('\itu')
figure, plot(T(:),max(real(Y(:,1:N)')))
%hold on, plot(T(:),max(real(Y(:,N+1:NN)')))
grid on
title(sprintf('Time,N=%3.0f, time = %8.6f ',N,T(end)))
xlabel('\itt'), ylabel('\itu')
K=length(T);
X1=ones(K,1)*x';Y1=T*ones(1,N);% per.
figure, surfc(X1,Y1,real(Y(:,1:N)))% per.
```

```
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('Surface,imag=%8.6f,Laika sl.sk.=%3.06f, . . .
Max=%8.6f',im,K,MA))
function F=SIST(t,y,AT)
F=AT*y;
```

We have following maximal values of solution $\max|U(t_f)|$ by $N = 80$ depending on the order of approximation:
1) $\tau = 0.5 - 0.3586 O(h^2), 0.3692 O(h^4), 0.3689 O(h^6), 0.3688 O(h^8)$, $0.3688 FDSES$ ($FDSES: 0.3528 N = 40, 0.3521 N = 20$, $FDSO(h^2): 0.3358$ N=40, 0.4661 N=20);
2) $\tau = 0.1 - 0.1319 O(h^2), 0.1354 O(h^4), 0.1357 O(h^6), 0.1357 O(h^8)$, $0.1357 FDSES$ ($FDSES: 0.1314 N = 40, 0.1322 N = 20$, $FDSO(h^2): 0.1199$ N=40, 0.1080 N=20).

In the Figs. 4.23-4.28 we can see the FDSES solutions by $N = 80$.

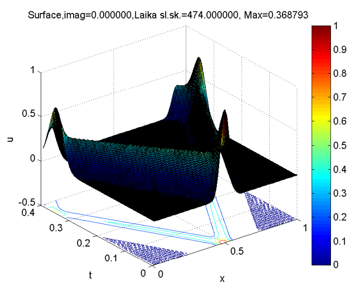


Fig. 4.23 FDSES solution- surface by $\tau = 0.5, N = 80$

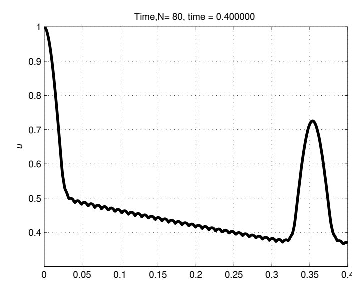


Fig. 4.24 FDSES solution depending on t by $\tau = 0.5, N = 80$

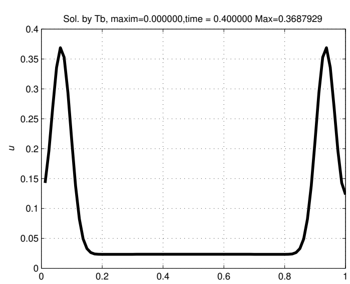


Fig. 4.25 FDSES solution by $\tau = 0.5, t_f = 0.4, N = 80$

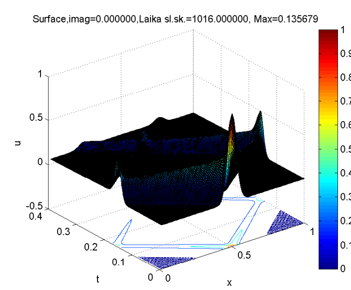


Fig. 4.26 FDSES solution- surface by $\tau = 0.1, N = 80$

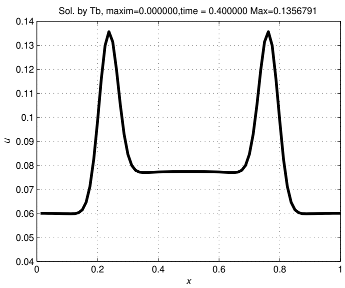


Fig. 4.27 FDSES solution by $\tau = 0.1, t_f = 0.4, N = 80$

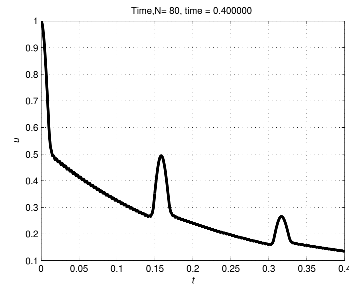


Fig. 4.28 FDSES solution depending on t by $\tau = 0.1, N = 80$

In Figs. 4.29, 4.30 we can see the FDSES and FDS $O(h^2)$ solutions by $t_f = 0.4, \tau = 0.5, N = 20$.

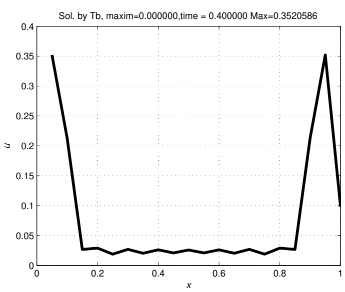


Fig. 4.29 FDSES solution by $\tau = 0.5, t_f = 0.4, N = 20$

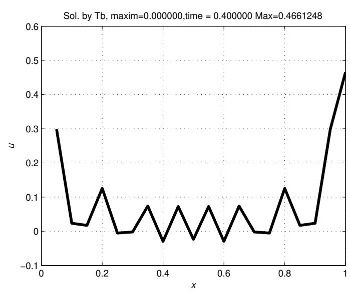


Fig. 4.30 FDS $O(h^2)$ solution by $\tau = 0.5, t_f = 0.4, N = 20$

We can see, that FDS methods give the solutions with oscillations. FDSES method is without oscilations and the solution is positive even if $N = 20$.

By replacing the Matlab operator
$[T,Y] = ode15s(@SIST,[0,Tb],y0,options,AT)$; with the operators
$[T1,Y1] = ode15s(@SIST,[0,Tb],y0,options,AT)$;
$[T,Y] = ode15s(@SIST,[Tb,0],Y1(end,:),options,AT)$;
we can solve the time inverse (retrospective) problem.
An example for $\tau = 0.5, t_f = 0.4, N = 80$ the maximal values of FDSES solution $\max|U(t_f)|$ for $t = 0$ is 0.999989 (exact value is $\sin^{100}(0.5\pi) = 1$).

4.7 Conclusions

We consider the homogenous 1-D hyperbolic heat conduction problem in plate and in the sphere with holes $(0 < r_0 < r < R)$ by radial symmetry. Using the tranformation the problem in the sphere with holes is reduced to the following hyperbolic heat conduction problem in the plate.
Using the finite differences of second order approximation for partial derivatives of second order respect to x we obtain the initial value problem for system of ordinary differential equations (ODEs) of second order in the matrix form with the standard 3-diagonal matrix A. For the difference sheme with **exact spectrum** (FDSES) the matrix A is represented in the form $A = WDW$, where $W = W^{-1}$ is the symmetrical orthogonal matrix with elements $\sqrt{\frac{2}{N}} \sin \frac{\pi i j}{N}$, $i, j = \overline{1, N-1}$. We are considered the **analytical solutions** using the spectral representation of matrix A in the form $A = WDW$.
For the approximations the derivatives with **matrix of derivatives** we can consider also nonuniform grid with the grid points of the roots of the Chebyshev . The numerical experiment is produced by MATLAB solver ”ode15s” .

Chapter 5
3-D HHC equation: A. Buikis, H. Kalis, I. Kangro, 2015 [81]

In this chapter we consider the following 3-D hyperbolic heat conduction problem:

$$\begin{cases} \tau_* \frac{\partial^2 T(x,y,z,t)}{\partial t^2} + c\rho \frac{\partial T(x,y,z,t)}{\partial t} = \frac{\partial}{\partial x}(k_x \frac{\partial T(x,y,z,t)}{\partial x}) + \\ \frac{\partial}{\partial y}(k_y \frac{\partial T(x,y,z,t)}{\partial y}) + \frac{\partial}{\partial z}(k_z \frac{\partial T(x,y,z,t)}{\partial z}) - \gamma_0 T(x,y,z,t) + f(x,y,z,t), \\ x \in (0,L_x), y \in (0,L_y), z \in (0,L_z), t \in (0,t_f), \\ \frac{\partial T(0,y,z,t)}{\partial x} = \frac{\partial T(x,0,z,t)}{\partial y} = \frac{\partial T(x,y,0,t)}{\partial z} = 0, \\ k_x \frac{\partial T(L_x,y,z,t)}{\partial x} + \alpha_x T(L_x,y,z,t) = 0, \\ k_y \frac{\partial T(x,L_y,z,t)}{\partial y} + \alpha_y T(x,L_y,z,t) = 0, \\ k_z \frac{\partial T(x,y,L_z,t)}{\partial z} + \alpha_z T(x,y,L_z,t) = 0, \\ T(x,y,z,0) = T_0(x,y,z), \frac{\partial T(x,y,z,0)}{\partial t} = V_0(x,y,z), \end{cases} \tag{5.1}$$

where ρ, c is the density and the heat capacity of the steel, k_x, k_y, k_z are the constant heat conductivity coefficients, $\alpha_x, \alpha_y, \alpha_z$ are the constant heat transfer coefficients, t_f is the final time, τ_* is the relaxation time, $T_0(x,y,z)$ is the given initial temperature, $\gamma_0 > 0$ is the given constant . For the inverse problem the function $V_0(x,y,z)$ is unknown and then we can used the aditional condition $T(x,y,z,t_f) = T_f(x,y,z)$, where T_f is given final temperature.

5.1 CAM with parabolic integral splines for the Solutions 3-D problem

Using **conservation averaged method (CAM) respect to** z **with quadratic polynomial** [9]
$T_z(x,y,t) = \frac{1}{L_z}\int_0^{L_z} T(x,y,z,t)dz$,
$T(x,y,z,t) = T_z(x,y,t) + m_z(x,y,t)(z-0.5L_z)+$
$e_z(x,y,t)((z-0.5L_z)^2 - L_z^2/12)$, with the unknown functions m_z, e_z, we can determined this functions from boundary conditions (5.1) in following form:
$m_z = e_z L_z = -\frac{0.5\alpha_z}{k_z+L_z\alpha_z/3}T_z(x,y,t)$. Then

$$T(x,y,z,t) = 0.5T_z(x,y,t)\frac{2k_z + L_z\alpha_z - \alpha_z z^2/L_z}{k_z + L_z\alpha_z/3}$$

and the initial-boundary value problem (5.1) is in following form

$$\begin{cases} \tau_* \frac{\partial^2 T_z(x,y,t)}{\partial t^2} + c\rho\frac{\partial T_z(x,y,t)}{\partial t} = \frac{\partial}{\partial x}(k_x\frac{\partial T_z(x,y,t)}{\partial x})+ \\ \frac{\partial}{\partial y}(k_y\frac{\partial T_z(x,y,t)}{\partial y}) - (\gamma_z+\gamma_0)T_z(x,y,t) + f_z(x,y,t), \\ x\in(0,L_x), y\in(0,L_y), t\in(0,t_f), \\ \frac{\partial T_z(0,y,t)}{\partial x} = \frac{\partial T_z(x,0,t)}{\partial y} = 0, \\ k_x\frac{\partial T_z(L_x,y,t)}{\partial x} + \alpha_x T_z(L_x,y,t) = 0, \\ k_y\frac{\partial T_z(x,L_y,t)}{\partial y} + \alpha_y T_z(x,L_y,t) = 0, \\ T_z(x,y,0) = T_{z,0}(x,y), \frac{\partial T_z(x,y,0)}{\partial t} = V_{z,0}(x,y), \end{cases} \tag{5.2}$$

where $\gamma_z = \frac{\alpha_z k_z}{L_z k_z + \alpha_z L_z^2/3}$,
$f_z(x,y,t) = \frac{1}{L_z}\int_0^{L_z} f(x,y,z,t)dz$, $T_{z,0}(x,y) = \frac{1}{L_z}\int_0^{L_z} T_0(x,y,z)dz$,
$V_{z,0}(x,y) = \frac{1}{L_z}\int_0^{L_z} V_0(x,y,z)dz$.

Using **averaged method respect to** y
$T_{z,y}(x,t) = \frac{1}{L_y}\int_0^{L_y} T_z(x,y,t)dy$,
$T_z(x,y,t) = T_{z,y}(x,t) + m_y(x,t)(y-0.5L_y) + e_y(x,t)((y-0.5L_y)^2 - L_y^2/12)$,
with the unknown functions m_y, e_y, we can determined this functions from boundary conditions (5.2) in following form:
$m_y = e_y L_y = -\frac{0.5\alpha_y}{k_y+L_y\alpha_y/3}T_{z,y}(x,t)$. Then

$$T_z(x,y,t) = 0.5T_{z,y}(x,t)\frac{2k_y + L_y\alpha_y - \alpha_y y^2/L_y}{k_y + L_y\alpha_y/3}$$

and the initial-boundary value problem (5.2) is in following form

$$\begin{cases} \tau_* \frac{\partial^2 T_{z,y}(x,t)}{\partial t^2} + c\rho \frac{\partial T_{z,y}(x,t)}{\partial t} = \frac{\partial}{\partial x}(k_x \frac{\partial T_{z,y}(x,t)}{\partial x}) - \\ (\gamma_z + \gamma_y + \gamma_0)T_{z,y}(x,t) + f_{z,y}(x,t), x \in (0,L_x), t \in (0,t_f), \\ \frac{\partial T_{z,y}(0,t)}{\partial x} = 0, k_x \frac{\partial T_{z,y}(L_x,t)}{\partial x} + \alpha_x T_{z,y}(L_x,t) = 0, \\ T_{z,y}(x,0) = T_{z,y,0}(x), \frac{\partial T_{z,y}(x,0)}{\partial t} = V_{z,y,0}(x), \end{cases} \tag{5.3}$$

where $\gamma_y = \frac{\alpha_y k_y}{L_y k_y + \alpha_y L_y^2/3}, f_{z,y}(x,y,t) = \frac{1}{L_y}\int_0^{L_y} f_z(x,y,t)dy,$
$T_{z,y,0}(x) = \frac{1}{L_y}\int_0^{L_y} T_{z,0}(x,y)dy$, $V_{z,y,0}(x) = \frac{1}{L_y}\int_0^{L_y} V_{0,z}(x,y)dy.$

It is possible make the averaging also **respect to** x
$T_{z,y,x}(t) = \frac{1}{L_x}\int_0^{L_x} T_{z,y}(x,t)dx,$
$T_{z,y}(x,t) = T_{z,y,x}(t) + m_x(t)(x - 0.5L_x) + e_x(t)((x - 0.5L_x)^2 - L_x^2/12),$
with the unknown functions m_x, e_x. We can determine this functions from BCs (5.3) in following form:
$m_x = e_x L_x = -\frac{0.5\alpha_x}{k_x + L_x\alpha_x/3}T_{z,y,x}(t)$. Then

$$T_{z,y}(x,t) = 0.5T_{z,y,x}(t)\frac{2k_x + L_x\alpha_x - \alpha_x x^2/L_x}{k_x + L_x\alpha_x/3}$$

and the initial-boundary value problem (5.3) is in following form

$$\begin{cases} \tau_* \frac{\partial^2 T_{z,y,x}(t)}{\partial t^2} + c\rho \frac{\partial T_{z,y,x}(t)}{\partial t} + \gamma T_{z,y,x}(t) = f_{z,y,x}(t), t \in (0,t_f), \\ T_{z,y,x}(0) = T_{z,y,x,0}, \frac{\partial T_{z,y,x}(0)}{\partial t} = V_{z,y,x,0}, \end{cases} \tag{5.4}$$

where $\gamma = \gamma_z + \gamma_y + \gamma_x + \gamma_0, \gamma_x = \frac{\alpha_x k_x}{L_x k_x + \alpha_x L_x^2/3}, f_{z,y,x}(t) = \frac{1}{L_x}\int_0^{L_x} f_{z,y}(x,t)dx,$
$T_{z,y,x,0} = \frac{1}{L_x}\int_0^{L_x} T_{z,y,0}(x)dx$, $V_{z,y,x,0} = \frac{1}{L_x}\int_0^{L_x} V_{0,z,y}(x)dx.$

Therefore we have from (5.4) the initial problem for ODEs of the second order. The solution of this problem is

$$\begin{cases} T_{z,y,x}(t) = exp(-0.5t/\tau)(C\sinh(\kappa t) + B\cosh(\kappa t)) + \\ \frac{1}{\kappa\tau}\int_0^t exp(-0.5\frac{1}{\tau}(t-\xi))\sinh(\kappa(t-\xi))G(\xi)d\xi, \end{cases} \tag{5.5}$$

where $\tau = \tau_*/(c\rho), \kappa = \sqrt{0.25/\tau^2 - \gamma_*/\tau}$, $B = T_{z,y,x,0},$
$C = \frac{1}{\kappa}(V_{z,y,x,0} + \frac{1}{2\tau}T_{z,y,x,0}),$

$\gamma_* = \gamma/(c\rho), \gamma = \gamma_z + \gamma_y + \gamma_x + \gamma_0, G(t) = f_{z,y,x}(t)/(c\rho)$. If $4\gamma_*\tau > 1$, then the hyperbolic functions to need replaced with the trigonometrical and the parameter κ with $\sqrt{\gamma_*/\tau - 0.25/\tau^2}$.

Using **averaged value with constant function**[9] we have the unknown functions $m_k = 0, e_k = 0, k = (z,y,x)$ and the equations (5.2-5.4) remain one's own form, where $\gamma_k = \frac{\alpha_k}{L_k}, k = (z,y,x)$.

5.2 Mathematical model of the modified 1-D problem

Similarly (5.3) we consider the following modified 1-D hyperbolic heat conduction problem in the plate:

$$\begin{cases} \tau\frac{\partial^2 T(x,t)}{\partial t^2} + \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}(\bar{k}\frac{\partial T(x,t)}{\partial x}) - \gamma_* T(x,t) + f(x,t), \\ x \in (0,L), t \in (0,t_f), \\ \frac{\partial T(0,t)}{\partial x} - \sigma_1 T(0,t) = 0, \frac{\partial T(L,t)}{\partial x} + \sigma_2 T(L,t) = 0, t \in (0,t_f), \\ T(x,0) = T_0(x), \frac{\partial T(x,0)}{\partial t} = V_0(x), x \in (0,L), \end{cases} \tag{5.6}$$

where $L = L_x, \bar{k} = k_x/(c\rho)$, τ is the relaxation time $(\tau < 1)$, $T_0(x)$ is the given initial temperature, $\sigma_1, \sigma_2 = \alpha_x/k_x, \gamma_* = \gamma/(c\rho), \gamma = \gamma_z + \gamma_y + \gamma_0$ are the constant coefficients (for boundary condition with symmetry $\sigma_1 = 0$), $f(x,t)$ is the source function.
For the inverse problem the function $V_0(x)$ is unknown and then we can used the aditional condition $T(x,t_f) = T_f(x)$, where T_f is given final temperature.

The following 1-D modified hyperbolic heat conduction problem in the sphere with holes $(0 < r_0 < r < R)$ by radial symmetry is:

$$\begin{cases} \tau\frac{\partial^2 T(r,t)}{\partial t^2} + \frac{\partial T(r,t)}{\partial t} = \frac{\bar{k}}{r}\frac{\partial^2 (rT(r,t))}{\partial r^2} - \gamma_* T(r,t) + f(r,t), \\ r \in (r_0,R), t \in (0,t_f), \\ \frac{\partial T(r_0,t)}{\partial r} = 0, \frac{\partial T(R,t)}{\partial r} + \alpha_2 T(R,t) = 0, t \in (0,t_f), \\ T(r,0) = T_0(r), \frac{\partial T(r,0)}{\partial t} = V_0(r), r \in (r_0,R), \end{cases} \tag{5.7}$$

where r is the polar coordinate, R, r_0 are the radius of the sphere and hole, $\alpha_2 = \alpha_r/k_r, \bar{k} = k_r/(c\rho)$, k_r, α_r are the corresponding heat conducting parameters.
For the inverse problem the function $V_0(r)$ is unknown and then we

can used the aditional condition $T(r,t_f) = T_f(r)$, where T_f is given temperature.
Using the tranformation $V = Tr, x = r - r_0$ we can the problem (5.7) reduced to the hyperbolic heat conduction problem in the plate:

$$\begin{cases} \tau \frac{\partial^2 V(x,t)}{\partial t^2} + \frac{\partial V(x,t)}{\partial t} = \frac{\partial}{\partial x}(\bar{k}\frac{\partial V(x,t)}{\partial x}) - \gamma_* V(x,t) + f(x+r_0,t), \\ x \in (0,L), t \in (0,t_f), \\ \frac{\partial V(0,t)}{\partial x} - \sigma_1 V(0,t) = 0, \frac{\partial V(L,t)}{\partial x} + \sigma_2 V(L,t) = 0, t \in (0,t_f), \\ V(x,0) = T_0(x+r_0)(x+r_0), \frac{\partial V(x,0)}{\partial t} = V_0(x+r_0)(x+r_0), x \in (0,L) \end{cases} \tag{5.8}$$

where $\sigma_1 = \frac{1}{r_0}, \sigma_2 = \alpha_2 - \frac{1}{R} \geq 0, L = R - r_0, T(r,t) = \frac{V(x+r_0,t)}{x+r_0}$.

In the case of full sphere $(r_0 \to 0)$ the first boundary condition is $V(0,t) = 0$ or $\sigma_1 = \infty$ and $T(0,t) = \frac{\partial V(0,t)}{\partial x}$.

We consider uniform grid in the space $x_j = jh, j = \overline{0,N}, Nh = L$. Using the finite differences of second order approximation for partial derivatives of second order respect to x we obtain from (5.6) the initial value problem for system of ODEs of second order in the following matrix form

$$\begin{cases} \tau \ddot{U}(t) + \dot{U}(t) + \bar{k}AU(t) + \gamma_* U(t) = F(t), \\ U(0) = U_0, \dot{U}(0) = V_0, \end{cases} \tag{5.9}$$

where A is the 3-diagonal matrix of $N+1$ order in the form

$$A = \frac{1}{h^2} \begin{pmatrix} 2+h\sigma_1 & -2 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ .. & .. & .. & \dots & .. & .. & \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -2 & 2+h\sigma_2 \end{pmatrix}$$

$U(t), \dot{U}(t), \ddot{U}(t), V_0, U_0, F(t)$ are the column-vectors of $N+1$ order with elements
$u_j(t) \approx T(x_j,t)),\ \dot{u}_j(t) \approx \frac{\partial T(x_j,t)}{\partial t},\ \ddot{u}_j(t) \approx \frac{\partial^2 T(x_j,t)}{\partial t^2}, v_j(0) = V_0(x_j)$ $u_j(0) = U_0(x_j), f_j(t) = f(x_j,t), j = \overline{0,N}$.

The corresponding spectral problems are solved in chapter 2.

5.3 The solution of the discrete problem

We can consider the **analytical solutions** of (5.9) using the spectral representation of matrix $A = WDW^T$. From transformation $V = W^T U (U = WV)$ follows the seperate system of ODEs

$$\begin{cases} \tau \ddot{V}(t) + \dot{V}(t) + \bar{k} D V(t) + \gamma_* V(t) = G(t), \\ V(0) = W^T U_0, \dot{V}(0) = W^T V_0, \end{cases} \tag{5.10}$$

where
$v(t), \dot{v}(t), \ddot{v}(t), \dot{v}(0), v(0), G(t) = W^T F(t)$ are the column-vectors of M order with elements $v_k(t), \dot{v}_k(t), \ddot{v}_k(t), \dot{v}_k(0), v_k(0), g_k(t) k = \overline{1,M}, M = N+1$.
The solution of this system is the function

$$\begin{cases} v_k(t) = exp(-0.5t/\tau)(\frac{\sinh(\kappa_k t)}{\kappa_k}(\dot{v}_k(0) + 0.5 v_k(0)/\tau) + \\ \cosh(\kappa_k t) v_k(0)) + \frac{1}{\tau \kappa_k} \int_0^t \exp(-\frac{t-\xi}{2\tau}) \sinh(\kappa(t-\xi)) g_k(\xi) d\xi, \end{cases} \tag{5.11}$$

where $d_k = \delta_k + \gamma_*/\bar{k}, \delta_k = \mu_k, \kappa_k = \sqrt{0.25/\tau^2 - \bar{k} d_k/\tau}$. If $4\bar{k} d_k \tau > 1$, then the hyperbolic functions are replaced with the trigonometrical and the parameter κ_k with $\sqrt{\bar{k} d_k/\tau - 0.25/\tau^2}$.
If $\kappa_k = 0$, then $v_k(t) = \exp(-0.5t/\tau)[t(\dot{v}_k(0) + 0.5 v_k(0)/\tau) + v_k(0)] + \frac{1}{\tau} \int_0^t \exp(-\frac{t-\xi}{2\tau})(t-\xi) g_k(\xi) d\xi$.

Note: in (5.10) the first $u_1(0), f_1(t), v_1(0)$ and last $u_M(0), f_M(t), v_M(0)$ components of vectors $U_0, F(t), V_0$ are divided with $\sqrt{2}$, but the components $u_1(t), u_M(t)$ of the solution vector $U(t)$ need to multiply with $\sqrt{2}$.

5.4 The methods for solving the inverse problem

For the inverse problem the vector V_0 in (5.9) is unknown and we have additional condition $U(t_f) = u_f$, where u_f is the vectors-column with elements $u_{f,k} = T_f(x_k), k = \overline{1,M}$.
The **analytical solutions** of this problem can be obtain from (5.11) replaced the second initial condition $\dot{V}(0) = W^T V_0$ with $V(t_f) = W^T u_f$. Then the solution is following

$$\begin{cases} v_k(t) = exp(-0.5t/\tau)[\frac{\sinh(\kappa_k t)}{\sinh(\kappa_k t_f)}(\exp(0.5t_f/\tau)v_k(t_f)- \\ v_k(0)\cosh(\kappa_k t_f) - \frac{1}{\tau\kappa_k}\int_0^{t_f}\exp(\frac{\xi}{2\tau})\sinh(\kappa(t_f-\xi))g_k(\xi)d\xi)+ \\ \frac{1}{\tau\kappa_k}\int_0^t \exp(\frac{\xi}{2\tau})\sinh(\kappa(t-\xi))g_k(\xi)d\xi + cosh(\kappa_k t)v_k(0)], \end{cases} \tag{5.12}$$

where $v_k(t_f)$ are the components of vector $V(t_f)$ (the **Note** for the vectors V_0, u_f, F is valid).
From (5.11) follows that the second conditions is in following form
$\dot{v}_k(0) = \frac{\kappa_k}{\sinh(\kappa_k t_f)}[exp(0.5t_f/\tau)v_k(t_f) - v_k(0)\cosh(\kappa_k t_f)-$
$\frac{1}{\tau\kappa_k}\int_0^{t_f}\exp(\frac{\xi}{2\tau})\sinh(\kappa(t_f-\xi))g_k(\xi)d\xi] - 0.5v_k(0)/\tau,$
$V_0 = W\dot{V}(0)$. If $v_k(t_f) = g_k(t) = 0$, then
$\dot{v}_k(0) = -\frac{\kappa_k}{\coth(\kappa_k t_f)}v_k(0) - 0.5v_k(0)/\tau,$

5.5 Some examples and numerical results

The numerical results for the problem (5.6, 5.7) ($\sigma_1 = 0$) are obtained for intensive Carbon steel quenching models with $\bar{k} = 0.000017713$, $\gamma = 0 : 10 : 10^7$ ($\bar{k} = \frac{k_x}{c\rho}, k_x = 60.5\frac{W}{mC}$ - the heat conductivity, $\rho = 7870\frac{kg}{mC}$ - the density of the steeL, $c = 434\frac{J}{kgC}$ - the heat capacity),$\alpha_x =$ 30 - the heat transfer coefficient) and $L = 1, \tau = 0.1; 0.5, T_0(x) = 600, t_f = 1; 3; 5, T_f(x) = 0, N = 20$. We have following results for V_0 (see the Table 5.1).
In the last row (*) of this table the obtaining profile of the temperature is non monotonic,but for $\tau = 0.5, t_f = 5$ we have the oscilations and the negative temperature (the parameterγ is too big) (see the Figs. 5.1, 5.2.)

For $\tau = 0.5, \gamma = 10$ we have $V_0 = -1202,98$ (Figs. 5.3, 5.4).
For $\tau = 0.1$ we have correspondly $V_0 = -6000,26$ (Figs. 5.5, 5.6).

The numerical results for the problem (5.5) are obtained by the above-mentioned parameters with $R = 8, r_0 = 1, T_0(r) = 600; 600\frac{r-r_0}{R-r_0}$, $t_f = 0.5; 5, T_f(r) = 0, N = 20, \gamma = 10$.
In Figs. 5.7, 5.8 are represented the results of the calculation by $\tau = 0.5, \tau = 0.1, T_0(r) = 600, t_f = 5$.

In the Figs. 5.9, 5.10 are presented the results of the calculation by $\tau = 0.5, T_0(r) = 600\frac{r-r_0}{R-r_0}, r_0 = 1, t_f = 5$, where the averaged value $V_0 = -600,1$.

Table 5.1 The values of V_0 depending on γ, τ

τ	γ	V_0	τ	γ	V_0
0.5	0	-1202.98	0.1	0	-6000.27
	10^2	-1202.96		10^2	-6000.25
	10^3	-1199.88		10^3	-6000.10
	10^4	-1198.30		10^4	-5998.52
	10^5	-1182.24		10^5	-5982.67
	10^6	-987.51		10^6	-5819.33
	$2*10^6$	-684.97		10^7	-2326.18
*	3.10^6	-798.29	*	11.10^6	272.90

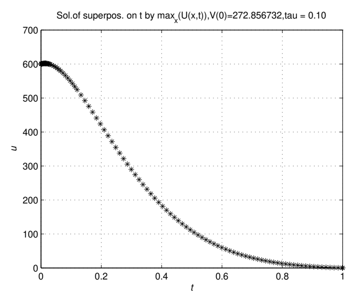


Fig. 5.1 Solution $max(U(x,t))$ depending on t by $\tau = 0.1, t_f = 1, \gamma = 11.10^6$

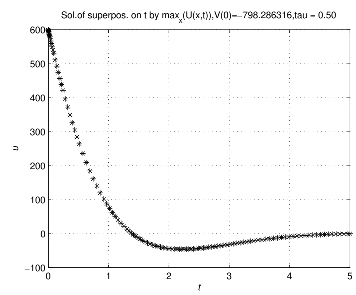


Fig. 5.2 Solution $max(U(x,t))$ depending on t by $\tau = 0.5, t_f = 5, \gamma = 3.10^6$

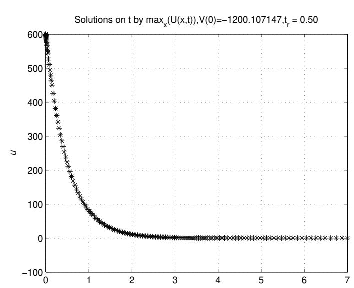


Fig. 5.3 Solution $max(U(x,t))$ depending on t by $\tau = 0.5, t_f = 5, \gamma = 10$

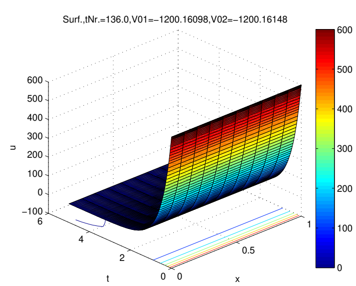


Fig. 5.4 Solution $U(x,t)$ by $\tau = 0.5, t_f = 5, \gamma = 10$

In the Figs. 5.11, 5.12 are presented the results of the calculation by $\tau = 0.1, T_0(r) = 600\frac{r-r_0}{R-r_0}, r_0 = 1, t_f = 0.5$, where the averaging value $V_0 = -3000, 5$.

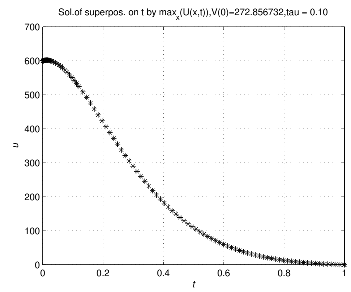


Fig. 5.5 Solution $max(U(x,t))$ depending on t by $\tau = 0.1, \gamma = 10$

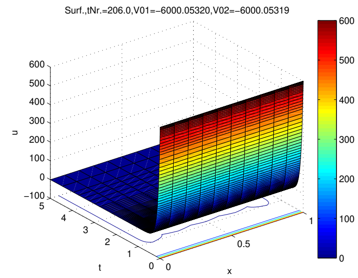


Fig. 5.6 Solution $U(x,t)$ by $\tau = 0.1, \gamma = 10$

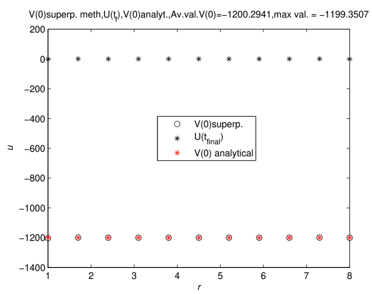


Fig. 5.7 Functions V_0, U_f depending on r by $\tau = 0.5, T_0(r) = 600$

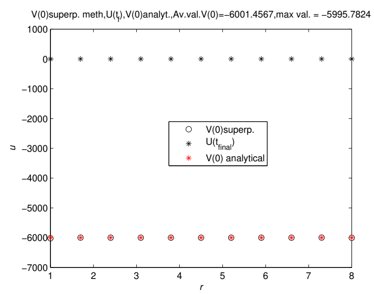


Fig. 5.8 Functions V_0, U_f depending on r by $\tau = 0.1, T_0(r) = 600$

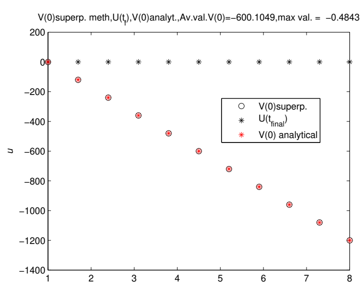


Fig. 5.9 Functions V_0, U_f depending on r by $\tau = 0.5, T_0(r) = 600\frac{r-r_0}{R-r_0}$

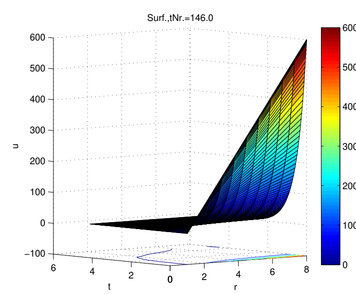


Fig. 5.10 Solution depending on t, r by $\tau = 0.5, T_0(r) = 600\frac{r-r_0}{R-r_0}$

5.6 MATLAB programm

The numerical results are obtaining with MATLAB using the following m. file **”hipSiltA(20)”** and **”ipas3v”**(see chapter 1):

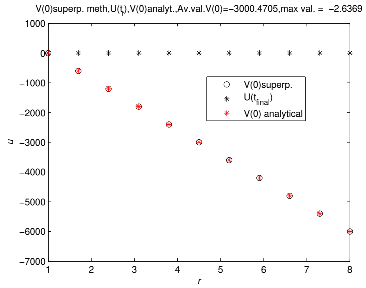


Fig. 5.11 Functions V_0, U_f depending on r by $\tau = 0.1, T_0(r) = 600\frac{r-r_0}{R-r_0}$

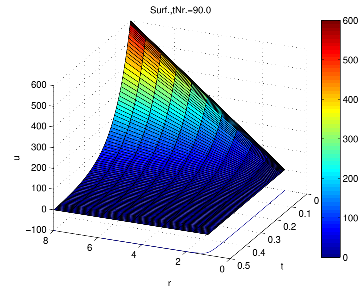


Fig. 5.12 Solution depending t, r by $\tau = 0.1, T_0(r) = 600\frac{r-r_0}{R-r_0}$

```
%tau*U_{tt}+U_t+a2 AU=0  MATLAB solvers and analytical
%t=Tb,U_{t=0}=600,U_{t(t=0)} =v_0=?,
% U_t=Tb =0, U_{x(x=0)}=0, U_{x(x=L)}+ alfa U_{x=L}=0
function hipSiltA(N)
N1=N+1; Tb=3;L=1;x=linspace(0,L,N1)';h=L/N;
N2=N-1;NT=[1:N1];NN=2*N1; tau=0.5;
k1=60.5;c1=434;r1=7870;a2=k1/(c1*r1)
%Carbon steel
a22=1/(c1*r1);alfax=30; alfa=alfax/c1;gamma=3000000;
g1=a22*(alfax/L+gamma);\% aver. with  constant
MK=50;b1=0.5/tau*sqrt(1-4*tau*g1);u0=600;
v0=-u0*(b1*coth(Tb*b1)+0.5/tau);t=linspace(0,Tb,MK);
u=exp(-0.5*t/tau).*(u0*cosh(t*b1)+(v0/b1+. . .
0.5*u0/(tau*b1))*sinh(t*b1));
figure,plot(t,u,'k*')
grid on
title(sprintf('Aver.const.u(t) on t by t=%4.2f,tau =%4.2f,. . .
v0=%8.4f ',Tb,tau,v0))
xlabel('t'), ylabel('u')
sig1=0;sig2=alfa;[lk0,lk,W]=ipas3v(N,sig1,sig2,L);
g1=a22*(alfax/(L+$L^2$*alfa/3)+gamma); %aver with kvadr. polin.
b1=0.5/tau*sqrt(1-4*tau*g1);u0=600;
v0=-u0*(b1*coth(Tb*b1)+0.5/tau);
u=exp(-0.5*t/tau).*(u0*cosh(t*b1)+(v0/b1+. . .
0.5*u0/(tau*b1))*sinh(t*b1));
Vx=v0/(2+2/3*alfa*L)*(2+L*alfa-alfa*x.^2/L);
figure,plot(x,Vx,'k*')
title(sprintf('Aver.pol.V0(x)on x by t=%4.2f,tau =%4.2f,. . .
v0=%8.4f ',Tb,tau,v0))
figure,plot(t,u,'k*')
title(sprintf('Aver.pol.u(t) on t by t=%4.2f,tau =%4.2f,. . .
v0=%8.4f ',Tb,tau,v0))
xlabel('t'), ylabel('u')
A2= a2*W*diag(lk)*W';
y1=600*ones(N1,1);y1(1)=y1(1)/sqrt(2);y1(N1)=y1(N1)/sqrt(2);
P=W'*y1;P1=zeros(MK,N1);P0=zeros(N1,1);
for k=1:N1
```

```
    b=sqrt(0.25/tau^2 -a22*(lk(k)/tau-gamma/tau));
P1(:,k)=P(k)*exp(-0.5*t'/tau).*...
(cosh(b*t')-sinh(b*t')*coth(b*Tb));
    P0(k)=-P(k)*(0.5/tau +b*coth(b*Tb));
end
P2=(W*P1')';V0=W*P0;V0(1)=V0(1)*sqrt(2);
V0(N1)=V0(N1)*sqrt(2);
P2(:,1)=P2(:,1)*sqrt(2);P2(:,N1)=P2(:,N1)*sqrt(2);
u2=P2(MK,1:N1);figure,plot(x,u2','*')
title(sprintf('Sol.on x by t=Tb=\%4.2f,tau = \%4.2f ',Tb,tau))
xlabel('x'), ylabel('u')
X4=ones(MK,1)*x';Y4=t'*ones(1,N1);
figure, surfc(X4,Y4,P2)
colorbar
xlabel('r'), ylabel('t'), zlabel('u')
title(sprintf('Surf.,tNr.=\%4.1f',MK))
figure, plot(x,V0,'*')
title (' Initial cond.V0')
Z1=zeros(N1,N1);E1=eye(N1,N1);
A=zeros(NN,NN);
y1=600*ones(N1,1);
y1(1)=y1(1)/sqrt(2);y1(N1)=y1(N1)/sqrt(2);
A=[Z1,E1;-(A2+gamma*a22*E1)/tau,-E1/tau];
y01=[y1;zeros(N1,1)];uT1=expm(A*Tb)*y01;
uT1(1)=uT1(1)*sqrt(2);uT1(N1)=uT1(N1)*sqrt(2);
y02=[zeros(N1,1);ones(N1,1)];y02(N1+1)=y02(N1+1)/sqrt(2);
y02(NN)=y02(NN)/sqrt(2);uT2=expm(A*Tb)*y02;
uT2(1)=uT2(1)*sqrt(2);uT2(N1)=uT2(N1)*sqrt(2);
MI2=(zeros(1:N1,1)-uT1(1:N1))./uT2(1:N1);
y1(1)=y1(1)*sqrt(2);y1(N1)=y1(N1)*sqrt(2);
y0=[y1;MI2];SM=sum(MI2)/N1;
figure,plot(x,MI2,'ko')
title(sprintf('V(0) depending on x by t=0,Av.val.=...
%8.4f,tau = %4.2f ',SM,tau))
xlabel('x'), ylabel('u')
options=odeset('RelTol', 1.0e-7);
[T1,Y1]=ode15s(@SIST,[0 Tb],y0,options,A);
u1=Y1(end,1:N1);u1(1)=u1(1)*sqrt(2);u1(N1)=u1(N1)*sqrt(2);
figure,plot(x,u1','ko')
title(sprintf('Sol.of superpos. on x by t=Tb=%4.2f,. . .
tau =\%4.2f ',Tb,tau))
xlabel('x'), ylabel('u')
figure,plot(T1(:),max(Y1(:,1:N1)'),'k*')
title(sprintf('Sol.of superpos. on t by $max_x$(U(x,t)),. . .
V(0)=%8.6f,tau = %4.2f ',SM,tau))
xlabel('t'), ylabel('u')
K=length(T1);
X11=ones(K,1)*x';Y11=T1*ones(1,N1);
figure, surfc(X11,Y11,Y1(:,1:N1))
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('Surf.,tNr.=%4.1f,gamma=%6.4f',K,gamma))
function F=SIST(t,y,A)
F=A*y;
```

5.7 CAM with hyperbolic type integral splines for the solutions 3-D problem in z-direction

Using **averaged method with respect to** z with hyperbolic trigonometric functions
$T(x,y,z,t) = T_z(x,y,t) + m_z(x,y,t)\frac{0.5L_z \sinh(a_z(z-0.5L_z))}{\sinh(0.5a_zL_z)}$
$+e_z(x,y,t)0.125\frac{\cosh(a_z(z-0.5L_z))-A_{0z}}{\sinh^2(0.25a_zL_z)}$,
where $T_z(x,y,t) = \frac{1}{L_z}\int_0^{L_z} T(x,y,z,t)dz$,
$A_{0z} = \frac{(sinh(0.5a_zL_z))}{0.5a_zL_z)}$.
We can see that the parameters $a_z > 0$ tend to zero then the limit is the integral parabolic spline (A. Buikis [9]):
$T(x,y,z,t) = T_z(x,y,t) + m_z(x,y,t)(z-0.5L_z) + e_z(x,y,t)(\frac{(z-0.5L_z)^2}{L_z^2} - \frac{1}{12})$.
The unknown functions $m_z(x,y,t), e_z(x,y,t)$, we can determined from boundary conditions (5.1) in z-direction:
$d_z m_z = k_z^* e_z, m_z = p_z e_z, e_z = -g_z T_z$,
where
$g_z = \alpha_z^*/(\alpha_z^*(0.5L_z p_z + b_z) + 2k_z^*), \alpha_z^* = \frac{\alpha_z}{k_z}$,
$d_z = 0.5L_z a_z \coth(0.5a_zL_z)$ $,k_z^* = 0.25a_z \coth(0.25a_zL_z), p_z = k_z^*/d_z$,
$b_z = \frac{\cosh(0.5a_zL_z)-A_{0z}}{8\sinh^2(0.25a_zL_z)}$.
We can use the optimal parameter $a_z = \sqrt{\frac{\gamma_0}{k_z}}$.

Now the initial-boundary value 2-D problem is in the form

$$\begin{cases} \tau_* \frac{\partial^2 T_z(x,y,t)}{\partial t^2} + c\rho \frac{\partial T_z(x,y,t)}{\partial t} = \frac{\partial}{\partial x}(k_x \frac{\partial T_z(x,y,t)}{\partial x}) + \\ \frac{\partial}{\partial y}(k_y \frac{\partial T_z(x,y,t)}{\partial y}) - (B_z g_z + \gamma_0)T_z(x,y,t) + f_z(x,y,t), \\ x \in (0,L_x), y \in (0,L_y), t \in (0,t_f), \\ \frac{\partial T_z(0,y,t)}{\partial x} = \frac{\partial T_z(x,0,t)}{\partial y} = 0, \\ k_x \frac{\partial T_z(L_x,y,t)}{\partial x} + \alpha_x T_z(L_x,y,t) = 0, \\ k_y \frac{\partial T_z(x,L_y,t)}{\partial y} + \alpha_y T_z(x,L_y,t) = 0, \\ T_z(x,y,0) = T_{z,0}(x,y), \frac{\partial T_z(x,y,0)}{\partial t} = V_{z,0}(x,y), \end{cases} \quad (5.13)$$

where $B_z = \frac{2k_z^*}{L_z}$, $f_z(x,y,t) = \frac{1}{L_z}\int_0^{L_z} f(x,y,z,t)dz$,
$T_{z,0}(x,y) = \frac{1}{L_z}\int_0^{L_z} T_0(x,y,z)dz$, $V_{z,0}(x,y) = \frac{1}{L_z}\int_0^{L_z} V_0(x,y,z)dz$.

5.8 The averaged method in y-direction

Using **averaged method with respect to** y
$T_{z,y}(x,t) = \frac{1}{L_y}\int_0^{L_y} T_z(x,y,t)dy$,
$T_z(x,y,t) = T_{z,y}(x,t) + m_{z,y}(x,t)\frac{0.5L_y \sinh(a_y(y-0.5L_y))}{\sinh(0.5a_yL_y)} +$
$e_{z,y}(x,t)0.125\frac{\cosh(a_y(y-0.5L_y))-A_{0y}}{\sinh^2(0.25a_yL_y)}$, $A_{0y} = \frac{(sinh(0.5a_yL_y))}{0.5a_yL_y)}$.
The unknown functions $m_{z,y}(x,t), e_{z,y}(x,t)$, we can determined this functions from boundary conditions (1.13) in following form:
$d_y m_{z,y} = k_y^* e_{z,y}, m_{z,y} = p_y e_{z,y}, e_{z,y} = -g_y T_{z,y}$,
where
$g_y = \alpha_y^*/(\alpha_y^*(0.5L_y p_y + b_y) + 2k_y^*), \alpha_y^* = \frac{\alpha_y}{k_y}$,
$d_y = 0.5L_y a_y \coth(0.5a_yL_y)$,$k_y^* = 0.25a_y \coth(0.25a_yL_y), p_y = k_y^*/d_y$,
$b_y = \frac{\cosh(0.5a_yL_y)-A_{0y}}{8\sinh^2(0.25a_yL_y)}$.
We can use the optimal parameter $a_y = \sqrt{\frac{\gamma_0+B_zg_z}{k_y}}$.

Then the initial-boundary value problem (5.13) is in following form

$$\begin{cases} \tau_*\frac{\partial^2 T_{z,y}(x,t)}{\partial t^2} + c\rho\frac{\partial T_{z,y}(x,t)}{\partial t} = \frac{\partial}{\partial x}(k_x\frac{\partial T_{z,y}(x,t)}{\partial x}) - \\ (B_zg_z + B_yg_y + \gamma_0)T_{z,y}(x,t) + f_{z,y}(x,t), x \in (0,L_x), t \in (0,t_f), \\ \frac{\partial T_{z,y}(0,t)}{\partial x} = 0, k_x\frac{\partial T_{z,y}(L_x,t)}{\partial x} + \alpha_x T_{z,y}(L_x,t) = 0, \\ T_{z,y}(x,0) = T_{z,y,0}, \frac{\partial T_{z,y}(x,0)}{\partial t} = V_{z,y,0}, \end{cases} \tag{5.14}$$

where $B_y = \frac{2k_y^*}{L_y}, f_{z,y}(x,y,t) = \frac{1}{L_y}\int_0^{L_y} f_z(x,y,t)dy$,
$T_{z,y,0}(x) = \frac{1}{L_y}\int_0^{L_y} T_{z,0}(x,y)dy$, $V_{z,y,0}(x) = \frac{1}{L_y}\int_0^{L_y} V_{0,z}(x,y)dy$.

5.9 The averaged method in x-direction

It is possible make the averaging also **respect with** x
$T_{z,y,x}(t) = \frac{1}{L_x}\int_0^{L_x} T_{z,y}(x,t)dx$,
$T_{z,y}(x,t) = T_{z,y,x}(t) + m_{z,y,x}(t)\frac{0.5L_x \sinh(a_x(x-0.5L_x))}{\sinh(0.5a_xL_x)} +$
$e_{z,y,x}(t)0.125\frac{\cosh(a_x(x-0.5L_x))-A_{0x}}{\sinh^2(0.25a_xL_x)}$, $A_{0x} = \frac{(sinh(0.5a_xL_x))}{0.5a_xL_x)}$,
where $d_x m_{z,y,x} = k_x^* e_{z,y,x}, m_{z,y,x} = p_x e_{z,y,x}, e_{z,y,x} = -g_x T_{z,y,x}$,
$g_x = \alpha_x^*/(\alpha_x^*(0.5L_x p_x + b_x) + 2k_x^*), \alpha_x^* = \frac{\alpha_x}{k_x}$,

$d_x = 0.5L_x a_x \coth(0.5a_x L_x)$,$k_x^* = 0.25a_x \coth(0.25a_x L_x), p_x = k_x^*/d_x$, $b_x = \frac{\cosh(0.5a_x L_x) - A_{0x}}{8\sinh^2(0.25a_x L_x)}$.

We can use the optimal parameter $a_x = \sqrt{\frac{\gamma_0 + B_z g_z + B_y g_y}{k_x}}$.

Then the initial-boundary value problem (5.14) is in the following form for ODEs of the second order:

$$\begin{cases} \tau_* \frac{\partial^2 T_{z,y,x}(t)}{\partial t^2} + c\rho \frac{\partial T_{z,y,x}(t)}{\partial t} + \gamma T_{z,y,x}(t) = f_{z,y,x}(t), t \in (0, t_f), \\ T_{z,y,x}(0) = T_{z,y,x,0}, \frac{\partial T_{z,y,x}(0)}{\partial t} = V_{z,y,x,0}, \end{cases} \tag{5.15}$$

where $\gamma = B_z g_z + B_y g_y + B_x g_x + \gamma_0, B_x = \frac{2k_x^*}{L_x}, f_{z,y,x}(t) = \frac{1}{L_y}\int_0^{L_y} f_{z,y}(x,t)dx$, $T_{z,y,x,0} = \frac{1}{L_y}\int_0^{L_y} T_{z,y,0}(x)dx$, $V_{z,y,x,0} = \frac{1}{L_y}\int_0^{L_y} V_{0,z,y}(x)dx$.

The solution of this problem is

$$\begin{cases} T_{z,y,x}(t) = exp(-0.5t/\tau)(C\sinh(\kappa t) + B\cosh(\kappa t)) + \\ \frac{1}{\kappa\tau}\int_0^t exp(-0.5\frac{1}{\tau}(t-\xi))\sinh(\kappa(t-\xi))G(\xi)d\xi, \end{cases} \tag{5.16}$$

where $\tau = \tau_*/(c\rho), \kappa = \sqrt{0.25/\tau^2 - \gamma_*/\tau}, B = T_{z,y,x,0}$,
$C = \frac{1}{\kappa}(V_{z,y,x,0} + \frac{1}{2\tau}T_{z,y,x,0})$,
$\gamma_* = \gamma/(c\rho), G(t) = f_{z,y,x}(t)/(c\rho)$.

If $4\gamma_*\tau > 1$, then the hyperbolic functions to need replaced with the trigonometrical and the parameter κ with $\sqrt{\gamma_*/\tau - 0.25/\tau^2}$.

If $4\gamma_*\tau = 1, \kappa = 0$, then $C\sinh(\kappa t) + B\cosh(\kappa t) = t(V_{z,y,x,0} + \frac{1}{2\tau}T_{z,y,x,0}) +$ $T_{z,y,x,0}$, and $\frac{1}{\kappa\tau}\int_0^t exp(-0.5\frac{1}{\tau}(t-\xi))\sinh(\kappa(t-\xi))G(\xi)d\xi =$
$\frac{1}{\tau}\int_0^t exp(-0.5\frac{1}{\tau}(t-\xi))(t-\xi)G(\xi)d\xi$.

Chapter 6
Periodical BCs: H. Kalis, M. Kokainis, A. Gedroics, 2015 [78]

The solutions of the linear initial-boundary value problem for heat transfer equations are obtained analytically and numerically. We define the finite difference scheme with exact spectrum (FDSES), where the finite difference matrix A is represented in the form form $A = PDP^*$ (P, D are the matrixes of finite difference eigenvectors and eigenvalues corespondently) and the elements of diagonal matrix D are replaced with the first eigenvalues from the differential operator, P^* is the conjugate matrix.

6.1 The mathematical model

We consider the linear initial-boundary heat transfer problem with the periodical boundary conditions in following form:

$$\begin{cases} \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}(\nu \frac{\partial T(x,t)}{\partial x}) + f(x,t), x \in (0,L), t \in (0,t_f), \\ T(0,t) = T(L,t), \frac{\partial T(0,t)}{\partial x} = \frac{\partial T(L,t)}{\partial x}, t \in (0,t_f), \\ T(x,0) = T_0(x), x \in (0,L), \end{cases} \tag{6.1}$$

where $\nu > 0$ is the constant parameter, t_f is the final time, T_0, f are given functions.

We consider uniform grid in the space $x_j = jh, j = \overline{0,N}, Nh = L$, where N is even number.
Using the finite differences of second order approximation for partial derivatives of second order respect to x we obtain the initial value problem for system of ordinary differential equations (ODEs) in the

following matrix form

$$\dot{U}(t) + \nu A U(t) = F(t), \quad U(0) = U_0, \tag{6.2}$$

where A is the 3-diagonal circulant matrix of N order in the form

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ 1 & -2 & 1 & \dots & 0 & 0 & 0 \\ .. & .. & .. & \dots & .. & .. & \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

$U(t), \dot{U}(t), U_0, F(t)$ are the column-vectors of N order.
Circulant matrix A can be described by the elements of a first row $h^2 A = [2, -1, 0, 0, \dots, -1]$. The 3-diagonal matrix A of N order can be represented with following difference operator of second order approximation

$$Ay = \begin{cases} -(y_{j+1} - 2y_j + y_{j-1})/h^2, & j = \overline{1,N}, \\ y_{N+1} = y_1, \; y_0 = y_N \end{cases} \tag{6.3}$$

where y is column-vector of N order with elements $y_j, j = \overline{1,N}$.
We use from two vectors y^1, y^2 following scalar product $[y^1, \bar{y}^2] = (\sum_{j=1}^{N} y_j^1 \bar{y}_j^2)$,
where $\bar{y}$ is the conjugate value of y.

6.2 The spectral problems

The corresponding discrete spectral problem $Ay^n = \mu_n y^n, n = \overline{1,N}$ with circulant matrix have following solution:

$$\begin{cases} y^n = C_n^{-1} (y_1^n, y_2^n, \dots, y_N^n)^T, \\ \mu_n = \frac{4}{h^2} \sin^2(\pi n h / L), \end{cases} \tag{6.4}$$

where $y_j^n = \phi_n(x_j) = \exp(2\pi i n x_j / L), j = \overline{1,N}, i = \sqrt{-1}$ are the components of column-vector y^n.

For the circulant matrix $C = [c_1 \; c_2 \; \cdots \; c_N]$
the eigenvalues μ_n can simply obtain in following form

$$\mu_n = \sum_{j=0}^{N-1} c_{j+1} \phi_n(x_j).$$

From this form follows the eigenvalues of matrix A for different order of approximation $O(h^k), k \geq 2$(see chapter1):
1) $k = 4, h^2 A = [-\frac{5}{2}, \frac{4}{3}, -\frac{1}{12}, 0, \cdots, 0, -\frac{1}{12}, \frac{4}{3}]$,
$h^2 \mu_n = -4(\sin^2(\pi nh/L) + \frac{1}{3}\sin^4(\pi nh/L))$,
2) $k = 6, h^2 A = [-\frac{49}{18}, \frac{3}{2}, -\frac{3}{20}, \frac{1}{90}, 0, \cdots, 0, \frac{1}{90}, -\frac{3}{20}, \frac{3}{2}]$,
$h^2 \mu_n = -4(\sin^2(\pi nh/L) + \frac{1}{3}\sin^4(\pi nh/L) + \frac{8}{45}\sin^6(\pi nh/L))$,
3) $k = 8, h^2 A = [-\frac{205}{72}, \frac{8}{5}, -\frac{1}{5}, \frac{8}{315}, -\frac{1}{560}, 0, \cdots, 0, -\frac{1}{560}, \frac{8}{315}, -\frac{1}{5}, \frac{8}{5}]$.
$h^2 \mu_n = -4(\sin^2(\pi nh/L) + \frac{1}{3}\sin^4(\pi nh/L) + \frac{8}{45}\sin^6(\pi nh/L) +$
$\frac{4}{35}\sin^8(\pi nh/L))$,

The constants $C_n^2 = N$ and we have the orthonormed eigenvectors y^n, y^m with the scalar product $[y^n, \bar{y}^m] = \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker symbol.
Therefore the matrix A can be represented in form $A = WDW^*$, where the column of the matrix W and the diagonal matrix D contains N orthonormed
eigenvectors $w^n = y^n$ and eigenvalues $\mu_n, n = \overline{1,N}$ correspondly. From $W^*W = E$ follows that $W^{-1} = W^*$.

The solution of the spectral problem for differential equations

$$-y''(x) = \lambda y(x), x \in (0,L), y(0) = y(L), y'(0) = y'(L),$$

is in following form:
$y_n(x) = \sqrt{L^{-1}}\phi_n(x) = \sqrt{L^{-1}}\exp(2\pi inx/L)$,
where $(y_n, \bar{y}_m) = \int_0^L y_n(x)\bar{y}_m(x)dx = \delta_{n,m}$ and $\lambda_n = (2\pi n/L)^2$ are the eigenvalues $n = 0, \pm 1, \pm 2. \cdots$. For the scalar product $h[y^n, \bar{y}^m]$ the integral $(y_n, \bar{y}_m)$ is approximated with trapezoidal formula and in the limit case if $h \to 0$ then follows that $\mu_n \to \lambda_n$.

6.3 The analytical solution

We can consider the **analytical solutions** of the system of ODEs (6.2) using the spectral representation of matrix $A = WDW^*$. From transformation $V = W^*U(U = WV)$ follows the seperate system of ODEs

$$\dot{V}(t)+\nu DV(t)=G(t), \quad V(0)=W^{*}U_{0}, \tag{6.5}$$

where $V(t),\dot{V}(t),V(0),G(t)=W^{*}F(t)$ are the column-vectors of N order with elements $v_k(t),\dot{v}_k(t),v_k(0),g_k(t)k=\overline{1,M}$.
The solution of this system is the function

$$v_k(t)=v_k(0)\exp(-\kappa_k t)+\int_0^t exp(-\kappa_k(t-\tau))g_k(\tau)d\tau, \tag{6.6}$$

where $\kappa_k=\nu\mu_k$.

We can used also the **Fourier method** for solving (6.1) in the form $T(x,t)=\sum_{k\in Z}v_k(t)y_k(x)$, where $y_k(x)$ are the orthonormed eigenvectors, $v_k(t)$ is the solution (6.6),with $v_k(0)=(T_0,\bar{y}_k)$.

For the FDSES the matrix A is represented in the form form $A=WDW^{*}$ and the diagonal matrix D contain the first N eigenvalues
$d_k=\lambda_k$, $k=\overline{1,N}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$ in following way:
1)$d_k=\lambda_k$ for $k=\overline{1,N_2}$, where $N_2=N/2$.
2)$d_k=\lambda_{N-k}$ for $k=\overline{N_2,N}$.
If $d_k=\mu_k$, then we have the method of finite difference approximation with matrix A.
The FDSES method is more stable as the method of finite difference by approximation with central difference (FDS), because the eigenvalues are larger $d_k>\mu_k$.
The results obtained with Fourier series contain on $x=0,x=L$ oscillations (Gibbs phenomena). For FDSES method these oscillations disappear.

If the functions $f(x,t),T_0(x)$ are proportional to the eigenvector $w_p(x)=\sqrt{1/L}\exp(2\pi ipx/L), f(x,t)=g(t)w_p(x), T_0(x)=a_0w_p(x)$, then the solution we can obtained in the form $T(x,t)=y(t)w_p(x)$, where for function $y(t)$ follows the ODEs
$\dot{y}(t)=-\nu\lambda_p y(t)+g(t)$ with $y(0)=a_0, \lambda_p=(\frac{2\pi p}{L})^2$.
We have the exact solution
$y(t)=\exp(-\nu\lambda_p t)a_0+\int_0^t\exp(-\bar{k}\lambda_p(t-\xi)g(\xi)d\xi$.

The solution we can also obtained in real form:
$T(x,t)=\sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L}+a_{ks}(t)\sin\frac{2\pi kx}{L})+\frac{a_{0c}(t)}{2}$,
$f(x,t)=\sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L}+b_{ks}(t)\sin\frac{2\pi kx}{L})+\frac{b_{0c}(t)}{2}$,
$b_{kc}(t)=\frac{2}{L}\int_0^L f(\xi,t)\cos\frac{2\pi k\xi}{L}d\xi$, $b_{ks}(t)=\frac{2}{L}\int_0^L f(\xi,t)\sin\frac{2\pi k\xi}{L}d\xi$,

where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (6.6) by
$a_{kc}(0) = \frac{2}{L}\int_0^L T_0(\xi)\cos\frac{2\pi k\xi}{L}d\xi$, $a_{ks}(0) = \frac{2}{L}\int_0^L T_0(\xi)\sin\frac{2\pi k\xi}{L}d\xi$,
$g_k(t) = b_{kc}(t)$ or $b_{ks}(t), \kappa_k = \nu\lambda_k$.

From Fourier series
$T(x,t) = \sum_{k=-\infty}^{\infty} a_k(t)w_k(x), f(x,t) = \sum_{k=-\infty}^{\infty} b_k(t)w_k(x)$,
$b_k(t) = (f, w_k)_*$, follows ODEs $\dot{a}_k(t) = -\nu\lambda_k a_k(t) + b_k(t)$
with $a_k(0) = (T_0, w_k)_*, \lambda_k = (\frac{2\pi k}{L})^2$. We have following solutions
$a_k(t) = \exp(-\nu\lambda_k t)a_k(0) + \int_0^t \exp(-\nu\lambda_k(t-\xi)b_k(\xi)d\xi$.
From orthonormal eigenvectors follows, that $b_p(t) = g(t), a_p(0) = a_0, b_k(t) = a_k(0) = 0, k \neq p$ and $a_p(t) = y(t)$.

From the discrete Fourier series
$T(x_j,t) = \sum_{k=-N/2}^{N/2} a_k(t)w_k(x_j)$,
$f(x_j,t) = \sum_{k=-N/2}^{N/2} b_k(t)w_k(x_j), w_k(x_j) = w_j^k = \sqrt{1/N}\exp(2\pi ikj/N)$
or in the vector form $U(t) = \sum_{k=-N/2}^{N/2} a_k(t)w^k$ we have ODEs
$\dot{a}_k(t) = -\nu\mu_k a_k(t) + b_k(t)$
with $a_k(0) = (U_0, w^{*k}), b_k(t) = (F(t), w^{*k})$.
We have following solutions
$a_k(t) = \exp(-\nu\mu_k t)a_k(0) + \int_0^t \exp(-\nu\mu_k(t-\xi)b_k(\xi)d\xi$.
From orthonormal eigenvectors follows, that $b_p(t) = g(t), a_p(0) = a_0, b_k(t) = a_k(0) = 0, k \neq p$
and $a_p(t) = y(t)$, if the eigenvalue μ_p are replaced with λ_p and $p \leq N/2$.

We can obtained the solution of the discrete problem also in the following real form
$u_j(t) = \sum_{k=1}^{N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$,
$f_j(t) = \sum_{k=1}^{N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{N}\sum_1^N f_j(t)\cos\frac{2\pi kj}{N}, b_{ks}(t) = \frac{2}{N}\sum_1^N f_j(t)\sin\frac{2\pi kj}{N}$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (6.6) by
$a_{kc}(0) = \frac{2}{N}\sum_1^N T_0(x_j)\cos\frac{2\pi kj}{N}$, $a_{ks}(0) = \frac{2}{N}\sum_1^N T_0(x_j)\sin\frac{2\pi kj}{N}$,
$g_k(t) = b_{kc}(t)$ or $b_{ks}(t)$,
$\kappa_k = \nu d_k$, $d_k = \mu_k$ for FDS, $d_k = \lambda_k$ for FDSES.

From the FDS the solution of the matrix equation (6.2) is
$U(t) = \exp(-\nu tA)U(0) + \int_0^t \exp(-\nu A(t-\xi)F(\xi)d\xi$.
Using the matrix A representation $A = WDW^*$ and transformation $V = W^*U$ follows that for every matrix function $f(A) = Wf(D)W^*$ and $V = \exp(-\nu tD)V(0) + \int_0^t \exp(-\nu D(t-\xi)G(\xi)d\xi$.

Therefore we have the solution in the form (6.6). If $p \leq N/2$ then the components
$v_k(0) = (W^*U_(0))_k = \sum_{j=1}^N w_j^{*k} T_0(x_j) = a_0 \sum_{j=1}^N w_j^{*k} w_p(x_j) = \frac{a_0}{\sqrt{h}}(w^{*k}, w^p)$,
$g_k(t) = (W^*F)_k = \frac{g(t)}{\sqrt{h}}(w^{*k}, w^p), k = \overline{1,N}$.
We get $v_p(0) = \frac{a_0}{\sqrt{h}}, g_p(t) = \frac{g(t)}{\sqrt{h}}, v_k(0) = g_k(t) = 0, k \neq p$
and from (6.6) follows $v_p(t) = \frac{1}{\sqrt{h}}(\exp(-\nu\mu_p t)a_0 + \int_0^t \exp(-\nu\mu_p(t-\xi)g(\xi)d\xi), v_k(t) = 0, k \neq p$.
For FDSES from $U = WV, w_j^k = \sqrt{h} w_k(x_j)$ and replaced the discrete eigenvalue μ_p with λ_p we obtain the exact solutions $T(x_j,t) = y(t)w_p(x_j), j = \overline{0,N}$.

If the functions $f(x,t), T_0(x)$ are proportional to the functions $f_1(x) = \sin(2\pi p_1 x/L), f_2(x) = \cos(2\pi p_2 x/L)$,
then using the expresions $f_1(x) = \frac{\sqrt{L}}{2i}(w_{p_1}(x) - w_{-p_1}(x)), f_2(x) = \frac{\sqrt{L}}{2}(w_{p_2}(x) + w_{-p_2}(x))$,
$w_{-p}(x_j) = w_{N-p}(x_j) = w_p^*(x_j), \mu_{-p} = \mu_p, \lambda_{-p} = \lambda_p$
we have the preliminary results and the exact solution for $\max(p_1, p_2) \leq N/2$.

Example 6.1. If $f(x,t) = g_1(t)f_1(x) + g_2(t)f_2(x), T_0(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$
then we have $b_k(t) = \pm\frac{g_1(t)}{2i}, k = \pm p_1$,
$b_k(t) = \frac{g_2(t)}{2}, k = \pm p_2, b_k(t) = 0, k \neq (\pm p_1, \pm p_2)$,
$a_k(0) = \pm\frac{\alpha_1}{2i}, k = \pm p_1, a_k(0) = \frac{\alpha_2}{2}, k = \pm p_2, a_k(0) = 0, k \neq (\pm p_1, \pm p_2)$.
Therefore,
$T(x,t) = a_{-p_1}(t)w_{-p_1}(t) + a_{p_1}(t)w_{p_1}(t) + a_{-p_2}(t)w_{-p_2}(t) + a_{p_2}(t)w_{p_2}(t) =$
$f_1(x)(\alpha_1 \exp(-\nu\lambda_{p_1}t) + \int_0^t \exp(-\nu\lambda_{p_1}(t-\xi))g_1(\xi)d\xi)+$
$f_2(x)(\alpha_2 \exp(-\nu\lambda_{p_2}t) + \int_0^t \exp(-\nu\lambda_{p_2}(t-\xi))g_2(\xi)d\xi)$.
In the discrete case we have the exact solution by replaced the discrete eigenvalues μ_{p_1}, μ_{p_2}, with $\lambda_{p_1}, \lambda_{p_2}$.

6.4 Example of nonlinear heat transfer equations

We shall consider the initial-boundary value problem for solving the following nonlinear heat transfer equation:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 (g(T))}{\partial x^2} + f(T),$$

where $g(T) = T^{\sigma+1}, f(T) = aT^{\beta}$ is nonlinear functions with $a > 0, \beta \geq 1, \sigma \geq 0, T(x,t) \geq 0, T_0(x) \geq 0$.
In paper [5] by $(a = 1)$ is proved with the first kind boundary conditions that
1) by $\beta < \sigma + 1$ exists global bounded solution for all t,
2) by $\beta \geq \sigma + 1$ exists global bounded solution
for sufficient small $||T_0||$, but for larger $||T_0||$, exists finite value of time T_* when $u(x,t) \to \infty$ if $t \to T_*$.
The initial value problem for ODEs (6.2) is in the form

$$\dot{U} + AG = F, U(0) = U_0,$$

where G, F are the vectors-column of N order with elements $g_k = g(u(x_k,t)), f_k = af(u(x_k,t)), k = \overline{1,N}$.
The numerical experiment with $L = 1$ and $T_0(x) = x(1-x) \geq 0$, is produced by MATLAB 7.4 solver "ode23s" by first kind boundary conditions [1].
For $a = 5, \sigma = \beta = 3, (\beta < \sigma + 1), t = 10, N = 6, 10, 20$ are obtained following maximal error using FDS and FDSES methods:
1) $N = 6 - 0,0125(FDS), 0.0011(FDSES)$;
2) $N = 10 - 0.0046(FDS), 0.0003(FDSES)$;
3) $N = 20 - 0.0013(FDS), 0.0001(FDSES)$.
In the Figs. 6.1 ,6.2, 6.3, 6.4 are represented 4 type solutions by $T_0(x) = \sin^{100}(\pi x), N = 50, \sigma = 3$ for periodical boundary conditions obtained:
1) $\beta = 5, a = 100$, the solution is "blow up" locally by $T_* = 5.481136$,
2) $\beta = 4, a = 100$, the solution is "blow up" globally by
$T_* = 0.2020261$,
3)$\beta = 5, a = 1$, the solutions tends to zero, if $t \to \infty$,
4) $\beta = 4, a = 0.01$, the solutions tends to the stationary limit.
The following MATLAB programm is with m.file **nelper.**

```
function nelper(N)
sigma=3;sigma1=sigma+1;beta=5;a=100;
N1=N+1; Tb1=0.05;Tb2=6.4811;L=1;x=linspace(0,L,N1)';
h=L/N;N2=N-1;NT=[1:N];
lk0=(2*pi/L*NT).^2; % exact eig-val.
lk2=4/h^2*(sin(pi*h*NT)).^2; %FDS O(h^2)
```

```
lk4=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%FDS O(h^4)
lk6=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
    8/45*(sin(pi*h*NT)).^6);%FDS O(h^6)
lk8=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
    8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%FDS O(h^8)
d=lk2; %FDS
%NH=N/2; d(1:NH)=lk0(1:NH);
%d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
figure
plot(NT,lk2,'-',NT,lk4,'-.', NT,lk6,'*',NT,lk8,'o',NT,d,'d')
legend('Eig-val O(h^2)','Eig-val O(h^4)','Eig-val O(h^6)',. . .
'Eig-val O(h^8)','Eig-val exact')
W=exp(2*pi*h*i*[1:N]'*[1:N]);x=x(2:N1);
A2=zeros(N,N);
%A2=A2+diag(ones(N2,1),1)+diag(ones(N2,1),-1)-2*diag(ones(N,1));
%A2(1,N)=1; A2(N,1)=1;A2=A2/(h^2):
%D=-h*(1./W)*A2*W;% control
A=real(-h*W*diag(d)*conj(W)); %FDS and FDSES
y0=sin(pi*x).^100;
options=odeset('RelTol', 1.0e-7);
[T1,Y1]=ode15s(@SIST,[0 Tb1],y0,options,A,sigma1,beta,a);
im=max(imag(Y1(end,:)));
figure,plot(x,y0,'k-')
hold on
plot(x,real(Y1(end,:)'),'k*')
[T2,Y2]=ode15s(@SIST,[Tb1 Tb2],...
Y1(end,:),options,A,sigma1,beta,a);
plot(x,real(Y2(end,:)'),'ko')
grid on
title(sprintf('beta=%2.0f,sigma=%2.0f,a=%3.1f,. . .
T1=%8.6f,T2=%8.6f',beta, sigma,a,T1(end),T2(end)))
xlabel('\itx'), ylabel('\itu')
legend('Sol.U(x,0)','Sol.U(x,T1)', 'Sol.U(x,T2)')
figure
T=[T1;T2];Y=[max(real(Y1(:,:)'))';max(real(Y2(:,:)'))'];
plot(T,Y)
grid on
title(sprintf('DV lab.aproks.DS,N=...
%3.0f, time = %8.6f ',N,T2(end)))
xlabel('\itt'), ylabel('\itu')
function F=SIST(t,y,A,sigma1,beta,a)
F=A*y.^sigma1+a*y.^beta;
```

We have following results by $N = 40, \beta = 5, \sigma = 3, a = 100:$
$T_* = 5.448350(FDS - O(h^2)), T_* = 5.536841(FDS - O(h^4)),$
$T_* = 5.539480(FDS - O(h^6)), T_* = 5.539669(FDS - O(h^4)), T_* = 5.539397(FDSES).$
The eigenvalues depending on the order of approximation and the results of FDSES we can see in Figs. 6.5, 6.6

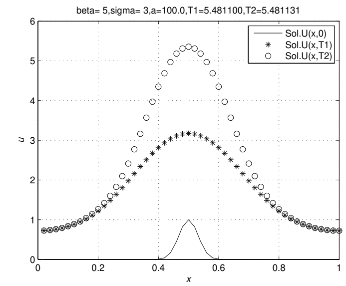


Fig. 6.1 $U \to \infty$ for $x = 0.5$, $\beta = 5, \sigma = 3, a = 100, T_* = 5.481136$

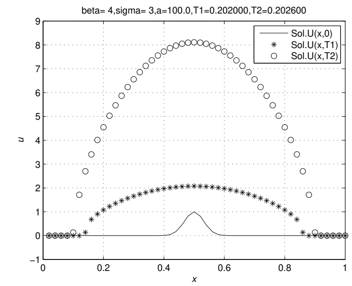


Fig. 6.2 $U \to \infty$ for $x \in (0,1)$, $\beta = 4, \sigma = 3, a = 100, T_* = 0.2020261$

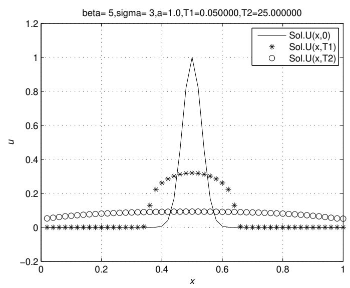


Fig. 6.3 $U \to 0$ if $t \to \infty$, for $\beta = 5, \sigma = 3, a = 1$

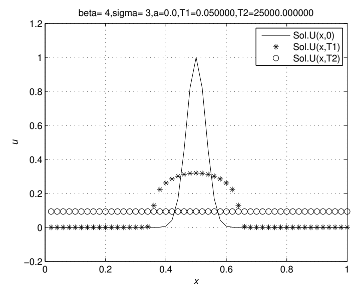


Fig. 6.4 $U \to$ stationary if $t \to \infty$, for $\beta = 4, \sigma = 3, a = 0.01$

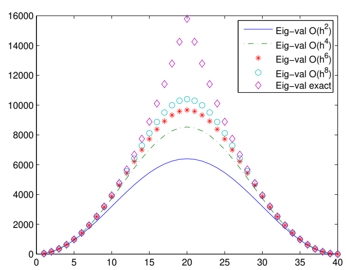


Fig. 6.5 Eig-values of matrix A depending on the order of approx. by $N = 40$

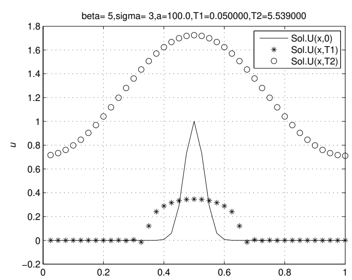


Fig. 6.6 The blow-up solution usind FDSES by $\beta = 5, \sigma = 3, a = 100$

6.5 Example of Burger's equation

For numerical experiments we consider following nonlinear initial-boundary problem for Burger's equations in following form:

$$\frac{\partial T(x,t)}{\partial t} = \nu \frac{\partial^2 T(x,t)}{\partial x^2} - T(x,t)\frac{\partial T(x,t)}{\partial x}),$$

where
1) $T_0(x) = 4\nu\pi\sin(2\pi x)/(2+cos(2\pi x))$, or
2) $T_0(x) = \sin^{100}(\pi x), x \in (0,1)$.
Using the tranformation $T = -2\nu\frac{\partial \ln(V)}{\partial x}$
we can be obtain homogenous linear heat transfer equation $\frac{\partial V(x,t)}{\partial t} = \nu\frac{\partial^2 V(x,t)}{\partial x^2}$.
For the first case of initial conduction we have the analytical solution:
$V(x,t) = 2 + \exp(-4\pi^2\nu t)\cos(2\pi x)$,
$T(x,t) = 4\nu\pi\exp(-4\pi^2\nu t)\sin(2\pi x)/V(x,t)$.

In this case the method of lines is in the form

$$\dot{U}(t) + \nu AU(t) = -0.5A_1U(t)^2, \quad U(0) = U_0,$$

where $A_1 = \frac{1}{4h}[0,1,\cdots,0,-1]$ is the 3-diagonal circulant matrix of N order.
The numerical experiment with $\nu = 1, t = 0.2$, is produced by MATLAB solver "ode15s" . In the Figs. 6.7, 6.8 we can see the results obtained by $N = 40$ for both type initial conditions by three moments of time $t = 0, t_1 = 0.01, t_2 > t_1$.

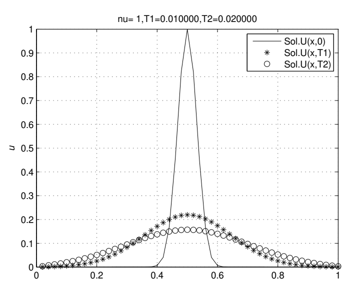


Fig. 6.7 Numerical solution by $N = 40, t_1 = 0.01, t_2 = 0.02$

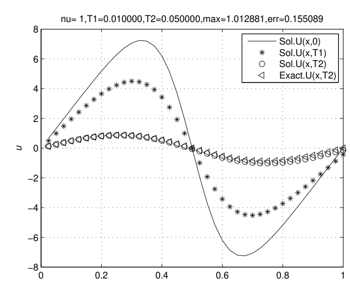


Fig. 6.8 Numerical and analytical solution by $N = 40, t_1 = 0.01, t_2 = 0.05, err = 0.155$

6.6 Example with linear heat transfer equation

For numerical calculation we consider the initial boundary value problem (6.1) with
$f = 0, \nu = L = 1, T_0 = \sin(2\pi x), T(x,t) = \sin(2\pi x)\exp(-4 * \pi^2 t)$.
We have following MATLAB m.file **SiltPer:**

```
%system ODE U_t+k AU=f with periodical BC
%t=Tb,u(x,t)=sin(2 pi x)exp(-4 pi^2 t),k=1,f=0,N-even
function SiltPer(N)
N1=N+1;MK=20; Tb=0.2;L=1;
x=linspace(0,L,N1)';t=linspace(0,Tb,MK);
h=L/N;N2=N-1;k=1;x=x(2:N1);
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+2*diag(ones(N,1));
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
NH=N/2; d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
A2=W*diag(d)*W1; %FDS or FDSES
y1=sin(2*pi*x); % init-cond
P=W1*y1;P1=zeros(MK,N);
for k=1:N
    b=d(k); %FDS or FDSES
    P1(:,k)=P(k)*exp(-b*t');
end
P2=(W*P1')';
prec=-sin(2*pi*x)*exp(-4*pi^2*t);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(MK,1)*x';Y1=t'*ones(1,N);
figure,plot(t',max(abs(P2(:,1:N)'-prec)),'k*')% max error on t
title(sprintf('err. Max-sol.an.on t, Max=%9.7f ',Ma1))
xlabel('\itt'), ylabel('\itu')
figure,plot(x,P2(end,1:N)','ko')
grid on
title(sprintf('Sol.an.on x by Tb.,Max=%9.7f ',Ma1))
xlabel('\itx'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
```

```
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('err. anal.,tNr.=%4.1f,max=%9.7f',MK,Ma1))
```

Using the operator **SiltPer(10)**we obtan following maximal errors ($t_f = 0.2$):
0.01133 (FDS-$O(h^2)$), 0.00057 (FDS- $O(h^4)$),
0.00004 (FDS -$O(h^6)$), 2.10^{-6} (FDS-$O(h^8)$), $!0^{-15}$ (FDSES).
By $N = 40$ the results are:
0.00074 (FDS-$O(h^2)$),2.10^{-6} (FDS- $O(h^4)$), 8.10^{-7} (FDS -$O(h^6)$),
3.10^{-11} (FDS-$O(h^8)$), 10^{-14} (FDSES), (see Figs. 6.9, 6.10)

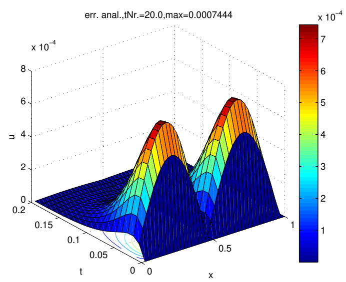


Fig. 6.9 Error with FDS by $N = 40, O(h^2)$

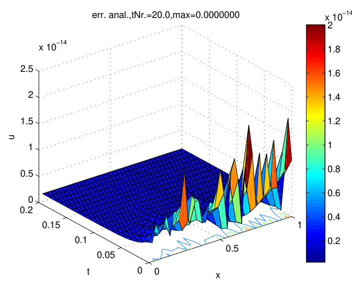


Fig. 6.10 Error with FDSES by $N = 40$

6.7 Example of heat transfer equation with the periodical BC

We consider the initial boundary value problem (6.1) by $L = 1, \nu = 1, f = 0, T_0(x) = \sin(2\pi mx)$, where m is integer in $(1,N)$.
Then the exact solution is $T(x,t) = \exp(-4\pi^2 m^2 t)\sin(2\pi mx)$.
The solution of (6.1) with the **Fourier method** can be obtained in form
$T(x,t) = \sum_{k=-\infty}^{\infty} v_k(t) w^k(x)$, where $v_k(t)$ is the solution of 6.6) in the form $v_k(t) = \exp(-\kappa_k t) v_k(0, \kappa_k = d_k = 4\pi^2 k^2$,
$v_k(0) = \int_0^1 T_0(x) w_*^k(x) dx = 0$, $v_k(t) = 0$ for $k \neq \pm m$, $v_{\pm m}(0) = \frac{0.5}{\pm i}$,
$v_{\pm m}(t) = \pm \frac{\exp(-(2\pi m)^2 t)}{2i}$,
$T(x,t) = v_{-m}(t) w^{-m}(x) + v_m(t) w^m(x) = \exp(-4\pi^2 m^2 t)\sin(2\pi mx)$.
Therefore we have using the Fourier method the exact solution.
We can consider the **analytical solutions** for FDS of (6.2) using the

spectral representation of matrix $A = WDW_*$. From transformation $V = W_*U(U = WV)$ follows the seperate system of ODEs (6.5).
The matrix solution of the system (6.5) is $V(t) = \exp(-Dt)V_0, D = diag(\mu_k)$ or in the form (6.6),
where $v_k(t) = \exp(-\kappa_k t)v_k(0, \kappa_k = \mu_k$,
$v_k(0) = (W^*U_0)_k = 0$, $v_k(t) = 0$ for $k \neq m, k \neq N-m$.
From $\mu_{N-k} = \mu_k, w^{N-k} = w_*^k$ follows
$v_m(0) = \frac{\sqrt{N}}{2i}, v_{N-m}(0) = -\frac{\sqrt{N}}{2i}$.
Therefore
$U(t) = \exp(-\mu_m t)U_0$, where $U_0 = (\sin(2\pi m x_1), \cdots, \sin(2\pi m x_N)^T$ is the column-vector of the N order,
$x_j = jh, j = \overline{1.N}, Nh = 1$.
The solution can be obtained in the matrix form
$U(t) = W\exp(-Dt)W^*U_0$.
For the FDSES $\mu_m = d_m = (2\pi m)^2$ and we have also the exact solution.

Using the **discrete Fourier** transformation
$U(t) = \sum_{k=1}^N a_k(t)w^k(Aw^k = \mu_k w^k)$, we get $a_k(t) = \exp(-\mu_k t)a_k(0)$, where
$a_k(0) = U_0.w_*^k = 0$ for $k \neq m, k \neq N-m$, $a_m(0) = \frac{\sqrt{N}}{2i}, a_{N-m}(0) = -\frac{\sqrt{N}}{2i}$.
We have $U(t) = a_m(t)w^m + a_{N-m}(t)w_*^m = \frac{\sqrt{N}}{2i}\exp(-\mu_m t)(w^m - w_*^m) = \exp(-\mu_m t)U_0$.
For numerical calculation we consider the initial boundary value problem (6.1) with $t_f = 0.05, L = 1, f = 0, T_0 = \sin(2\pi m x)$, for $m = 1;2;3;4, N = 10$.
We have following MATLAB m.file **Silt m:**

```
%t=Tb,u(x,t)=sin(2 pi m x)exp(-(2 pi m)^2 t),m<N-even
function Siltm(N)
N1=N+1;MK=10;m=4; Tb=0.05;L=1;
x=linspace(0,L,N1)';t=linspace(0,Tb,MK);
h=L/N;N2=N-1;x=x(2:N1);
%A2=A2-diag(ones(N2,1),1)-...
%diag(ones(N2,1),-1)+2*diag(ones(N,1));
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2}
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
```

```
%8/45*(sin(pi*h*NT)).^6);%O(h^6)
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
%8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
%NH=N/2; d(1:NH)=lk0(1:NH);
%d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L);
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L);
A2=W*diag(d)*W1; %FDS or FDSES
y1=sin(2*pi*m*x); % init-cond
P=zeros(N,1);P=W1*y1;P1=zeros(MK,N);
for k=1:N
    b=d(k); %FDS or FDSES
    P1(:,k)=exp(-b*t')*P(k);
    end
P2=(W*P1.').';% this is transponation operator
P21=W*diag(exp(-d*t(end)))*W1*y1;%okei !
prec=sin(2*pi*m*x)*exp(-(2*pi*m)^2*t);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
X1=ones(MK,1)*x';Y1=t'*ones(1,N);
figure,plot(t',max(abs(P2(:,1:N).'-prec)),'k*')% max error on t
title(sprintf('err. Max-sol.an.on t, Max=%9.7f ',Ma1))
xlabel('\itt'), ylabel('\itu')
figure,plot(x,P21,'ko',x,prec(1:N,end),'*',x,P2(end,1:N),'-')
%figure,plot(x,P2(end,1:N)','ko')
grid on
title(sprintf('Sol.an.on x by Tb.,Max=%9.7f ',Ma1))
xlabel('\itx'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('t'), zlabel('u')
title(sprintf('err. anal.,tNr.=%4.1f,max=%9.7f',MK,Ma1))
```

Using the operator **Siltm(10)**we obtan following maximal errors (see Tab. 6.1):

Table 6.1 The FDS maximal error depending on order of approximation and m by $N = 10$

Method	m=1	m=2	m=3	m=4
$O(h^2)$	0.0115	0.0457	0.0899	0.0990
$O(h^4)$	5.10^{-4}	0.0084	0.0290	0.0410
$O(h^6)$	3.10^{-5}	0.0019	0.0126	0.0234
$O(h^8)$)	2.10^{-6}	5.10^{-4}	0.0060	0.0153
FDSES	1.10^{-15}	6.10^{-16}	1.10^{-15}	7.10^{-16}

In the Figs. 6.11, 6.12 we can see the FDSES exact solutions by $m = 4$ and $N = 10, N = 40$.

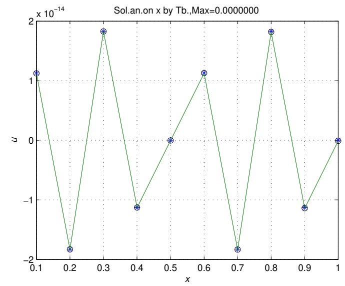


Fig. 6.11 FDSES solutions by $N = 10, m = 4, t_f = 0.05$

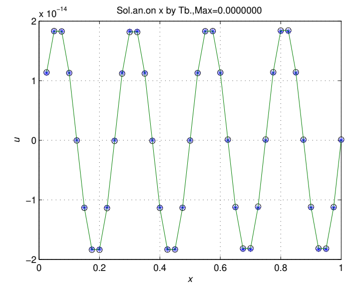


Fig. 6.12 FDSES solutions by $N = 40, m = 4, t_f = 0.05$

6.8 The mathematical model for heat transfer equations with convection

We consider the linear heat transfer equation in the following form:

$$\frac{\partial T(x,t)}{\partial t} = \nu \frac{\partial^2 T(x,t)}{\partial x^2} + a \frac{\partial T(x,t)}{\partial x} + f(x,t) \tag{6.7}$$

with the periodical boundary conditions(6.1) (a=const).
We can used the **Fourier method** for solving the initial-boundary value problem in the form
$T(x,t) = \sum_{k\in Z} a_k(t) w^k(x), f(x,t) = \sum_{k\in Z} b_k(t) w^k(x)$,
where $w^k(x)$ are the orthonormed eigenvectors, $b_k(t) = (f, w_*^k(x))$(see chapter 1).
Then for the unknown functions $a_k(t)$ get the complex initial value problem for ODEs of first order:

$$\begin{cases} \dot{a}_k(t) + a_k(t)\lambda_k = b_k(t), \\ a_k(0) = \frac{1}{L}\int_0^L T_0(s)\exp\frac{-2i\pi ks}{L} ds, \\ b_k(t) = \frac{1}{L}\int_0^L f(s,t)\exp\frac{-2i\pi ks}{L} ds, \end{cases} \tag{6.8}$$

where $\lambda_k = \nu\left(\frac{2\pi k}{L}\right)^2 - ai\frac{2\pi k}{L}$.
The solution of (6.8) is

$$a_k(t) = \exp(-\lambda_k t) a_k(0) + \int_0^t \exp(\lambda_k)(t-s)) b_k(s) ds.$$

The solution with the Fourier method can be obtained in real form when
$f(x,t) = \sum_{k=1}^{\infty}(b_k(t)w^k(x) + b_{-k}(t)w^{-k}(x)) + \frac{b_0(t)}{\sqrt{L}} =$
$\frac{1}{2}\sum_{k=1}^{\infty}((b_k(t)+b_k(t))(w^k(x)+w_*^k(x)) + (b_k(t)-b_{-k}(t))(w^k(x)-w_*^k(x))) +$
$\frac{b_0(t)}{\sqrt{L}} = \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2}$,
where
$b_{kc}(t) = \frac{1}{\sqrt{L}}(b_k(t)+b_{-k}(t)) = \frac{1}{\sqrt{L}}\int_0^L f(x,t)(w_*^k(x)+w^k(x))dx =$
$\frac{2}{L}\int_0^L f(s,t)\cos\frac{2\pi ks}{L}ds$,
$b_{ks}(t) = \frac{i}{\sqrt{L}}(b_k(t)-b_{-k}(t)) = \frac{i}{\sqrt{L}}\int_0^L f(x,t)(w_*^k(x)-w^k(x))dx =$
$\frac{2}{L}\int_0^L f(s,t)\sin\frac{2\pi ks}{L}ds.$
Therefore the solution we can obtain also in real form:
$T(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
where $a_{kc}(t), a_{ks}(t)$ are unknown functions.
From $f(x,t) = \frac{\partial T(x,t)}{\partial t} - (\frac{\partial^2 T(x,t)}{\partial x^2} + a\frac{\partial T(x,t)}{\partial x})$
follows
$f(x,t) = \sum_{k=1}^{\infty}(\dot{a}_{kc}(t)\cos\frac{2\pi kx}{L} + \dot{a}_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{\dot{a}_{0c}(t)}{2} +$
$\sum_{k=1}^{\infty}((a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k))\cos\frac{2\pi kx}{L}$
$+(a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k))\sin\frac{2\pi kx}{L})$,
because $(a_k(t)\lambda_k + a_{-k}(t)\lambda_{-k})/\sqrt{L} = a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k)$,
$i(a_k(t)\lambda_k - a_{-k}(t)\lambda_{-k})/\sqrt{L} = a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k)$,
where $a_{kc}(t) = \frac{a_k(t)+a_{-k}(t)}{\sqrt{L}}, a_{ks}(t) = \frac{i(a_k(t)-a_{-k}(t)}{\sqrt{L}}$
are the coefficients in the expression from the solution $T(x,t)$.
Therefore we obtain the initial boundary value problem for the system of two ODEs:

$$\begin{cases} \dot{a}_{kc}(t) + a_{kc}(t)Re(\lambda_k) + a_{ks}(t)Im(\lambda_k) = b_{kc}(t), \\ \dot{a}_{ks}(t) + a_{ks}(t)Re(\lambda_k) - a_{kc}(t)Im(\lambda_k) = b_{ks}(t), \\ a_{kc}(0) = \frac{2}{L}\int_0^L T_0(s)\cos\frac{2\pi ks}{L}ds, \\ a_{ks}(0) = \frac{2}{L}\int_0^L T_0(s)\sin\frac{2\pi ks}{L}ds. \end{cases} \tag{6.9}$$

6.8.1 Solutions of the system of two ODEs

In the matrix form we have

$$\dot{A}_k(t) + \Lambda_k A_k(t) = B_k(t), A_k(0) = A_{k0}, \tag{6.10}$$

where
$\Lambda_k = \begin{pmatrix} Re(\lambda_k) & Im(\lambda_k) \\ -Im(\lambda_k) & Re(\lambda_k) \end{pmatrix}$ is the matrix of second order,
$A_k(t), B_k(t), A_{k0}$ are the column-vectors with elements
$(a_{kc}(t), a_{ks}(t)), (b_{kc}(t), b_{ks}(t)), (a_{kc}(0), a_{ks}(0))$.
We can represented the matrix Λ_k in the form $\Lambda_k = PDP^{-1}$,
where $P = \begin{pmatrix} 0.5 & -i \\ -0.5i & 1) \end{pmatrix}$, $P^{-1} = \begin{pmatrix} 1 & i \\ 0.5i & 0.5) \end{pmatrix}$, $D = \begin{pmatrix} \lambda_k^* & 0 \\ 0 & \lambda_k) \end{pmatrix}$,
$\lambda_k^* = Re(\lambda_k) - iIm(\lambda_k)$, $Re(\lambda_k) = \nu \frac{4\pi^2 k^2}{L^2}, Im(\lambda_k) = -\frac{2\pi k a}{L}$.
Then the matrix solution of (6.10) $A_k(t) = \exp(-\Lambda t)A_{k0} + \int_0^t \exp(-\Lambda(t-s)B_k(s)ds$
with the transformations $\tilde{A}_k(t) = P^{-1}A_k(t), A_k(t) = P\tilde{A}_k(t)$ we can obtain in the form

$$\tilde{A}_k(t) = \exp(-Dt)\tilde{A}_{k0} + \int_0^t \exp(-D(t-s)\tilde{B}_k(s)ds,$$

where $\tilde{A}_{k0} = P^{-1}A_{k0}, \tilde{B}_k(t) = P^{-1}B_k(t)$.
For this seperable we can determine the elements $\tilde{a}_{kc}(t), \tilde{a}_{ks}(t)$ of the column-vector $\tilde{A}_k$
depending on the elements $\tilde{a}_{kc}(0) = a_{kc}(0) + ia_{ks}(0), \tilde{a}_{ks}(0) = 0.5ia_{kc}(0) + 0.5a_{ks}(0)$,
$\tilde{b}_{kc}(t) = b_{kc}(t) + ib_{ks}(t), \tilde{b}_{ks}(t) = 0.5ib_{kc}(t) + 0.5b_{ks}(t)$
of the column-vectors
$\tilde{A}_{k0}, \tilde{B}_k(t)$ we obtain the solution of the ODEs system (6.8) in form

$$\begin{cases} a_{kc}(t) = \exp(-Re(\lambda_k)t)(a_{kc}(0)\cos(Im(\lambda_k)t) - a_{ks}(0)\sin(Im(\lambda_k)t)) + \\ \int_0^t \exp(-Re(\lambda_k)(t-s))(b_{kc}(s)\cos(Im(\lambda_k)(t-s)) - \\ b_{ks}(s)\sin(Im(\lambda_k)(t-s))ds, \\ a_{ks}(t) = \exp(-Re(\lambda_k)t)(a_{kc}(0)\sin(Im(\lambda_k)t) + a_{ks}(0)\cos(Im(\lambda_k)t)) + \\ \int_0^t \exp(-Re(\lambda_k)(t-s))(b_{kc}(s)\sin(Im(\lambda_k)(t-s)) + \\ b_{ks}(s)\cos(Im(\lambda_k)(t-s))ds. \end{cases} \tag{6.11}$$

For $k = 0$ we obtain $a_{0c}(t) = \int_0^t b_{0c}(s)ds + a_{0c}(0)$.

In the limit case $t \to \infty$ if the source function $f = f(x)$ not depending on t the we obtain from (6.10) the stacionary solution (see chapter1).

If $a = 0$ then we have the expressions (6.9 -6.11) with $(Re(\lambda_k) = \nu\left(\frac{2k\pi}{L}\right)^2, Im(\lambda_k) = 0.$

6.8.2 *Example with special date*

For numerical calculation we consider the initial boundary value problem with $f(x,t) = -2\pi a\cos(2\pi x)\exp(-4\pi^2 t), T_0 = \sin(\pi x), T(x,t) = \sin(\pi x)\exp(-4\pi^2 t), \nu = 1.$
From (6.8) follows:
$T(x,t) = a_1(t)\exp(2\pi ix) + a_{-1}(t)\exp(-2\pi ix), a_1(0) = \frac{1}{2i},$
$a_{-1}(0) = -\frac{1}{2i}, b_1(t) = b_{-1}(t) = -\pi a\exp(-4\pi^2 t),$
$\lambda_1 = 4\pi^2 - 2\pi ai, \lambda_{-1} = 4\pi^2 + 2\pi ai,$
$a_1(t) = \exp(-\lambda_1 t)/(2i) + (\exp(-2\pi ait) - 1)/(2i),$
$a_{-1}(t) = -\exp(-\lambda_{-1}t)/(2i) - (\exp(2\pi ait) - 1)/(2i).$
Therefore
$T(x,t) = (\exp(\lambda_1 t)\exp(2\pi ix) - \exp(\lambda_{-1}t)\exp(-2\pi ix))/(2i) +$
$\exp(-4\pi^2 t)(\sin(2\pi x) - \sin(2\pi(at+x))) = \exp(-4\pi^2 t)sin(2\pi x).$

From (6.11) we have:
$T(x,t) = a_{1c}\cos(2\pi x) + a_{1s}\sin(2\pi x), a_{1c}(0) = 0, a_{1s}(0) = 1,$
$b_{1s}(0) = 0, b_{1c}(0) = -2\pi a\exp(-4\pi^t), a_{1c}(t) = \exp(-\pi^2 t)(\sin(2\pi at) - \sin(2\pi at)),$
$a_{1s}(t) = \exp(-\pi^2 t)(\cos(2\pi at) + 1 - \cos(2\pi at)),$
$T(x,t) = \exp(-\pi^2 t)\sin(2\pi x).$

6.8.3 *Discrete problem with the Iljin FDS*

For the **discrete problem** we have the system of N ODEs in the form of (6.2)

$$\dot{U}(t) + \tilde{A}U(t) = F(t), \quad U(0) = U_0,$$

where the circulant matrix $\tilde{A} = \frac{\nu}{h^2}[2\gamma, -(\gamma+\alpha), 0, 0, ...0, -(\gamma-\alpha)]$,
with the eigenvalues $\tilde{\mu}_k = \frac{4\nu}{h^2}(\sin(k\pi/N)^2(\gamma - i\alpha\cot\frac{k\pi}{N}),$
$\gamma = \alpha\coth(\alpha), \alpha = \frac{ah}{2\nu}$ (the order of approximation $O(h^2)$)
and with the elements of the orthonormed eigenvectors
$w_j^k = \sqrt{\frac{1}{N}}\exp(2\pi ikj/N), w_{*j}^k = \sqrt{\frac{1}{N}}\exp(-2\pi ikj/N)k, j = \overline{1,N}.$

If $h \to 0$, then $\tilde{\mu}_k \to \lambda_k$.
For the column-vector $F(t)$ elements $f_j(t)$ we similarly from the chapter 1 obtain $f_j(t) = \sum_{k=1}^{*N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
where
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^{N} f_j(t)\cos\frac{2\pi kj}{N}, b_{ks}(t) = \frac{2}{N}\sum_{j=1}^{N} f_j(t)\sin\frac{2\pi kj}{N}, k = \overline{1,N_2}$,
$b_0(t) = b_N(t) = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} f_j(t),\ b_{0c}(t) = b_{Nc}(t) = \frac{2}{\sqrt{N}}b_0(t), b_{N_2,c}(t) = \frac{2}{\sqrt{N}}$
$b_{N_2}(t) = \frac{2}{N}\sum_{j=1}^{N}\cos(j\pi),\ N_2 = \frac{N}{2}, b_{N_2s}(t) = b_{Ns}(t) = 0,\ \sum_{k=1}^{*N_2}\beta_k = \sum_{k=1}^{N_2-1}\beta_k + \frac{\beta_{N/2}}{2}$.
For the solution
$u_j(t) = \sum_{k=1}^{*N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$
and
$u_j(0) = \sum_{k=1}^{*N_2}(a_{kc}(0)\cos\frac{2\pi kj}{N} + a_{ks}(0)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(0)}{2}$
with $a_{kc}(0) = \frac{2}{N}\sum_{j=1}^{N} u_j(0)\cos\frac{2\pi kj}{N},\ a_{ks}(0) = \frac{2}{N}\sum_{j=1}^{N} u_j(0)\sin\frac{2\pi kj}{N}$
we need determine the unknown functions $a_{kc}(t), a_{ks}(t)$ of the expressions
$f_j(t) = \dot{u}_j + (\tilde{A}u)_j = \sum_{k=1}^{*N_2}(\dot{a}_{kc}(t)\cos\frac{2\pi kj}{N} + \dot{a}_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{\dot{a}_{Nc}(t)}{2} + \sum_{k=1}^{*N_2}(a_{kc}(t)Re(\tilde{\mu}_k) + a_{ks}(t)Im(\tilde{\mu}_k))\cos\frac{2\pi kj}{N} + (a_{ks}(t)Re(\tilde{\mu}_k) - a_{kc}(t)Im(\tilde{\mu}_k))\sin\frac{2\pi kj}{N})$.

Therefore, for the determine the functions $a_{kc}(t), a_{ks}(t)$ we obtain the systems of ODEs (6.9,6.10) and the solution (6.11), where the eigenvalues λ_k are replaced with the discreate eigenvalues $\tilde{\mu}_k, k = \overline{1,N}$.

If $a = 0$ then $Re(\tilde{\mu}_k) = \frac{4\nu}{h^2}(\sin(k\pi/N)^2, Im(\tilde{\mu}_k) = 0$.

6.8.4 *Discrete problem in multi-points stencil*

Using the **multi-points stencil** with the order of approximation $O(h^{2n}), n = 1,2,3,\ldots$ (see chapter 1)we have
$\tilde{A} = \nu A - aA^0, \tilde{\mu}_k = \nu\mu_k - ai|\mu_k^0|$, where μ_k, μ_k^0 are the eigenvalues from circulant matrices A, A^0.
The **complex** expressions we can obtain from following matrices representation
$A = WDW^*, A^0 = WD^0W^*, D = diag(\mu_k), D^0 = diag(\mu_k^0)$.
Then the system of ODEs is

$$\dot{V}(t)+(\nu D-aD^0)V(t)=\tilde{F}(t),$$

where $V(t)=W^*U(t),U(t)=WV(t),\tilde{F}(t)=W^*F(t)$ or
$\dot{v}_k(t)+(\nu\mu_k-a\mu_k^0)v_k(t)=\tilde{f}_k(t)$ ($v_k(t),\tilde{f}_k(t)$
are the elements of column-vectors $V(t),\tilde{F}(t)$.)
We have followng solution:

$$v_k(t)=\exp(-\tilde{\mu}_k t)v_k(0)+\int_0^t\exp(-\tilde{\mu}_k(t-\tau)\tilde{f}_k(\tau)d\tau, k=\overline{1,N-1}$$

where $\tilde{\mu}_k=\nu\mu_k-a\mu_k^0, v_N(t)=v_N(0)+\int_0^t\tilde{f}_N(\tau)d\tau, v_k(0)=(W_*U(0))_k$.
The **real** solutions we can easily obtain from following expressions:
$Aw^k=\mu_k w^k, Aw_*^k=\mu_k w_*^k, A^0w^k=i|\mu_k^0|w^k, A^0w_*^k=-i|\mu_k^0|w_*^k$
or $A\cos_k=\mu_k\cos_k, A\sin_k=\mu_k\sin_k,\ A^0\cos_k=-|\mu_k^0|\sin_k, A^0\sin_k=|\mu_k^0|\cos_k$,
where $\cos_k,\sin_k$ are N-order column-vectors with the elements
$\cos\frac{2\pi kj}{N},\sin\frac{2\pi kj}{N}, j=\overline{1,N}$.
We have following properties for the scalar products:
$\frac{2}{N}\cos_k\cos_m=\delta_{k,m}$,
$\frac{2}{N}\sin_k\sin_m=\delta_{k,m},\frac{2}{N}\cos_k\sin_m=0$. Then $\dot{U}(t)=\sum_{k=1}^{*N_2}(\dot{a}_{kc}(t)\cos_k+\dot{a}_{ks}(t)\sin_k)+\frac{\dot{a}_{0c}(t)}{2}\cos_0$,
$U(t)=\sum_{k=1}^{*N_2}(a_{kc}(t)\cos_k+a_{ks}(t)\sin_k)+\frac{a_{0c}(t)}{2}\cos_0$,
$AU(t)=\sum_{k=1}^{*N_2}\mu_k(a_{kc}(t)\cos_k+a_{ks}(t)\sin_k), A\cos_0=0$,
$A^0U(t)=\sum_{k=1}^{*N_2}|\mu_k^0|(-a_{kc}(t)\sin_k+a_{ks}(t)\cos_k), A^0\cos_0=0$,
$F(t)=\sum_{k=1}^{*N_2}(b_{kc}(t)\cos_k+b_{ks}(t)\sin_k)+\frac{b_{0c}(t)}{2}\cos_0$,
$b_{kc}(t)=\frac{2}{N}F(t)\cos_k, b_{ks}(t)=\frac{2}{N}F(t)\sin_k$,
$U(0)=\sum_{k=1}^{*N_2}(a_{kc}(0)\cos_k+b_{ks}(0)\sin_k)+\frac{b_{0c}(0)}{2}\cos_0$,
where the unknow functions $a_{kc}(t),a_{ks}(t)$ are obtained from following ODEs

$$\begin{cases}\dot{a}_{kc}(t)+\nu\mu_k a_{kc}(t)-a|\mu_k^0|a_{ks}(t)=b_{kc}(t),\\ \dot{a}_{ks}(t)+\nu\mu_k a_{ks}(t)+a|\mu_k^0|a_{kc}(t)=b_{ks}(t),\\ a_{kc}(0)=\frac{2}{N}U_0\cos_k, a_{ks}(0)=\frac{2}{N}U_0\sin_k.\end{cases}\tag{6.12}$$

This problem is equal to (6.9), where $Re(\lambda_k)=\nu\mu_k, Im(\lambda_k)=-a|\mu_k^0|$.
Using the eigenvalues λ_k in (6.12) we have FDSES.
For **initial data** proportional to fixed frequency $k_0\le N/2$:
$U_0=a_c\cos_{k_0}+a_s\sin_{k_0}, F(t)=b_c(t)\cos_{k_0}+b_s(t)\sin_{k_0}$

we have
$a_{kc}(0) = a_c\delta_{k_0,k}, a_{ks}(0) = a_s\delta_{k_0,k}$,
$b_{kc}(t) = b_c(t)\delta_{k_0,k}, b_{ks}(t) = b_s(t)\delta_{k_0,k}$, and
$U(t) = a_{k_0c}(t)\cos_{k_0} + a_{k_0s}(t)\sin_{k_0}$,
where $a_{k_0c}(t), a_{k_0s}(t)$ are the solutions of ODEs (6.12) in the form (6.11), where $Re(\lambda_k) = \nu\mu_k, Im(\lambda_k) = -a|\mu_k^0|$.

6.8.5 Example with Euler-Newton FDS for solving the Cauchy problem

We consider the discrete homogenous heat transfer problem in multi-points stencil with the periodical boundary conditions (6.2). Using Euler-Newton FDS we have in every time step ($t_n = n\tau, n = 1,2,...$)

$$(U^{n+1} - U^n)/\tau = -\Gamma AU^n, n \geq 0, \Gamma = (\exp(-A\tau) - E)(-A\tau)^{-1}), \tag{6.13}$$

where
$U^n = U(t_n), U^0 = U_0, A, E$ are the N-order circular and unit matrices.
From (6.13) follows $U^{n+1} = \exp(-A\tau)U^n$ or $U^n = \exp(-At_n)U_0$.
For matrix A representation follows
$\exp(-A\tau)W = \exp(-D\tau)W$ or $\exp(-A\tau)w^k = \exp(-\mu_k\tau)w^k, k = \overline{1,N}$.
Then $V^n = W^*U^n, (U^n = WV^n), U^{n+1} = WV^{n+1}$,
$V^{n+1} = \exp(-D\tau)V^n, V^0 = W^*U_0$.
For real solutions we have
$\exp(-A\tau)\cos_k = \exp(-\mu_k\tau)\cos_k, \exp(-A\tau)\sin_k = \exp(-\mu_k\tau)\sin_k$
and $U^{n+1} = \sum_{k=1}^{*N_2}(a_{kc}^{n+1}\cos_k + a_{ks}^{n+1}\sin_k) + \frac{a_{0c}^{n+1}}{2}\cos_0$,
$U^n = \sum_{k=1}^{*N_2}(a_{kc}^n\cos_k + a_{ks}^n\sin_k) + \frac{a_{0c}^n}{2}\cos_0$,
$\exp(-A\tau)U^n) = \sum_{k=1}^{*N_2}\exp(-\mu_k\tau)(a_{kc}^n\cos_k + a_{ks}^n\sin_k) + \frac{a_{0c}^n}{2}\cos_0$,
$U_0 = \sum_{k=1}^{*N_2}(a_{kc}^0\cos_k + a_{ks}^0\sin_k) + \frac{a_{0c}^0}{2}\cos_0$,
$a_{kc}^0 = \frac{2}{N}U_0\cos_k, a_{ks}^0 = \frac{2}{N}U_0\sin_k$.
Therefore, we get
$a_{kc}^{n+1} = exp(-\mu_k\tau)a_{kc}^n, a_{ks}^{n+1} = exp(-\mu_k\tau)a_{ks}^n, n = 0,1,2,...,$
or $a_{kc}^n = exp(-\mu_k t_n)a_{kc}^0, a_{ks}^n = exp(-\mu_k t_n)a_{ks}^0$,
$U^n = \sum_{k=1}^{*N_2} exp(-\mu_k t_n)(a_{kc}^0\cos_k + a_{ks}^0\sin_k) + \frac{a_{0c}^0}{2}\cos_0$.

6.9 The system of parabolic type equations with the periodic BCs

We consider the initial - boundary problem of linear M-order system in following form:

$$\begin{cases} \frac{\partial T_m(x,t)}{\partial t} = \sum_{s=1}^{M} \frac{\partial}{\partial x}(k_{m,s}\frac{\partial T_m(x,t)}{\partial x}) + f_m(x,t), x \in (0,L), t \in (0,t_f), \\ T_m(0,t) = T_m(L,t), \frac{\partial T_m(0,t)}{\partial x} = \frac{\partial T_m(L,t)}{\partial x}, t \in (0,t_f), \\ T_m(x,0) = T_{m,0}(x), x \in (0,L), m = \overline{1,M}, \end{cases} \tag{6.14}$$

where $\tilde{K}$ is the positive definite matrix with the elements $k_{m,s}$,$T_{m,0}, f_m(x,t)$ are given functions.
This system we can rewriten in the matrix form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}(\tilde{K}\frac{\partial u(x,t)}{\partial x}) + f(x,t), x \in (0,L), t \in (0,t_f), \\ u(0,t) = u(L,t), \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x}, t \in (0,t_f), \\ u(x,0) = u_0(x), x \in (0,L), \end{cases} \tag{6.15}$$

where u, f are column-vectors with elements$T_m, f_m, m = \overline{1,M}$.
Using the Fourier series the solution we can obtained in the following form:
$u(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
$f(x,t) = \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{L}\int_0^L f(\xi,t)\cos\frac{2\pi k\xi}{L}d\xi, b_{ks}(t) = \frac{2}{L}\int_0^L f(\xi,t)\sin\frac{2\pi k\xi}{L}d\xi$,
where the column-vectors $a_{kc}(t), a_{ks}(t)$ of the M order are the corresponding solutions of the following differential equations

$$\begin{cases} \dot{a}_{kc}(t) + \lambda_k\tilde{K}a_{kc}(t) = b_{kc}(t), a_{kc}(0) = \frac{2}{L}\int_0^L u_0\xi)\cos\frac{2\pi k\xi}{L}d\xi, \\ \dot{a}_{ks}(t) + \lambda_k\tilde{K}a_{ks}(t) = b_{ks}(t), a_{ks}(0) = \frac{2}{L}\int_0^L u_0\xi)\sin\frac{2\pi k\xi}{L}d\xi, \end{cases} \tag{6.16}$$

where $b_{kc}(t), b_{ks}(t)$ are the column-vectors of M order.
The solution of this system is the vector functions

$$\begin{cases} a_{kc}(t) = \exp(-\lambda_k\tilde{K}t)a_{kc}(0) + \int_0^t exp(-\lambda_k\tilde{K}(t-\tau))b_{kc}(\tau)d\tau, \\ a_{ks}(t) = \exp(-\lambda_k\tilde{K}t)a_{ks}(0) + \int_0^t exp(-\lambda_k\tilde{K}(t-\tau))b_{ks}(\tau)d\tau. \end{cases} \tag{6.17}$$

For the **discrete problem** ($O(h^{2n})$ order of approximation) we have the system of ODEs in the following form:

$$\begin{cases} \dot{v}_j(t) = \tilde{K}\Lambda v(t) + f_j(t), t \in [0,t_f], \\ v_j(0) = u_0(x_j), x_j = jh, Nh = L, j = \overline{1,N}, \end{cases} \tag{6.18}$$

where the column-vectors of the M order $v_j(t) \approx u(x_j,t), f_j(t) = f(x_j,t)$,
the expression of the finite difference operator in multi-points stencil with 2n+1-points $(N \geq 2n+1)$
$\Lambda v_j = \frac{1}{h^2}(C_n(v_{j-n} + v_{j+n}) + ... + C_1(v_{j-1} + v_{j+1}) + C_0 v_j)$,
$C_0 = -\sum_{p=1}^{n} C_p$, $C_p = \frac{2(n!)^2(-1)^{p-1}}{p^2(n-p)!(n+p)!}, p = \overline{1,n}$ (see section 1).
We have following matrix representation for circulant matrix
$(A = -\Lambda) = -\frac{1}{h^2}[C_0, C_1, ..., C_n, 0, ..., 0, C_n, ..., C_1]$,
with the eigenvalues
$\mu_k = \frac{4}{h^2}\sum_{m=1}^{n} P_m \sin^{2m}(\pi k/N), P_m = \frac{2((m-1!))^2 4^{m-1}}{(2m)!}$.
Using the discrete Fourier method the solution we can obtained in the following form:
$v_j(t) = \sum_{k=1}^{N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$,
$f_j(t) = \sum_{k=1}^{N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^{N} f_j(t)\cos\frac{2\pi kj}{N}$, $b_{ks}(t) = \frac{2}{N}\sum_{j=1}^{N} f_j(t)\sin\frac{2\pi kj}{N}$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (6.16,6.17 , λ_k is replaced to μ_k) with
$a_{kc}(0) = \frac{2}{N}\sum_{j=1}^{N} u_0(x_j)\cos\frac{2\pi kj}{N}$, $a_{ks}(0) = \frac{2}{N}\sum_{j=1}^{N} u_0(x_j)\sin\frac{2\pi kj}{N}$.
For FDSES, replaced the discrete eigenvalue μ_k with λ_k we obtain the exact solutions for initial data with the frequency $\leq N/2$.

6.10 The system of parabolic type equations with convection

We consider the initial-boundary problem of linear M-order system in following form:

$$\begin{cases} \frac{\partial T_m}{\partial t} = \sum_{s=1}^{M}(\frac{\partial}{\partial x}(k_{m,s}\frac{\partial T_m}{\partial x}) + p_{m,s}\frac{\partial T_m}{\partial x}) + f_m, \\ T_m(0,t) = T_m(L,t), \frac{\partial T_m(0,t)}{\partial x} = \frac{\partial T_m(L,t)}{\partial x}, \\ T_m(x,0) = T_{m,0}(x), x \in (0,L), m = \overline{1,M}, \end{cases} \tag{6.19}$$

where $\tilde{K}$ is the positive definite M-order matrix with different positive eigenvalues $\mu_K > 0$ and the elements $k_{m,s}$, P is the real M-order matrix with different real eigenvalues μ_P and the elements $p_{m,s}, m,s = \overline{1,M}$..

This system we can rewriten in the matrix form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}(\tilde{K}\frac{\partial u(x,t)}{\partial x}) + P\frac{\partial u(x,t)}{\partial x} + f(x,t), \\ u(0,t) = u(L,t), \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x}, t \in (0,t_f), \\ u(x,0) = u_0(x), x \in (0,L), \end{cases} \tag{6.20}$$

where u, f are column-vectors with elements$T_m, f_m, m = \overline{1,M}$.
Using the Fourier series the solution we can obtained in the following form:
$u(x,t) = \sum_{k=1}^{\infty}(a_{kc}(t)\cos\frac{2\pi kx}{L} + a_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(t)}{2}$,
$f(x,t) = \sum_{k=1}^{\infty}(b_{kc}(t)\cos\frac{2\pi kx}{L} + b_{ks}(t)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{L}\int_0^L f(\xi,t)\cos\frac{2\pi k\xi}{L}d\xi$, $b_{ks}(t) = \frac{2}{L}\int_0^L f(\xi,t)\sin\frac{2\pi k\xi}{L}d\xi$,
where the column-vectors $a_{kc}(t), a_{ks}(t)$ of the M order are the corresponding solutions of the following differential equations

$$\begin{cases} \dot{a}_{kc}(t) + \lambda_k\tilde{K}a_{kc}(t) - \frac{2\pi k}{L}Pa_{ks}(t) = b_{kc}(t), \\ \dot{a}_{ks}(t) + \lambda_k\tilde{K}a_{ks}(t) + \frac{2\pi k}{L}Pa_{kc}(t) = b_{ks}(t), \end{cases} \tag{6.21}$$

where $b_{kc}(t), b_{ks}(t)$ are the column-vectors of M order,
$a_{kc}(0) = \frac{2}{L}\int_0^L u_0\xi)\cos\frac{2\pi k\xi}{L}d\xi$, $a_{ks}(0) = \frac{2}{L}\int_0^L u_0\xi)\sin\frac{2\pi k\xi}{L}d\xi$.
The solution of this system we can obtained in following form:

$$a_k(t) = \exp(-Rt)a_k(0) + \int_0^t exp(-R(t-\tau))b_k(\tau)d\tau, \tag{6.22}$$

where a_k, b_k are the column-vectors and $R = \begin{pmatrix} \lambda_k\tilde{K} & -\frac{2\pi k}{L}P \\ \frac{2\pi k}{L}P & \lambda_k\tilde{K} \end{pmatrix}$
the matrix of the $2M$ order.
For the **discrete problem** ($O(h^{2n})$ order of approximation) we have the system of ODEs in the following form:

$$\begin{cases} \dot{v}_j(t) = \tilde{K}\Lambda v_j(t) + P\Lambda^0 v_j(t) + f_j(t), t \in [0,t_f], \\ v_j(0) = u_0(x_j), x_j = jh, Nh = L, j = \overline{1,N}, \end{cases} \tag{6.23}$$

where the column-vectors of the M order $v_j(t) \approx u(x_j,t), f_j(t) = f(x_j,t)$,
the expressions of the finite difference operators in multi-points stencil with 2n+1-points $(N \geq 2n+1)$

$\Lambda v_j = \frac{1}{h^2}(C_n(v_{j-n}+v_{j+n})+...+C_1(v_{j-1}+v_{j+1})+C_0 v_j)$,
$C_0 = -\sum_{p=1}^n C_p, C_p = \frac{2(n!)^2(-1)^{p-1}}{p^2(n-p)!(n+p)!}, p = \overline{1,n}$
$\Lambda^0 v_j = \frac{1}{h}(c_n(v_{j+n}-v_{j-n})+...+c_1(v_{j+1}-v_{j-1}))$,
$c_p = \frac{(n!)^2(-1)^{p-1}}{p(n-p)!(n+p)!}, p = \overline{1,n}$ (see section 1).
We have following matrix representation for circulant matrices
$A = -\Lambda = -\frac{1}{h^2}[C_0,C_1,...,C_n,0,...,0,C_n,...,C_1]$,
with the eigenvalues
$\mu_k = \frac{4}{h^2}\sum_{p=1}^n Q_p \sin^{2p}(\pi k/N), Q_p = \frac{2((p-1!))^2 4^{p-1}}{(2p)!}$
and $A^0 = \Lambda^0 = \frac{1}{h}[0,c_1,...,c_n,0,...,0,-c_n,...,-c_1]$,
with the eigenvalues $\mu_k^0 = \frac{2i}{h}\sum_{p=1}^n c_p \sin\frac{2\pi pk}{N}$.
Using the discrete Fourier method the solution we can obtained in the following form:
$v_j(t) = \sum_{k=1}^{N_2}(a_{kc}(t)\cos\frac{2\pi kj}{N} + a_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(t)}{2}$,
$f_j(t) = \sum_{k=1}^{N_2}(b_{kc}(t)\cos\frac{2\pi kj}{N} + b_{ks}(t)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(t)}{2}$,
$b_{kc}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\cos\frac{2\pi kj}{N}$, $b_{ks}(t) = \frac{2}{N}\sum_{j=1}^N f_j(t)\sin\frac{2\pi kj}{N}$,
where $a_{kc}(t), a_{ks}(t)$ are the corresponding solutions of (6.21,6.22) and $\lambda_k, \frac{2\pi k}{L}$ are replaced with $\mu_k, Im(\mu_k^0)$,
$a_{kc}(0) = \frac{2}{N}\sum_{j=1}^N u_0(x_j)\cos\frac{2\pi kj}{N}$, $a_{ks}(0) = \frac{2}{N}\sum_{j=1}^N u_0(x_j)\sin\frac{2\pi kj}{N}$.
This real form we can obtain also from following expressions
$A\cos_k = \mu_k\cos_k, A^0\cos_k = -|\mu_k^0|\sin_k, A\sin_k = \mu_k\sin_k, A^0\sin_k = |\mu_k^0|\cos_k$,
where $\sin_k, \cos_k$ are N-order column-vectors
with the elements $\sin\frac{2\pi kj}{N}, \cos\frac{2\pi kj}{N}$
and using the orthonormed conditions
$\sum_{j=1}^N \sin_k\cos_s = 0, \sum_{j=1}^N \sin_k\sin_s = \sum_{j=1}^N \cos_k\cos_s = \frac{N}{2}\delta_{k,s}$.
Then for fixed frequency k in the initial data the solution can be write in the form $u(t) = d_s(t)\sin_k + d_c(t)\cos_k$, of vector-functions, where $d_s(t), d_c(t)$ are unknown the time-depending vector-functions.
Then $\dot{u}(t) = \dot{d}_s(t)\sin_k + \dot{d}_c(t)\cos_k$, $Au(t) = \mu_k(d_s(t)\sin_k + d_c(t)\cos_k)$, $A^0 u(t) = |\mu_k^0|(d_s(t)\cos_k - d_c(t)\sin_k)$.
For FDSES, replaced the discrete eigenvalue $\mu_k, Im(\mu_k^0)$ with $\lambda_k, \frac{2\pi k}{L}$ we obtain the exact solutions for initial data with the frequency $\leq N/2$.

6.11 Stability of approximations for time- dependent problems

The time-dependent difference equation (6.23) using the MN-order column-vector $v(t)$ with the elements $v_j^m(t), j=\overline{1,N}, m=\overline{1,M}$ in the form

$$\dot{v}(t) = (-\tilde{K}\bigotimes A + P\bigotimes A^0)v(t) + f(t) \tag{6.24}$$

serves as an approximation to the differential problem (6.20), in the sence that any smooth solution $u(t)$ satisfies the approximation (6.23) modulo a small local truncation error $\Psi(h,t) = O(h^{2n})$:

$$\frac{\partial u(t)}{\partial t} = (-\tilde{K}\bigotimes A)u(t) + (P\bigotimes B)u(t) + f(t) + \Psi(h,t), \tag{6.25}$$

where MN- order matrices

$$\tilde{K}\bigotimes A = \begin{pmatrix} k_{1,1}A & \cdots & k_{1,M}A \\ \cdots & \cdots & \cdots \\ k_{M,1}A & \cdots & k_{M,M}A \end{pmatrix}, P\bigotimes A^0 = \begin{pmatrix} p_{1,1}A^0 & \cdots & p_{1,M}A^0 \\ \cdots & \cdots & \cdots \\ p_{M,1}A^0 & \cdots & p_{M,M}A^0 \end{pmatrix},$$

are Kronecker tensor product, $u(t), u(0), f(t)$ are MN column-vectors with the elements
$u_j^m(t), u_j^m(0), f_j^m, m=\overline{1,M}, j=\overline{1,N}$.
Matrices can be defined with the representation $A = WDW^*, A^0 = WD^0W^*$, and solved numerically with the Matlab operator "kron".
In order to link the local order of accuracy with he desired global convergence rate of the approximation, one has to verify stability. We say that approximation (6.23) is stable, if for all sufficiently small h the following estimate holds

$$||\exp(Bt)|| \le C_{t_f}, 0 \le t \le t_f, C_{t_f}, \tag{6.26}$$

where $B = -\tilde{K}\otimes A + P\otimes A^0, C_{t_f} > 0-$ is constant.
If eigenvalues of matrices $\tilde{K}, P$ are $\lambda_s(\tilde{K}) > 0, \lambda_s(P), s=\overline{1,M}$ then exsist the transformation of M-order matrices W_K, W_P, and the representation $\tilde{K} = W_K D_K W_K^{-1}, P = W_P D_P W_P^{-1}$,
where $D_K = diag(\lambda_s(\tilde{K})), D_P = diag(\lambda_s(P))$
are the diagonal matrices. From properties of Kroneker tensor product follows $(W^* = W^{-1})$:
the eigenvalues $\lambda(B)$ of matrix B are

$-\mu_k\lambda_s(\tilde{K}) + \mu_k^0\lambda_s(P), k = \overline{1,N}, s = \overline{1,M}$ with the $Re(\lambda(B)) \le 0$ and the system of ODEs is stable.

6.12 System of nonlinear parabolic type equations

We consider the nonlinear system of M- heat transfer equation with periodical BCs in the following form:

$$\begin{cases} \frac{\partial T_m}{\partial t} = \sum_{s=1}^{M}(\frac{\partial}{\partial x}(k_{m,s}\frac{\partial g_{1,m}(T_m)}{\partial x}) + p_{m,s}\frac{\partial g_{2,m}(T_m}{\partial x}) + f_m g_{3,m}(T_m), \\ T_m(x,0) = T_{m,0}(x), x \in (0,L), m = \overline{1,M} \end{cases} \tag{6.27}$$

or

$$\frac{\partial T(x,t)}{\partial t} = \tilde{K}\frac{\partial^2 g_1(T(x,t))}{\partial x^2} + P\frac{\partial g_2(T(x,t))}{\partial x} + f(x)g_3(T(x,t)), \tag{6.28}$$

where $g_1(T), g_2(T), g_3(T)$ are the M-order column-vectors with the elements
$g_{1,m}(T_m), g_{2,m}(T_m), g_{3,m}(T_m), m = \overline{1,M}$,
We have following discrete form

$$\dot{u}(t) = (-\tilde{K}\bigotimes A)g_1(u(t)) + (P\bigotimes A^0)g_2(u(t)) + fg_3(u(t)). \tag{6.29}$$

An example for $M = 2$ we consider nonlinear power functions
$g_{1,1} = T^{\alpha_1}, g_{1,2} = T^{\alpha_2}, g_{2,1} = T^{\beta_1}, g_{2,2} = T^{\beta_2}$,
$g_{3,1} = T^{\gamma_1}, g_{3,2} = T^{\gamma_2}$.

1) If $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 3, \gamma_1 = \gamma_2 = 2$ (see Figs. 6.13, 6.14 with $t_f = 10$, we have stationary symmetric, periodic oscilations in the space), then for the maximal and minimal values of solutions u^1, u^2 (the minimal value is equal to maximal with oposite sign) depening on t and the values of the solutions depening on x by $t = t_f$, we have:
$N = 40$:
$0.17752; 0.10843(O(h^2)), 0.17755; 0.10847(O(h^4))$,
$0.17686; 0.10791(O(h^6))$,
$0.17715; 0.10822(O(h^8)), 0.17715; 0.10822(FDSES)$,
if N=80, then for FDSES and $O(h^8))$: 0.17723; 0.10827,
but for $O(h^2))$ 0.17732; 0.10832.

2) In the next Figs. 6.15, 6.16 for $\gamma_1 = \gamma_2 = 0, t_f = 0.1$(solutions tends fast to stationary).

3) In the next Figs. 6.17-6.20 for $\gamma_1 = \gamma_2 = 2, T_b = 10, L = 2;3;4$ (for L=3 the solutions tends slowly to stationary only by $t_f = 20$.)

4) For different value of $\alpha_1 = 3, \alpha_2 = 5, \beta_1 = \beta_2 = 3, \gamma_1 = \gamma_2 = 2, t_f = 10, L = 2$ results are represented in Fig. 6.21.
If $\alpha_1 = 5, \alpha_2 = 3$, then we have Fig. 6.22. In this case need be changed the following Matlab operators:
PDSper2- additions

```
[T1,Y1]=ode15s(@SIST1,[0,Tb],yy0,options,A1,A2,K,P,F2,N);
function F=SIST1(t,yy,A1,A2,K,P,F1,N)
F=-kron(K,A1)*[yy(1:N).^5; yy(N+1:2*N).^3]+. . .
kron(P,A2)*[yy(1:N).^3; yy(N+1:2*N).^3]+...
F1.*[yy(1:N).^2; yy(N+1:2*N).^2];
```

In Figs. 6.23, 6.24 are represented the results by $\alpha_1 = 5, \alpha_2 = 3$ for matrix $0.1\tilde{K}$ with the eigenvalues $0.1, 0.5$. We can see the oscilations in time ($N = 40, t_f = 5$).
We obtain following maximal values depending on $O(h^{2n}), n = 1,2,3,4$ and for FDSES:
$1.5461; 0.9288(O(h^2)), 1.4914; 0.8911(O(h^4))$,
$1.5061; 0.8969(O(h^6)), 1.4919; 0.8894(O(h^8)), 1.5081; 0.8945(FDSES)$.

5) Interesing results are obtained if $\tilde{K} = 0, \beta_1 = \beta_2 = 1$ and when the functions $g_{3,1}(T), g_{3,2}(T)$ are the trigonometric functions $\sin(T)$ or $\cos(T)$.
For $g_{3,1}(T) = g_{3,2}(T) = \sin(T), N = 80, L = t_f = 1$ we have following maximal and minimal values of solution depending on the order of approximations:
$2.71, -1.96; 2.14, -1.28(O(h^2)), 2.40, -1.93; 1.76, -1.33(O(h^4))$,
$2.39, -1.92; 1.73, -1.34(O(h^6))$,
$2.38, -1.88; 1.73, -1.32(O(h^8)), 2.38, -1.86; 1.72, -1.32(FDSES)$ and $(O(h^{20}))$ If $N = 20$ then for FDSES we have:
$2.37, -1.83; 1.73, -1.31$.

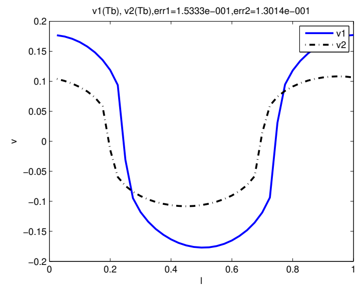


Fig. 6.13 Solutions by $t_f = 10, N = 40$ depending on x

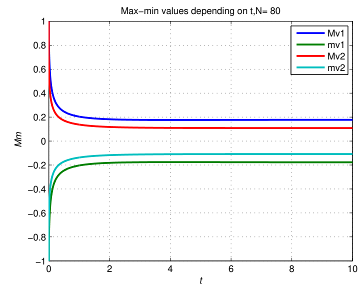


Fig. 6.14 Maximal and minimal values depending on t

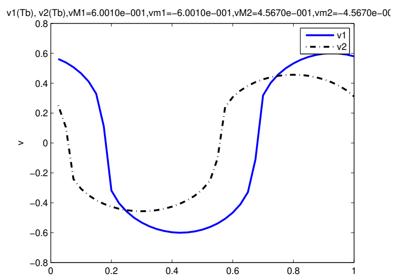


Fig. 6.15 Solutions by $t_f = 0.1, N = 40$ depending on x

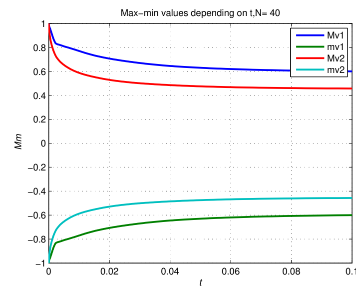


Fig. 6.16 Maximal and minimal values depending on t

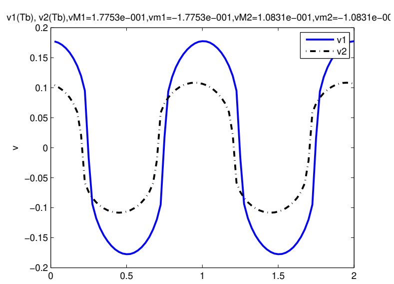


Fig. 6.17 Solutions by $t_f = 10, N = 80, L = 2$ depending on x

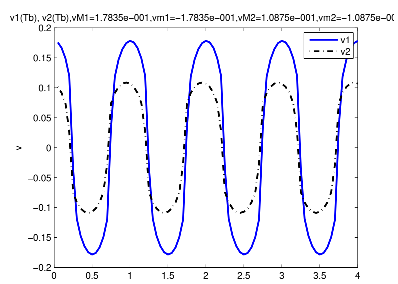


Fig. 6.18 Solutions by $t_f = 10, N = 80, L = 4$ depending on x

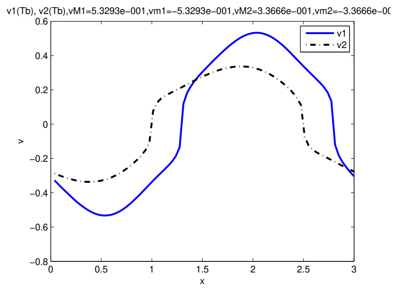


Fig. 6.19 Solutions by $t_f = 20, N = 80, L = 3$ depending on x

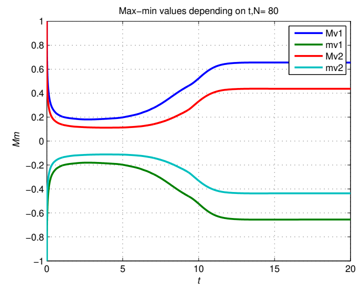


Fig. 6.20 Maximal and minimal values depending on t

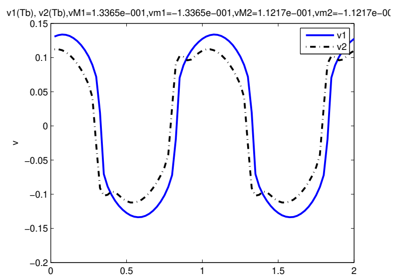


Fig. 6.21 Solutions by $t_f = 10, N = 80, L = 2, \alpha = 3; 5$ depending on x

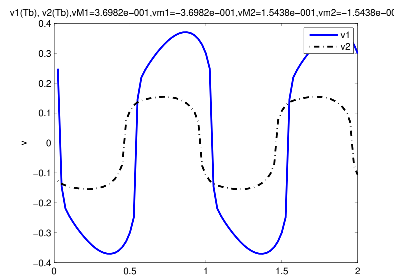


Fig. 6.22 Solutions by $t_f = 10, N = 80, L = 2, \alpha = 5; 3$ depending on x

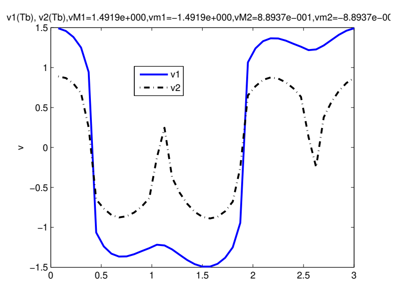


Fig. 6.23 Solutions by $t_f = 5, N = 40, L = 3, \alpha = 5; 3$ depending on x

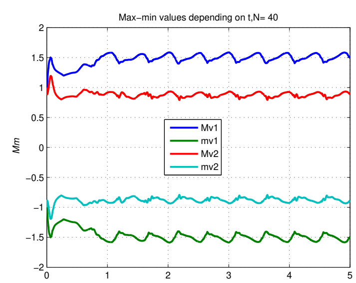


Fig. 6.24 Maximal and minimal values depending on t by $L = 3$

Chapter 7
Poisson equation: H. Kalis, I. Kangro, 2015 [83]

The solutions of the linear boundary value problem for Poisson equations are obtained analytically and numerically. Using the method of lines (lines are parallel to y axis) for periodical BC we define the FD-SES, where the finite difference matrix A is represented in the form form $A = WDW^*$
(W, D is the matrixes of finite difference eigenvectors and eigenvalues corespondently, W^* is the conjugate matrix and the elements of diagonal matrix D are replaced with the first eigenvalues from the differential operator.

7.1 The mathematical model

We consider the boundary value problem for Poisson equation with the periodical BCs in the x direction:

$$\begin{cases} \frac{\partial^2 T(x,y)}{\partial y^2} + \frac{\partial^2 T(x,y)}{\partial x^2} = f(x,y), x \in (0,L), y \in (0,H), \\ T(0,y) = T(L,y), \frac{\partial T(0,y)}{\partial x} = \frac{\partial T(L,y)}{\partial x}, y \in (0,H), \\ T(x,0) = T_l(x), T(x,H) = T_r(x), x \in (0,L), \end{cases} \tag{7.1}$$

where $T_l(x), T_r(x)$ are given BC function in the y direction. Similarly we can consider the problem with the periodical BCs in the both x and y directions:

$$\begin{cases} \frac{\partial^2 T(x,y)}{\partial y^2} + \frac{\partial^2 T(x,y)}{\partial x^2} = f(x,y), x \in (0,L), y \in (0,H), \\ T(0,y) = T(L,y), \frac{\partial T(0,y)}{\partial x} = \frac{\partial T(L,y)}{\partial x}, y \in (0,H), \\ T(x,0) = T(x,H), \frac{\partial T(x,0)}{\partial y} = \frac{\partial T(x,H)}{\partial y}, x \in (0,L). \end{cases} \quad (7.2)$$

This problem has unique solutions by $\int_0^H \int_0^L f(x,y)dxdy = 0$,
$T(x_0,y_0) = T_0$, where $x_0 \in [0,L], y_0 \in [0,H], T_0$ are fixed constants.
We consider uniform grid in the space $x_j = jh, j = \overline{0,N}, Nh = L$, where N is even number.
Using the finite differences of second order approximation for partial derivatives of second order respect to x we obtain the boundary value problem for system of ordinary differential equations (ODEs) in the following matrix form

$$\ddot{U}(y) - AU(y) = F(y), U(0) = U_l, U(H) = U_r, \quad (7.3)$$

$$\ddot{U}(y) - AU(y) = F(y), U(0) = U(H), \dot{U}(0) = \dot{U}(H), \quad (7.4)$$

where A is the 3-diagonal circulant matrix of N order,
$A = \frac{1}{h^2}[2,-1,0,0,...,-1]$ (see chapter 1), $U(y), \ddot{U}(y), F(y), U_l, U_r$ are the column-vectors of N order with the elements $u_j(y) \approx T(x_j,y))$,
$\ddot{u}_j(y) \approx \frac{\partial^2 T(x_j,y)}{\partial y^2}$, $f_j(y) = f(x_j,y),)$
$u_j(0) = T_l(x_j), u_j(H) = T_r(x_j), j = \overline{0,N}$.

The corresponding discrete spectral problem $Aw^n = \mu_n w^n, n = \overline{1,N}$ with circulant matrix have following solution:

$$\begin{cases} w^n = \sqrt{1/N}(w_1^n, w_2^n, ..., w_N^n)^T, \\ \mu_n = \frac{4}{h^2}\sin^2(\pi nh/L), \end{cases} \quad (7.5)$$

where $w_j^n = \phi_n(x_j) = \exp(2\pi i n x_j/L), j = \overline{1,N}, i = \sqrt{-1}$ are the components of orthonormed eigenvector w^n. The eigenvalues of matrix A for different order of approximation $O(h^k), k \geq 2$ are (see chapter1):
1) $k = 4, h^2$
$A = [-\frac{5}{2}, \frac{4}{3}, -\frac{1}{12}, 0, \cdots, 0, -\frac{1}{12}, \frac{4}{3}]$,
$h^2\mu_n = -4(\sin^2(\pi nh/L) + \frac{1}{3}\sin^4(\pi nh/L))$,
2) $k = 6, h^2$
$A = [-\frac{49}{18}, \frac{3}{2}, -\frac{3}{20}, \frac{1}{90}, 0, \cdots, 0, \frac{1}{90}, -\frac{3}{20}, \frac{3}{2}]$,
$h^2\mu_n = -4(\sin^2(\pi nh/L) + \frac{1}{3}\sin^4(\pi nh/L) + \frac{8}{45}\sin^6(\pi nh/L))$, 3) $k = 8, h^2$

$A = [-\frac{205}{72}, \frac{8}{5}, -\frac{1}{5}, \frac{8}{315}, -\frac{1}{560}, 0, \cdots, 0, -\frac{1}{560}, \frac{8}{315}, -\frac{1}{5}, \frac{8}{5}]$.
$h^2\mu_n = -4(\sin^2(\pi nh/L) + \frac{1}{3}\sin^4(\pi nh/L) +$
$\frac{8}{45}\sin^6(\pi nh/L) + \frac{4}{35}\sin^8(\pi nh/L))$,
Therefore the matrix A can be represented in form $A = WDW^*$, where the column of the matrix W and the diagonal matrix D contains N orthonormed eigenvectors w^n and eigenvalues $\mu_n, n = \overline{1,N}$ ($W^*W = E$, $W^{-1} = W^*$).
The solution of the spectral problem for differential equations

$$-w''(x) = \lambda w(x), x \in (0,L), w(0) = w(L), w'(0) = w'(L),$$

is in following form:
$w_n(x) = L^{-1}\phi_n(x) = L^{-1}\exp(2\pi inx/L), \lambda_n = (2\pi n/L)^2, n > 0$.

7.2 The analytical solution

We can consider the **analytical solutions** of the system of ODEs (7.3,7.4) using the spectral representation of matrix $A = WDW^*$. From transformation $V = W^*U (U = WV)$ follows the seperate system of ODEs

$$\begin{cases} \ddot{V}(y) - DV(y) = G(y), V(0) = W^*U_l, V(H) = W^*U_r, \\ \ddot{V}(y) - DV(y) = G(y), V(0) = V(H), \dot{V}(0) = \dot{V}(H), \end{cases} \tag{7.6}$$

where $V(y), \ddot{V}(y), \dot{V}(0), \dot{V}(H), V(0), V(H), G(y) = W^*F(y)$
are the column-vectors of N order with elements
$v_k(y), \ddot{v}_k(y), \dot{v}_k(0), \dot{v}_k(H), v_k(0), v_k(H), g_k(y) k = \overline{1,N}$.
The solution of the system (7.6) is the sum of two solutions function (with homogenous equation and with homogenous BC)

$$\begin{cases} v_k(y) = (\sinh(\kappa_k H))^{-1}[v_k(0)\sinh(\kappa_k(H-y)) + v_k(H)\sinh(\kappa_k y)] - \\ \int_0^H G_k(\xi, y) g_k(\xi) d\xi, \end{cases} \tag{7.7}$$

where $\kappa_k = \sqrt{\mu_k}, G_k(\xi, y)$ is the Green function in following way:

$$G_k(\xi, y) = \begin{cases} \frac{\sinh(\kappa_k(H-y))\sinh(\kappa_k\xi)}{\kappa_k \sinh(\kappa_k H)}, \ 0 \le \xi \le y, \\ \frac{\sinh(\kappa_k(H-\xi))\sinh(\kappa_k y)}{\kappa_k \sinh(\kappa_k H)}, y \le \xi \le H. \end{cases}$$

For $\mu_N = 0$ from 7.7) follows

$$v_N(y) = v_N(0) + \frac{y}{H}(v_N(H) - v_N(0)) - \int_0^H G_N(\xi, y) g_N(\xi) d\xi,$$

where

$$G_N(\xi, y) = \begin{cases} \frac{(H-y)\xi}{H}, \ 0 \le \xi \le y, \\ \frac{(H-\xi)y}{H}, \ y \le \xi \le H. \end{cases}$$

The general solution is following

$$v_k(y) = C_k \sinh(\kappa_k y) + B_k \cosh(\kappa_k y) + \frac{1}{\kappa_k}\int_0^y g_k(\xi)\sinh(\kappa_k(y-\xi))d\xi,$$

where C_k, B_k are constants. Using the BCs we have
$C_k = \frac{0.5}{\kappa_k \sinh(0.5\kappa_k H)} \int_0^H g_k(\xi)\sinh(\kappa_k(\xi - 0.5H))d\xi,$
$B_k = -\frac{0.5}{\kappa_k \sinh(0.5\kappa_k H)} \int_0^H g_k(\xi)\cosh(\kappa_k(\xi - 0.5H))d\xi.$
In this case the analytical solution (7.7) is

$$v_k(y) = -0.5(\kappa_k \sinh(0.5\kappa_k H))^{-1} \int_0^H G_k(\xi, y) g_k(\xi) d\xi, \qquad (7.8)$$

where $G_k(\xi, y)$ is the Green function in following way:

$$G_k(\xi, y) = \begin{cases} \cosh(\kappa_k(H/2 - y + \xi)), \ 0 \le \xi \le y, \\ \cosh(\kappa_k(H/2 - \xi + y)), \ y \le \xi \le H. \end{cases}$$

We can used also the **Fourier method** for solving (7.1) in the form $T(x,y) = \sum_{k\in Z} v_k(t) w_k(x)$, where $w_k(x)$ are the orthonormed eigenfunctions, $v_k(y)$ is the solution (7.7),with $v_k(0) = (T_l, w_k^*), v_k(H) = (T_r, w_k^*)$.

The solution we can also obtained in real form:

$$\begin{cases} T(x,y) = \sum_{k=1}^{\infty}(a_{kc}(y)\cos\frac{2\pi kx}{L} + a_{ks}(y)\sin\frac{2\pi kx}{L}) + \frac{a_{0c}(y)}{2}, \\ f(x,y) = \sum_{k=1}^{\infty}(b_{kc}(y)\cos\frac{2\pi kx}{L} + b_{ks}(y)\sin\frac{2\pi kx}{L}) + \frac{b_{0c}(y)}{2}, \\ b_{kc}(y) = \frac{2}{L}\int_0^L f(\xi,y)\cos\frac{2\pi k\xi}{L}d\xi, b_{ks}(y) = \frac{2}{L}\int_0^L f(\xi,y)\sin\frac{2\pi k\xi}{L}d\xi, \end{cases} \qquad (7.9)$$

where $a_{kc}(y), a_{ks}(y)$ are the corresponding solutions of (7.7, 7.8) by
$a_{kc}(0) = \frac{2}{L}\int_0^L T_l(\xi)\cos\frac{2\pi k\xi}{L}d\xi$, $a_{ks}(0) = \frac{2}{L}\int_0^L T_l(\xi)\sin\frac{2\pi k\xi}{L}d\xi$,
$a_{kc}(H) = \frac{2}{L}\int_0^L T_r(\xi)\cos\frac{2\pi k\xi}{L}d\xi$, $a_{ks}(H) = \frac{2}{L}\int_0^L T_r(\xi)\sin\frac{2\pi k\xi}{L}d\xi$,
$g_k(y) = b_{kc}(y)$ or $b_{ks}(y)$.

Similarly we can obtained the solution of the discrete problem also in the real form:

$$\begin{cases} u_j(y) = \sum_{k=1}^{*N_2}(a_{kc}(y)\cos\frac{2\pi kj}{N} + a_{ks}(y)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(y)}{2}, \\ f_j(y) = \sum_{k=1}^{N_2}(b_{kc}(y)\cos\frac{2\pi kj}{N} + b_{ks}(y)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(y)}{2}, \\ b_{kc}(y) = \frac{2}{N}\sum_1^N f_j(y)\cos\frac{2\pi kj}{N}, b_{ks}(y) = \frac{2}{N}\sum_1^N f_j(y)\sin\frac{2\pi kj}{N}, \end{cases} \tag{7.10}$$

where $a_{kc}(y), a_{ks}(y)$ are the corresponding solutions of (7.7, 7.8) by
$a_{kc}(0) = \frac{2}{N}\sum_1^N T_l(x_j)\cos\frac{2\pi kj}{N}$, $a_{ks}(0) = \frac{2}{N}\sum_1^N T_l(x_j)\sin\frac{2\pi kj}{N}$,
$a_{kc}(H) = \frac{2}{N}\sum_1^N T_r(x_j)\cos\frac{2\pi kj}{N}$, $a_{ks}(H) = \frac{2}{N}\sum_1^N T_r(x_j)\sin\frac{2\pi kj}{N}$,
$g_k(y) = b_{kc}(y)$ or $b_{ks}(y), \sum_{k=1}^{*N_2}\beta_k = \sum_{k=1}^{N_2-1}\beta_k + \frac{\beta_{N/2}}{2}$.
(for FDSES μ_k are replaced with λ_k).

For the FDSES the matrix A is represented in the form form $A = WDW^*$ and the diagonal matrix D contain the first N eigenvalues $d_k = \lambda_k$, $k=\overline{1,N}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$ in following way:
1)$d_k = \lambda_k$ for $k = \overline{1,N_2}$, where $N_2 = N/2$.
2)$d_k = \lambda_{N-k}$ for $k = \overline{N_2,N}$.
If $d_k = \mu_k$, then we have the method of FDS.
The FDSES method is more stable as the method of finite difference by approximation with central difference (FDS), because the eigenvalues are larger $d_k > \mu_k$.

7.3 The analytical solution in the matrix form

For homogenous equations $(F = 0)$ (7.3) we can obtain the solution in the form

$$U_1(y) = \sinh^{-1}(A_1H)[\sinh(A_1y)U_r + \sinh(A_1(H-y))U_l],$$

where $A_1 = \sqrt{A}$.
For homogenous BC $(U_l = U_r = 0)$ the solution is
$U_2(y) = -\int_0^H G(\xi,y)F(\xi)d\xi$,

where

$$G(\xi,y)=\begin{cases}\sinh(A_1(H-y))\sinh(A_1\xi)(A_1\sinh(A_1H))^{-1},\ 0\le\xi\le y,\\ \sinh(A_1(H-\xi))\sinh(A_1y)(A_1\sinh(A_1H))^{-1},\ y\le\xi\le H\end{cases}$$

is Green matrix-function.
The solution of the problem (7.3) is $U(y)=U_1(y)+U_2(y)$. Multiply this solution left with $W^{-1}=W^*$, and using the expressions $A=WDW^*, f(A)=Wf(D)W^*, W^*U_l=V(0), W^*U_r=V(H), W^*F=G(y)$ (f is every function) we obtain the column-vector $V(y)$ with components (7.7). If $det(A)=det(A_1)=0$ then we can not direct obtained the solution of the matrix form.
For periodical BCs are given also in the y direction, then from (7.4) we have the following vector-solution

$$U(y)=-0.5A_1^{-1}\sinh^{-1}(0.5A_1H)\int_0^H G(\xi,y)F(\xi)d\xi,$$

where

$$G(\xi,y)=\begin{cases}\cosh(A_1(H/2-y+\xi)),\ 0\le\xi\le y,\\ \cosh(A_1(H/2+y-\xi)),\ y\le\xi\le H\end{cases}$$

For $\mu_N=0$ from 7.7) follows

$$v_N(y)=v_N(0)+\frac{y}{H}(v_N(H)-v_N(0))-\int_0^H G_N(\xi,y)g_N(\xi)d\xi,$$

where

$$G_N(\xi,y)=\begin{cases}\frac{(H-y)\xi}{H},\ 0\le\xi\le y,\\ \frac{(H-\xi)y}{H},\ y\le\xi\le H.\end{cases}$$

For $\mu_N=0$ from 7.8) follows $v_N(y)=0$.

7.4 Some examples and numerical results

7.4.1 The boundary value problem with the periodical BC in one direction

For numerical calculation we consider the boundary value problem (7.1) with $H = L = 1, f = 0, T_l = 0, T_r(x) = \sinh(2\pi)\cos(2\pi x), T(x,y) = \sinh(2\pi y)\cos(2\pi x)$.
Using the Fourier method we obtain $v_k(0) = 0, v_k(H) = 0$ for $k \neq \pm 1$, $v_{\pm 1}(H) = \frac{\sinh(2\pi)}{2i}$, $v_{\pm 1}(y) = \frac{\sinh(2\pi y)}{2}$, $T(x,y) = \cos(2\pi x)\sinh(2\pi y)$.
We have following MATLAB m.file **PuasPer:**

```
%system ODE U_yy-AU=f with periodical BC
%t=Tb,u(x,y)=cos(2 pi x)sinh(2 pi y),f=0,N-even
function PuasPer(N,M)
N1=N+1; H=1;L=1;x=linspace(0,L,N1)';y=linspace(0,H,M);
h=L/N;N2=N-1;x=x(2:N1);
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+...
%2*diag(ones(N,1));
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A, control
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
%NH=N/2; d(1:NH)=lk0(1:NH);
%d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
A2=W*diag(d)*W1; %FDS or FDSES
yr=sinh(2*pi*L)*cos(2*pi*x);
yl=zeros(N,1); % bound-cond
P=W1*yl;P1=zeros(M,N);P0=W1*yr;
for k=1:N2
    b=sqrt(d(k)); %FDS or FDSES
P1(:,k)=P(k)*cosh(b*y')+(P0(k)-P(k)*cosh(b*H))/...
sinh(b*H)*sinh(b*y');
end
P1(:,N)=P(N)+(P0(N)-P(N))*y'/H;
P2=(W*P1')';
prec=cos(2*pi*x)*sinh(2*pi*y);% exact
Ma1=max(max(abs(P2-prec')));%max error an.
```

```
X1=ones(M,1)*x';Y1=y'*ones(1,N);
figure,plot(y',max(abs(P2(:,1:N)'-prec)),'k*')% max error on y
title(sprintf('err. Max-sol.an.on y, Max=%9.7f ',Ma1))
xlabel('\ity'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec'))% error anl.
colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('err. anal.,yNr.=%4.1f,max=%9.7f',M,Ma1))
```

Using the operator **PuasPer(40,10)** we obtan maximal errors:
0.0956 (FDS-$O(h^2)$),(see Fig. 7.1)
0.00031 (FDS- $O(h^4)$), 1.2 10^{-6} (FDS -$O(h^6)$), 5. 10^{-9} (FDS-$O(h^8)$), 6.10^{-12} (FDSES) (see Fig. 7.2).
By $N = M = 10$ we obtain 3.10^{-4} (FDS-$O(h^8)$), 10^{-12} (FDSES).

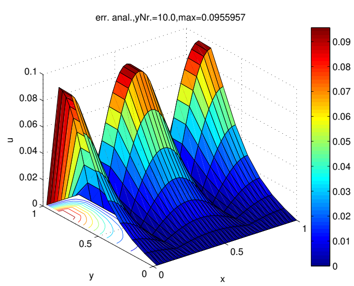


Fig. 7.1 Error with FDS by $N = 40, M = 10, O(h^2)$

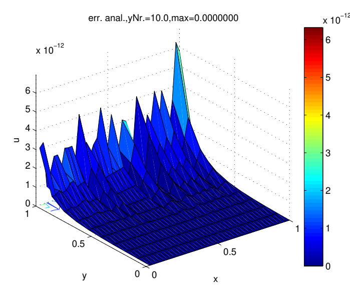


Fig. 7.2 Error with FDSES by $N = 40, M = 10$

7.4.2 The matrix- solution of boundary value problem with the periodical BC in two direction

Using the periodical BCs in both directions with right sides function $f(x,y) = -8\pi^2\cos(2\pi x)\cos(2\pi y)$ by $L = H = 1$ we have the exact solution $T(x,y) = \cos(2\pi x)\cos(2\pi y)$.
From the approximated solution follows that $U(y) = 8\pi^2\cos(2\pi y)(A_1^2 + 4\pi^2 E)^{-1}g$,
where E is the unit matrix of the N order,
$g = (\cos(2\pi x_1), \cos(2\pi x_2), \cdots, \cos(2\pi x_N))^T$ is the column-vector of the N order.
From (7.8) follows that

$g_k(\xi) = -8\pi^2 \frac{cos(2\pi\xi)}{\sqrt{N}} \sum_{j=1}^{N} \exp(-2\pi j x_k)\cos(2\pi x_j) = -8\pi^2 \frac{\sqrt{N}}{2}\cos(2\pi\xi)$
for $k = 1$ and $k = N-1$.
For other numbers of k we have $g_k(\xi) = 0$. We use the integrals $\int \cosh(a\xi + b)\cos(c\xi)d\xi =$
$\frac{a}{a^2+c^2}\sinh(a\xi + b)\cos(c\xi) + \frac{c}{a^2+c^2}\cosh(a\xi + b)\sin(c\xi)$,
where $a = \pm\kappa_k, b = H/2 \pm y, c = 2\pi$.
Then
$v_1(y) = v_{N-1} = 4\pi^2\sqrt{N}\cos(2\pi y)/(\kappa^2 + 4\pi^2)$, where $\kappa = \kappa_1 = \kappa_{N-1}$.
Therefore the components of the approximated solution U are
$u_j(y) = \frac{1}{\sqrt{N}} v_1(y)(\exp(2\pi x_j) + \exp(2\pi x_j(N-1))) = \frac{8\pi^2}{\kappa_1^2+4\pi^2}\cos(2\pi y)\cos(2\pi x_j)$.
We have following MATLAB m.file **”puas4”**(N=M) for matrix solution:

```
% u_yy= -u_xx+f, period. BC in x and y direc.
%u(x,y)=cos(2px)cos(2py)- exact sol.
function puas4(N)
N2=N-1;N1=N+1;h=1/N;L=1;H=1;
x=linspace(0,L,N+1);NT=(1:N)'/L;
y=linspace(0,H,N+1);
x=x(2:N1);y=y(2:N1)';
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2; %exact eig-val.
d=lk; %FDS
%NH=N/2; d(1:NH)=lk0(1:NH);% FDSES
%d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x/L)';
B1=W*diag(-d)*W1; %FDS or FDSES
%B1=zeros(N,N);
%B1=B1-1/12*diag(ones(N2-1,1),2)+4/3*diag(ones(N2,1),1)...
%-1/12*diag(ones(N2-1,1),-2)+. . .
%4/3*diag(ones(N2,1),-1)-5/2*diag(ones(N,1));
%B1(1,N)=4/3; B1(N,1)=4/3;B1(1,N2)=-1/12; B1(2,N)=-1/12;
%B1(N,2)=-1/12; B1(N2,1)=-1/12;% control O(h^4)
%B1=B1-2*diag(ones(N,1))+diag(ones(N2,1),-1)+...
% diag(ones(N2,1),1);
%B1(1,N)=1;B1(N,1)=1; %control O(h^2)
%B1=B1/(h^2);
e1=eye(N);
B2=sqrtm(-B1);B3=inv(B2^2+4*pi^2*e1);
B=8*pi^2*B3;prec=cos(2*pi*x')*cos(2*pi*y');
```

```
for iy=1:N
y1=y(iy,1);
g0=cos(2*pi*x)*cos(2*pi*y1);
u(:,iy)=B*g0';
end
Amax=max(max(abs(u'-prec')));
X=ones(N,1)* x; Y=y*ones(1,N);
surfc(X,Y,abs(u'-prec'))
colormap(gray),
colorbar
xlabel('x'), ylabel('y'), zlabel('u')
view(135,45)
title(sprintf('Period.BC in both direc. FDS O(h^2),. . .
h=%4.2f, Error=%8.6e',h,Amax))
```

Using the operator **puas4(10)**we obtan following maximal errors:
0.016502 (FDS-$O(h^2)$), (see Fig. 7.3);
0.000837 (FDS- $O(h^4)$), (see Fig. 7.4); 52. 10^{-6} (FDS -$O(h^6)$), (see Fig. 7.5);
4. 10^{-6} (FDS-$O(h^8)$),(see Fig. 7.6); 5.10^{-15} (FDSES) (see Fig. 7.7).
In the Fig. 7.8 is represented the exact solution.

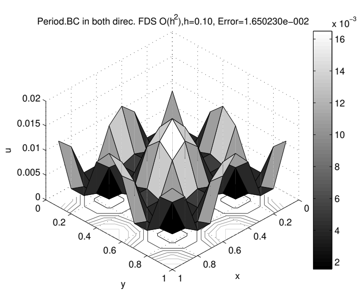


Fig. 7.3 Error with FDS by $N = 10, O(h^2)$

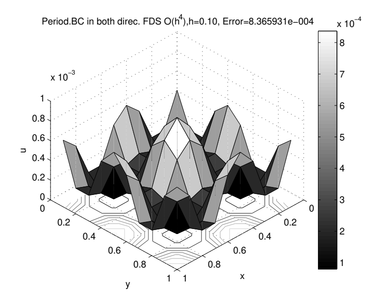


Fig. 7.4 Error with FDS by $N = 10, O(h^4)$

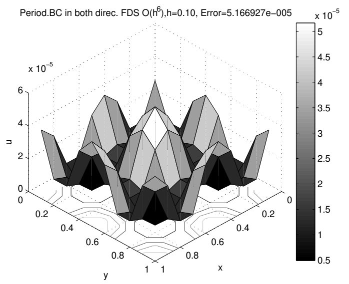


Fig. 7.5 Error with FDS by $N = 10, O(h^6)$

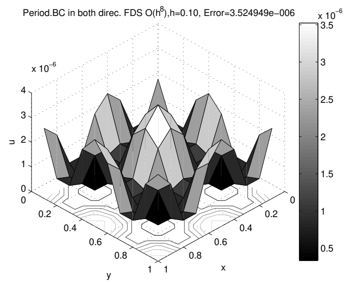


Fig. 7.6 Error with FDS by $N = 10, O(h^8)$

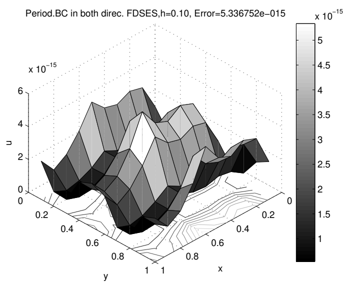


Fig. 7.7 Error with FDSES by $N = 10$

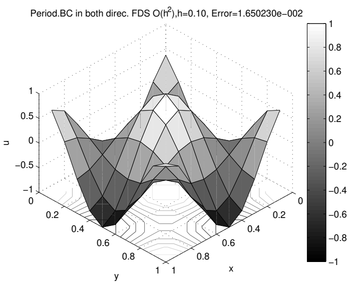


Fig. 7.8 Exact solution by $N = 10$

Using ihe MATLAB operators

```
B1=zeros(N,N);
B1=B1-1/12*diag(ones(N2-1,1),2)+4/3*diag(ones(N2,1),1)...
-1/12*diag(ones(N2-1,1),-2)+. . .
4/3*diag(ones(N2,1),-1)-5/2*diag(ones(N,1));
B1(1,N)=4/3; B1(N,1)=4/3;B1(1,N2)=-1/12; B1(2,N)=-1/12;
B1(N,2)=-1/12; B1(N2,1)=-1/12;% control O(h^4)
B1=B1-2*diag(ones(N,1))+diag(ones(N2,1),-1)+...
diag(ones(N2,1),1);
B1(1,N)=1;B1(N,1)=1; %control O(h^2)
B1=B1/(h^2);
```

in the m.file **puas4** for approximation orders $O(h^2), O(h^4)$ the results are remained.

7.4.3 The analytical solution of boundary value problem with the periodical BC in two direction

In the m.file **PuasPer2** is used the analytical solution (7.8) with the quadrature trapezoid formula for calculation of integrals
$\int_0^{y_j} F_1(y_j,t)dt \approx \frac{h_1}{2}(F_1(y_j,0)+F_1(y_j,y_j))+h_1\sum_{m=1}^{j-1}F_1(y_j,y_m),\, j=\overline{1,M}$,
$\int_{y_j}^{H} F_2(y_j,t)dt \approx \frac{h_1}{2}(F_2)(y_j,H)+F_2(y_j,y_j))+h_1\sum_{m=j+1}^{M-1}F_2(y_j,y_m)$,
$j=\overline{1,M-1}, \int_{y_M}^{H} F_2(y_j,t)dt=0$,
where $F_1(y,t)=\cosh(\kappa_k(0.5*H-y+t))g_k(t), F_2(y,t)=\cosh(\kappa_k(0.5*H+y-t))g_k(t), y_j=j*h_1, h_1=H/M$

```
%system ODE U_yy-AU=f with periodical BC in 2 direct,.an. sol.
%u(x,y)=cos(2 pi x)cos(2 pi y),f=-8 \pi^2 u(x,y),N-even
function PuasPer2(N,M)
N1=N+1; M1=M+1;H=1;L=1;x=linspace(0,L,N1)';
y=linspace(0,H,M1);
h=L/N;N2=N-1;M2=M-1;x=x(2:N1);y=y(2:M1)';h1=H/M;
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+...
%2*diag(ones(N,1));
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A,O(h^2)
NT=(1:N)'/L;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
Ck=sqrt(h/L);
lk0=(2*(1:N)'*pi/L).^2;
d=lk; %FDS
NH=N/2; d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
A2=W*diag(d)*W1; %FDS or FDSES
f=-8*pi^2*cos(2*pi*y)*cos(2*pi*x');g=W1*f';
gk=zeros(M,1);
P1=zeros(M,N);
for k=1:N
    gk(:)=g(k,:);
    b=sqrt(d(k)); %FDS or FDSES
    for j=1:M
    if j==M v2(j)=0;end
    s1=0; for j1=1:j-1
s1=s1+F1(j*h1,j1*h1,b,H)*gk(j1); end ;
v1(j)=0.5*h1*(F1(j*h1,0,b,H)*gk(M)+F1(j*h1,j*h1,b,H)*gk(j))...
+h1*s1;
    s2=0;for j1=j+1:M-1
```

```
s2=s2+F2(j*h1,j1*h1,b,H)*gk(j1); end
    if j≠M
v2(j)=0.5*h1*(F2(j*h1,H,b,H)*gk(M)+F2(j*h1,j*h1,b,H)*gk(j))...
+h1*s2;end
if k ≠ N  P1(j,k)=-0.5 /(b*sinh(0.5*H*b))*(v1(j)+v2(j));end
    if k==N P1(j,k)=0; end
    end; end
P2=(W*P1')';
P2=P2-P2(M,N)+1;
prec=cos(2*pi*y)*cos(2*pi*x');im =max(max(abs(imag(P2))))
Ma1=max(max(abs(P2-prec)));%max error an.
X1=ones(M,1)*x';Y1=y*ones(1,N);
figure,plot(y,max(abs(P2(:,1:N)'-prec')),'k*')% max error on y
title(sprintf('err. Max-sol.an.on y, Max=%9.7f ',Ma1))
xlabel('\ity'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec))% error anl.
colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('err. anal.,yNr.=%4.1f,max=%9.7f',M,Ma1))
function f=F1(y,t,b,H)
f=cosh(b*(0.5*H -y +t));
function f=F2(y,t,b,H)
f=cosh(b*(0.5*H +y -t));
```

Using the operator **PuaPer2(10,200)**we obtain following maximal errors :
0.0333 (FDS-$O(h^2)$), 0.0020 (FDS- $O(h^4)$), 0.00043 (FDS -$O(h^6)$), 0.00034 (FDS-$O(h^8)$), 0.00033 (FDSES).
For $N = 10, FDS - O(h^2)$ and diferent M follows: 0.0351 (M=80), 0.0335 (M=160), 0.0333 (M=200).

7.4.4 The Kronecker-tensor solution of the problem with the periodical BC in two direction

In the m.file **PuasTen2** is used the Kroneker -tensor method for the solution (7.2) .We consider also uniform grid in the y direction $y_m = mh_1, m = \overline{0,M}, Mh_1 = H$, where M is even number.
Using the finite differences of second order approximation for partial derivatives of second order respect to x, y we obtain the system of linear algebraical equations of the NxM order in the following matrix form

$$Au = -g, \tag{7.11}$$

where $A = E_2 \otimes A_1 + A_2 \otimes E_1$ is the block wise matrix of the $N.M$ order determined with the Kronecker tensor product in following form

$C = B^1 \otimes B^2$, where B^1, B^2 are the square matrices correspondly of N, M orders and the matrix C of $N.M$ order is following

$$C = \begin{pmatrix} b^1_{1,1}B^2 & b^1_{1,2}B^2 & b^1_{1,3}B^2 & \dots & b^1_{1,N-2}B^2 & b^1_{1,N-1}B^2 & b^1_{1,N}B^2 \\ b^1_{2,1}B^2 & b^1_{2,2}B^2 & b^1_{2,3}B^2 & \dots & b^1_{2,N-2}B^2 & b^1_{2,N-1}B^2 & b^1_{2,N}B^2 \\ .. & .. & .. & \dots & .. & .. & \\ b^1_{N,1}B^2 & b^1_{N,2}B^2 & b^1_{N,3}B^2 & \dots & b^1_{N,N-2}B^2 & b^1_{N,N-1}B^2 & b^1_{N,N}B^2 \end{pmatrix}$$

A_1, A_2 are circulant matrices $A_1 = \frac{1}{h^2}[2\ -1\ 0 \dots 0\ 0\ -1]$, $A_2 = \frac{1}{h_1^2}[2\ -1\ 0 \dots 0\ 0\ -1]; E_1, E_2$ are the unit matrices with the N, M order correspondly, u, g are the column-vectors of $N.M$ order with following elements
$u_{j,m} \approx T(x_j, y_m), g_{j,m} = f(x_j, y_m), j = \overline{1,N}, m = \overline{1,M}$,
Using the matrices spectral representation $A_k = W_k D_k W_k^*, k = 1;2$ and the properties of Kronecker product
$AC \otimes BD = (A \otimes B)(C \otimes D)$,
$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ we get

$$A = (W_2 \otimes W_1)(E_2 \otimes D_1 + D_2 \otimes E_1)(W_2^* \otimes W_1^*)$$

and $u = -A^{-1}g, (W_k^{-1} = W_k^*, (W_k^*)^{-1} = W_k)$ where

$$A^{-1} = (W_2 \otimes W_1)(E_2 \otimes D_1 + D_2 \otimes E_1)^{-1}(W_2^* \otimes W_1^*)$$

For the solution of the problem $Au = g$ analytically we use the transformation
$W^*u = v$ or $u = Wv$, where $W = W_2 \otimes W_1, W^* = W_2^* \otimes W_1^*$.
Then $Dv = -W_* g$ or $d_{j,m}v_{j,m} = -(W_* g)_{j,m}, j = \overline{1,N}, m = \overline{1,M}$,
where $D = E_2 \otimes D_1 + D_2 \otimes E_1$.
For $j = N, M = M$ we have $d_{N,M} = 0$, the value $v_{N,M}$ is indeterminable and we can take $v_{N,M} = 0$. The solution is in the form $u = Wv$.

```
%system ODE U_yy-AU=f with periodical BC in 2 direct,
%u(x,y)=cos(2 pi x)cos(2 pi y),f=-8 \pi^2 u(x,y),N,M-even
%Kroneker-Tensor algorithm
function PuasTen2(N,M)
N1=N+1; M1=M+1;H=1;L=1;x=linspace(0,L,N1)';y=linspace(0,H,M1);
h=L/N;N2=N-1;M2=M-1;x=x(2:N1);y=y(2:M1)';h1=H/M;
NM=N*M;NM2=NM-1;
%A2=A2-diag(ones(N2,1),1)-diag(ones(N2,1),-1)+...
%2*diag(ones(N,1));
```

```
%A2(1,N)=-1; A2(N,1)=-1; A2=A2/h^2; %matrix A,O(h^2), control
NT=(1:N)'/L;MT=(1:M)'/H;
lk=4/h^2*(sin(pi*h*NT)).^2; %O(h^2)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);%O(h^4)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6);%O(h^6)
lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+...
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);%O(h^8)
lk1=4/h1^2*(sin(pi*h1*MT)).^2; %O(h1^2)
lk1=4/h1^2*((sin(pi*h1*MT)).^2+1/3*(sin(pi*h1*MT)).^4);%O(h1^4)
lk1=4/h1^2*((sin(pi*h1*MT)).^2+1/3*(sin(pi*h1*MT)).^4+...
8/45*(sin(pi*h1*MT)).^6);%O(h1^6)
lk1=4/h1^2*((sin(pi*h1*MT)).^2+1/3*(sin(pi*h1*MT)).^4+...
8/45*(sin(pi*h1*MT)).^6+4/35*(sin(pi*h1*MT)).^8);%O(h1^8)
Ck=sqrt(h/L);
Ck1=sqrt(h1/H);
lk0=(2*(1:N)'*pi/L).^2;
lk01=(2*(1:M)'*pi/H).^2;
d=lk; %FDS-x
d1=lk1; %FDS-y
NH=N/2; d(1:NH)=lk0(1:NH);
d(NH:N2)=lk0(NH:-1:1);d(N)=0;%FDSES-x
MH=M/2; d1(1:MH)=lk01(1:MH);
d1(MH:M2)=lk01(MH:-1:1);d1(M)=0;%FDSES-y
W=Ck*exp(2*pi*i*(1:N)'*x'/L)';
W1=Ck*exp(-2*pi*i*(1:N)'*x'/L)';
Wy=Ck1*exp(2*pi*i*(1:M)'*y'/H)';
Wy1=Ck1*exp(-2*pi*i*(1:M)'*y'/H)';
Wxy=kron(Wy,W);Wxy1=kron(Wy1,W1);
A1=W*diag(d)*W1; %FDS or FDSES,control
A2=Wy*diag(d1)*Wy1; %FDS or FDSES
f=8*pi^2*cos(2*pi*y)*cos(2*pi*x');
f=reshape(f',NM,1);g=Wxy1*f;
dd=zeros(NM,1);gg=zeros(NM,1);P2=zeros(NM,1);
for j=1:M j1=1+(j-1)*N;j2=j1+N2; dd(j1:j2)=d(:)+d1(j);end
for ji=1:NM2 gg(ji)=g(ji)/dd(ji); end
gg(NM)=0;
P2=Wxy*gg;P2=reshape(P2,N,M)';
P2=P2-P2(M,N)+1;
prec=cos(2*pi*y)*cos(2*pi*x');im =max(max(abs(imag(P2))))
Ma1=max(max(abs(P2-prec)));%max error an.
X1=ones(M,1)*x';Y1=y*ones(1,N);
figure,plot(y,max(abs(P2(:,1:N)'-prec)),'k*')% max error on y
title(sprintf('err. Max-sol.an.on y, Max=%9.7f ',Ma1))
xlabel('\ity'), ylabel('\itu')
figure, surfc(X1,Y1,abs(P2-prec))% error anl.
colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('err. anal.,yNr.=%4.1f,max=%9.7f',M,Ma1))
```

Using the operator **PuaTen2(10,10)** we obtain by different order of FDS and of FDSES following maximal errors:
1)$O(h_1^2)$: 0.06711 (FDS-$O(h^2)$), 0.0347 (FDS- $O(h^4)$), 0.0331 (FDS

-$O(h^6)$),0.03301 (FDS-$O(h^8)$), 0.03300 (FDSES),
2)$O(h_1^4)$: 0.0347 (FDS-$O(h^2)$), 0.00335 (FDS- $O(h^4)$), 0.00178 (FDS -$O(h^6)$),0.00168 (FDS-$O(h^8)$), 0.00167 (FDSES),
3)$O(h_1^6)$: 0.0331 (FDS-$O(h^2)$), 0.00178 (FDS- $O(h^4)$), 0.00021 (FDS -$O(h^6)$),0.00011 (FDS-$O(h^8)$), 0.00010 (FDSES),
4)$O(h_1^8)$: 0.03301 (FDS-$O(h^2)$), 0.00168 (FDS- $O(h^4)$), 0.00011 (FDS -$O(h^6)$),0.00001 (FDS-$O(h^8)$), 7.10^{-6} (FDSES),
5)FDSES: 0.03300 (FDS-$O(h^2)$), 0.00167 (FDS- $O(h^4)$), 0.00010 (FDS -$O(h^6)$), 7.10^{-6} (FDS-$O(h^8)$), 10^{-15} (FDSES).
For $FDS - O(h^2) + O(h_1^2)$ and FDSES the errors are represented in the Figs. 7.9, 7.10.

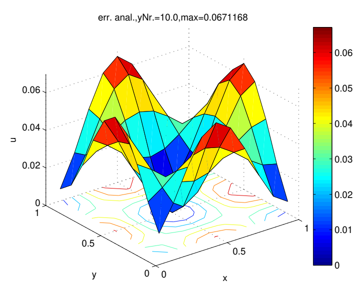


Fig. 7.9 Error with FDS of $O(h^2 + h_1^2)$ by $N = M = 10$

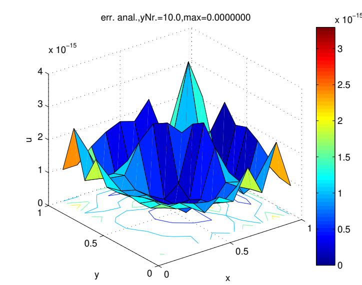


Fig. 7.10 Error with FDSES in (x,y) direction by $N = M = 10$

7.5 Equations with convections

We consider the boundary value problem for PDE with convection terms and with the periodical BCs in the x direction:

$$\begin{cases} \frac{\partial^2 T(x,y)}{\partial y^2} + a_2 \frac{\partial T(x,y)}{\partial y} + \nu \frac{\partial^2 T(x,y)}{\partial x^2} + \\ a_1 \frac{\partial T(x,y)}{\partial x} = f(x,y), x \in (0,L), y \in (0,H), \\ T(0,y) = T(L,y), \frac{\partial T(0,y)}{\partial x} = \frac{\partial T(L,y)}{\partial x}, y \in (0,H), \\ T(x,0) = T_l(x), T(x,H) = T_r(x), x \in (0,L), \end{cases} \tag{7.12}$$

where $T_l(x), T_r(x)$ are given BC function in the y direction and $\nu > 0, a_1, a_2$ are constants.

7.5.1 Convection in the y direction

If $a_1 = 0$ then the solution for Fourier series is in the real form (7.9), where the coefficients a_{kc}, a_{ks} we can obtain from following seperate system of ODEs:

$$\begin{cases} \ddot{a}_{kc}(y) + a_2\dot{a}_{kc}(y) - \nu\lambda_k a_{kc}(y) = b_{kc}(y), k = \overline{0,\infty}, \\ \ddot{a}_{ks}(y) + a_2\dot{a}_{ks}(y) - \nu\lambda_k a_{ks}(y) = b_{ks}(y), k = \overline{1,\infty}, \end{cases} \tag{7.13}$$

where $\lambda_k = \left(\frac{2\pi k}{L}\right)^2$, $a_{kc}(0) = \frac{2}{L}\int_0^L T_l(\xi)\cos\frac{2\pi k\xi}{L}d\xi$,
$a_{ks}(0) = \frac{2}{L}\int_0^L T_l(\xi)\sin\frac{2\pi k\xi}{L}d\xi$, $a_{kc}(H) = \frac{2}{L}\int_0^L T_r(\xi)\cos\frac{2\pi k\xi}{L}d\xi$,
$a_{ks}(H) = \frac{2}{L}\int_0^L T_r(\xi)\sin\frac{2\pi k\xi}{L}d\xi$,
$a_{0c}(0) = \frac{2}{L}\int_0^L T_l(\xi)d\xi, a_{0c}(H) = \frac{2}{L}\int_0^L T_r(\xi)d\xi$.
The solution of ODEs is

$$\begin{cases} a_{kc}(y) = \exp(-a_2y/2)(\sinh(\kappa_kH))^{-1}[a_{kc}(0)\sinh(\kappa_k(H-y)) + \\ \exp(a_2H/2)a_{kc}(H)\sinh(\kappa_ky) - \frac{1}{\kappa_k}\int_0^H G_k(\xi,y)b_{kc}(\xi)d\xi], \\ a_{ks}(y) = \exp(-a_2y/2)(\sinh(\kappa_kH))^{-1}[a_{ks}(0)\sinh(\kappa_k(H-y)) + \\ \exp(a_2H/2)a_{ks}(H)\sinh(\kappa_ky) - \frac{1}{\kappa_k}\int_0^H G_k(\xi,y)b_{ks}(\xi)d\xi], \end{cases} \tag{7.14}$$

where $\kappa_k = \sqrt{a_2^2/4 + \nu\lambda_k}, \kappa_0 = a_2/2, G_k(\xi,y)$ is the Green function in following way:

$$G_k(\xi,y) = \begin{cases} \sinh(\kappa_k(H-y))\sinh(\kappa_k\xi),\ 0 \le \xi \le y, \\ \sinh(\kappa_k(H-\xi))\sinh(\kappa_ky),\ y \le \xi \le H. \end{cases}$$

Similarly we can obtained the solution of the discrete problem also in the real form (7.10, 7.14), where $\kappa_k = \sqrt{a_2^2/4 + \nu\mu_k}, k = \overline{1,N/2}, \mu_k$ are the eigenvalues with different order of approximation in multi-points stencil (for FDSES μ_k are replaced with λ_k). For $a_2 = 0, \nu = 1$ we have the solutions (7.7).

7.5.2 Convection in the two directions

If $a_1 \neq 0$ then for Fourier coeficients we have following non seperate system of ODEs:

$$\begin{cases} \ddot{a}_{kc}(y) + a_2\dot{a}_{kc}(y) - \nu\lambda_k a_{kc}(y) + a_1\lambda_k^0 a_{ks} = b_{kc}(y), k = \overline{0,\infty}, \\ \ddot{a}_{ks}(y) + a_2\dot{a}_{ks}(y) - \nu\lambda_k a_{ks}(y) - a_1\lambda_k^0 a_{kc} = b_{ks}(y), k = \overline{1,\infty}, \end{cases} \tag{7.15}$$

where $\lambda_k^0 = \frac{2\pi k}{L} = \sqrt{\lambda_k}$. For the discrete problem in the system (7.15) the eigenvalues λ_k, λ_k^0 need replaced with the discrete eigenvalues $\mu_k, |\mu_k^0|$ (see chapter 1).

7.6 Poisson equation in polar coordinats

We consider the boundary value problem for Poisson equation in ring with the periodical BCs in the ϕ direction:

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(\phi,r)}{\partial r} \right) + \frac{\partial^2 T(\phi,r)}{\partial \phi^2} = f(\phi,r), \phi \in (0,2\pi), r \in (r_1,r_2), \\ T(0,r) = T(2\pi,r), \frac{\partial T(0,r)}{\partial \phi} = \frac{\partial T(2\pi,r)}{\partial \phi}, r \in (r_1,r_2), \\ T(\phi,r_1) = T_1(\phi), T(\phi,r_2) = T_2(\phi), \phi \in (0,2\pi), \end{cases} \tag{7.16}$$

where $T_1(\phi), T_2(\phi), f(\phi,r)$ are given periodical functions in the ϕ direction.
Using uniform grid
$\phi_j = jh, h = 2\pi/N, j = \overline{1,N}$ we obtain the boundary value problem for ODEs in the following matrix form

$$\ddot{U}(r) + \frac{1}{r}\dot{U}(r) - AU(r) = F(r), U(r_1) = U_1, U(r_2) = U_2, \tag{7.17}$$

where A is the 3-diagonal circulant matrix of N order,
$A = \frac{1}{h^2}[2,-1,0,0,...,-1]$ (in three point sencil or in multi-point stencil in chapter 1) ,
$U(r), \dot{U}(r), \ddot{U}(r), F(r), U_1, U_2$ are the column-vectors of N order with the elements $u_j(r) \approx T(\phi_j, r)$),
$\dot{u}_j(r) \approx \frac{\partial T(\phi_j,r)}{\partial r}$,,
$\ddot{u}_j(r) \approx \frac{\partial^2 T(\phi_j,r)}{\partial r^2}$, $f_j(r) = f(x_j,r)$,)
$u_j(r_1) = T_l(\phi_j), u_j(r_2) = T_r(\phi_j), j = \overline{0,N}$.

The corresponding discrete spectral problem $Aw^n = \mu_n w^n, n = \overline{1,N}$ with circulant matrix have following solution:

$$\begin{cases} w^n = \sqrt{1/N}(w_1^n, w_2^n, ..., w_N^n)^T, \\ \mu_n = \frac{4}{h^2}\sin^2(n\pi/N), \end{cases} \quad (7.18)$$

where $w_j^n = \exp(inx_j), j = \overline{1,N}, i = \sqrt{-1}$ are the components of orthonormed eigenvector w^n.
The solution of the spectral problem for differential equations

$$-w''(\phi) = \lambda w(\phi), \phi \in (0, 2\pi), w(0) = w(2\pi), w'(0) = w'(\phi),$$

is in following form:
$w_n(x) = \sqrt{\frac{1}{2\pi}}\exp(inx), \lambda_n = n^2.$
We can consider the **analytical solutions** of the system of ODEs (7.16,7.17)
using the spectral representation of matrix $A = WDW^*$.
From transformation $V = W^*U(U = WV)$ follows from 7.17 the seperate system of ODEs

$$\ddot{V}(r) + \frac{1}{r}\dot{V}(r) - DV(r) = G(r), V(r_1) = W^*U_1, V(r_2) = W^*U_2, \quad (7.19)$$

where $V(r), \ddot{V}(r), \dot{V}(r), V(r_1), V(r_2), G(r) = W^*F(r)$ are the column-vectors of N order with elements
$v_k(r), \ddot{v}_k(r), \dot{v}_k(r), v_k(r_1), v_k(r_2), g_k(r), D = diag(\mu_k), k = \overline{1,N}.$
The solution of the system (7.19) is

$$v_k(r) = C_k r^{p_k} + B_k r^{-p_k} + q_k(r),$$

where C_k, B_k are constants, $p_k = \sqrt{\mu_k}, q_k(r)$ is the particular solution.
For obtaining $q_k(r)$ we use the transformation $x = \ln(r) \in [x_1, x_2], r = \exp(x), x_1 = \ln(r_1), x_2 = \ln(r_2)$.
Then we have the equation
$v_k''(x) - \mu_k v_k(x) = \exp(2x)g_k(x)$, with the solution
$C_1 \sinh(p_k x) + C_2 \cosh(p_k x) + \frac{1}{p_k}\int_{x_1}^{x_2}\sinh(p_k(x-t))g_k(t)dt$
(C_1, C_2 are constants) and from $t = \ln(\xi)$ follows
$q_k(r) = \frac{1}{2p_k}\int_{r_1}^{r_2}\xi((\frac{r}{\xi})^{p_k} - (\frac{r}{\xi})^{-p_k})g_k(\xi)d\xi.$
The constantsC_k, B_k we can determine from given BC values $v_k(r_1), v_k(r_2)$.

We can used also the **Fourier method** for solving (7.16) in the form $T(\phi, r) = \sum_{k\in Z} v_k(r)w_k(\phi)$, where $w_k(\phi)$ are the orthonormed eigenfunctions, $v_k(r)$ is the solution (7.19), with $v_k(r_1) = (T_1, w_k^*), v_k(r_2) =$

$(T_2, w_k^*), p_k = k$.
The solution we can also obtained in real form:
$T(\phi, r) = \sum_{k=1}^{\infty}(a_{kc}(r)\cos(k\phi) + a_{ks}(r)\sin(k\phi) + \frac{a_{0c}(r)}{2}$,
$f(\phi, r) = \sum_{k=1}^{\infty}(b_{kc}(r)\cos(k\phi) + b_{ks}(r)\sin(k\phi) + \frac{b_{0c}(r)}{2}$,
$b_{kc}(r) = \frac{1}{\pi}\int_0^{2\pi} f(\xi, r)\cos(k\xi)d\xi, b_{ks}(r) = \frac{1}{\pi}\int_0^{2\pi} f(\xi, r)\sin(k\xi)d\xi$,
where $a_{kc}(r), a_{ks}(r)$ are the corresponding solutions of (7.19) by
$a_{kc}(r_1) = \frac{1}{\pi}\int_0^{2\pi} T_1(\xi)\cos(k\xi)d\xi, a_{ks}(r_1) = \frac{1}{\pi}\int_0^{2\pi} T_1(\xi)\sin(k\xi)d\xi$,
$a_{kc}(r_2) = \frac{1}{\pi}\int_0^{2\pi} T_2(\xi)\cos(k\xi)d\xi, a_{ks}(r_2) = \frac{1}{\pi}\int_0^{2\pi} T_2(\xi)\sin(k\xi)d\xi$,
$g_k(r) = b_{kc}(r)$ or $b_{ks}(r), p_k = k$.

Similarly we can obtained the solution of the discrete problem also in the real form:
$u_j(r) = \sum_{k=1}^{*N_2}(a_{kc}(r)\cos\frac{2\pi kj}{N} + a_{ks}(r)\sin\frac{2\pi kj}{N}) + \frac{a_{0c}(r)}{2}$,
$f_j(r) = \sum_{k=1}^{*N_2}(b_{kc}(r)\cos\frac{2\pi kj}{N} + b_{ks}(r)\sin\frac{2\pi kj}{N}) + \frac{b_{0c}(r)}{2}$,
$b_{kc}(y) = \frac{2}{N}\sum_1^N f_j(r)\cos\frac{2\pi kj}{N}, b_{ks}(r) = \frac{2}{N}\sum_1^N f_j(r)\sin\frac{2\pi kj}{N}$,
where $a_{kc}(r), a_{ks}(r)$ are the corresponding solutions of (7.19) by
$a_{kc}(r_1) = \frac{2}{N}\sum_{j=1}^N T_1(x_j)\cos\frac{2\pi kj}{N}, a_{ks}(r_1) = \frac{2}{N}\sum_{j=1}^N T_1(x_j)\sin\frac{2\pi kj}{N}$,
$a_{kc}(r_2) = \frac{2}{N}\sum_{j=1}^N T_2(x_j)\cos\frac{2\pi kj}{N}, a_{ks}(r_2) = \frac{2}{N}\sum_{j=1}^N T_2(x_j)\sin\frac{2\pi kj}{N}$,
$g_k(r) = b_{kc}(r)$ or
$b_{ks}(r), p_k = \sqrt{\mu_k}, N_2 = N/2, \sum_{k=1}^{*N_2}\beta_k = \sum_{k=1}^{N_2-1}\beta_k + \frac{\beta_{N_2}}{2}$.
(for FDSES μ_k are replaced with $\lambda_k = k^2$).

For the FDSES the matrix A is represented in the form $A = WDW^*$ and the diagonal matrix D contain the first N eigenvalues $d_k = k^2$, $k=\overline{1,N}$ from the differential operator $(-\frac{\partial^2}{\partial phi})$ in following way:
1)$d_k = k^2$ for $k = \overline{1, N_2}$,
2)$d_k = (N-k)^2$ for $k = \overline{N_2, N}$.
If $d_k = \mu_k$, then we have the method of FDS.

7.7 Poisson equation with the BCs of first kind

For BCs of first kind we consider special boundary value problem (7.1) with the homogenous BC in the x direction

$$T(0, y) = T(L, y) = 0, y \in [0, H].$$

We have the discrete problem (7.3) with the standart 3-diagonal matrix A of the $M = N-1$ order. We use the matrix representation $A = WDW, W^* = W$ where the diagonal matrix D contain the discrete eigenvalues $\mu_k = \frac{4}{h^2}\sin^2\frac{\pi k}{2N}$ and the column of the matrix W is equal to the orthonormed eigenvectors w^k with the elements
$w_j^k = w_k(x_j) = \sqrt{\frac{2}{N}}\sin\frac{\pi k j}{N}$, $k, j=\overline{1,M}$.
Then we have the seperate system of ODEs (7.6) with the solution (7.7).
For the FDSES the matrix A is represented in the form form $A = WDW$ and the diagonal matrix D contain the first M eigenvalues
$d_k = \lambda_k = (\frac{\pi k}{L})^2$, $k=\overline{1,M}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$.
Then in the solution (7.7) needs replaced μ_k with λ_k.

For special data
$T_l(x) = a_1 w_{p_1}(x), T_r(x) = a_2 w_{p_2}(x), f(x,t) = g(t) w_{p_3}(x)$ we have the exact solution for Fourier method and for FDSES by $M \geq \max(p_1, p_2, p_3)$ in the form

$$T(x,y) = a_1 w_{p_1}(x) v_{p_1}(y) + a_2 w_{p_2}(x) v_{p_2}(y) + w_{p_3} v_{p_3},$$

where
$v_{p_1}(y) = \sinh(\kappa_{p_1} H))^{-1}\sinh(\kappa_{p_1}(H-y))$,
$v_{p_2}(y) = \sinh(\kappa_{p_2} H))^{-1}\sinh(\kappa_{p_2} y)$,
$v_{p_3}(y) = -\int_0^H G_{p_3}(\xi, y) g(\xi) d\xi$,
$G_p(\xi, y)$ is the Green function in following way:

$$G_p(\xi,y) = \begin{cases} \frac{\sinh(\kappa_p(H-y))\sinh(\kappa_p \xi)}{\kappa_p \sinh(\kappa_p H)}, & 0 \leq \xi \leq y, \\ \frac{\sinh(\kappa_p(H-\xi))\sinh(\kappa_p y)}{\kappa_p \sinh(\kappa_p H)}, & y \leq \xi \leq H. \end{cases}$$

For FDS $(x = x_j, j = \overline{1,M})$ $\kappa_k = \sqrt{\mu_k}$,
but for FDSES $\kappa_k = \sqrt{\lambda_k}, k = (p_1, p_2, p_3)$.

7.8 Conclusions

The solutions of the boundary value problem for Poisson equations are obtained analytically and numerically,
using the method of lines (lines are parallel to y axis). For periodical

BCs we define the FDSES, where the finite difference matrix A is represented in the form form $A = WDW^*$,
where W, D is the matrixes of finite difference eigenvectors and eigenvalues corespondently, W^* is the conjugate matrix and the elements of diagonal matrix D are replaced with the first eigenvalues from differential operator.
We consider uniform grid in the space $x_j = jh, j = \overline{0,N}, Nh = L$, where N is even number.
Using the finite differences of second order approximation for partial derivatives of second order with respect to x we obtain the boundary value problem for system of ordinary differential equations (ODEs) in the matrix form. We are considered the **analytical solutions** of the system of ODEs using the spectral representation of matrix $A = WDW^*$.

Chapter 8
Difussion equation: H. Kalis, A. Buikis, I. Kangro, 2016 [34]

8.1 Introduction

We consider the 2-D stationary boundary value problem for diffusion equation with piece-wise constant coefficients in multi-layered domain. In one direction(x-axes direction) we have the homogenous boundary condition of the first kind (BC) or periodical boundary conditions (PBC).

We define the finite difference scheme with exact spectrum (FD-SES) ([2]1975, [14] 2011) ,using the finite difference matrix A in the form $A = WDW^T$ (W, D are the matrixes of finite difference eigenvectors and eigenvalues), where the elements of the diagonal matrix D are replaced with the first K eigenvalues from the differential operator of the second order ($K = N-1$ for BCs of first kind, $K = N$ for PBCs, $N+1$ is the number of mesh points in uniform grid) . The solution is obtained analytically and numerically. We consider the method of lines and Fourier methods for solving the corresponding problems with homogenous BCs and PBCs.
A.Buikis ([9] ,[12] 1994) consider different assumptions for averaging methods. With the help of this spline is reduce the 3-D problem of mathematical physics with piece-wise coefficients to 2-D problems for system of equations.

H.Kalis ([13],1997) developed an effective finite-difference method for solving a problem of the above type. This method may be considered as a generalization of the metod of finite volumes for layered systems. This procedure allows to reduce the 2-D problem to a system of 1-D problems.

8.2 A mathematical model in 2-D domain

The process of diffusion is consider in 2-D domain

$$\Omega = \{(x,y) : 0 \leq x \leq l, 0 \leq y \leq L\}.$$

The domain Ω consist of multilauer medium. We will consider the stationary 2-D problem of the linear diffusion theory for multilayered piece-wise homogenous materials of M layers in the form

$$\Omega_j = \{(x,y) : x \in (0,l), y \in (y_{j-1}, y_j)\}, i = \overline{1,M},$$

where $h_j = y_j - y_{j-1}$ is the height of layer $\Omega_j, y_0 = 0, y_M = L$. We will find the distribution of concentrations $u_j = u_j(x,y)$ in every layer Ω_j at the point $(x,y) \in \Omega_j$ by solving the following partial differential equation (PDE):

$$k_j \partial^2 u_j / \partial x^2 + k_j \partial^2 u_j / \partial y^2 + f_j(x,y) = 0, \tag{8.1}$$

where k_j are constant diffusions cefficients, $u_j = u_j(x,y)$ – the concentrations functions in every layer, $f_j(x,y)$ - the fixed sours function. The values u_j and the flux functions $k_j \partial u_j / \partial y$ must be continues on the contact lines between the layers $y = y_j, j = \overline{1, M-1}$:

$$\begin{aligned} &u_j(x,y_j) = u_{j+1}(x,y_j), \\ &k_j \partial u_j(x,y_j)/\partial y = k_{j+1} \partial u_{j+1}(x,y_j)/\partial y. \end{aligned} \tag{8.2}$$

We assume that the layered material is bounded above and below with the plane surfaces $y = 0, y = L$ with fixed boundary conditions of third kind in following form:

$$\begin{aligned} &\gamma_1 k_1 \partial u_1(x,0)/\partial y - \alpha_1(u_1(x,0) - T_1(x)) = 0, \\ &\gamma_2 k_M \partial u_M(x,L)/\partial y + \alpha_2(u_M(x,L) - T_2(x)) = 0, \end{aligned} \tag{8.3}$$

where $\gamma_1^2 + \alpha_1^2 \neq 0, \gamma_2^2 + \alpha_2^2 \neq 0$, T_1, T_2 are given functions. For $\gamma_1 = \gamma_2 = 0$ we have the BC of first kind. We have two form of fixed BCs in the x,y directions:
1) the periodical conditions by $x = 0, x = l$ in the form

$$u_j(0,y) = u_j(l,y), \partial u_j(0,y)/\partial x = \partial u_j(l,y)/\partial x, \tag{8.4}$$

2) the homogenous BC of first kind

$$u_j(0,y) = u_j(l,y) = 0, j = \overline{1,M}. \tag{8.5}$$

8.3 Analytical solution with Fourier methods

For analytical solution of the problem (8.1)-(8.5) we will consider Fourier series methods in following forms:
1) for BC (8.5) -
$u_j(x,y) = \sum_{k=1}^{\infty} a_{j,k}(y) X_k(x), f_j(x,y) = \sum_{k=1}^{\infty} b_{j,k}(y) X_k(x)$,
$T_1(x) = \sum_{k=1}^{\infty} c_{1,k} X_k(x)$ $T_2(x) = \sum_{k=1}^{\infty} c_{2,k} X_k(x)$,
where $X_k(x) = \sqrt{\frac{2}{L}} \sin \frac{k\pi x}{l}$ are the orthonormed eigenvectors
$(X_k, X_m) = \int_0^l X_k(x) X_m(x) dx = \delta_{k,m}$ with eigenvalues
$\lambda_k = (\frac{k\pi}{l})^2, (-\frac{d^2 X_k(x)}{dx^2} = \lambda_k X_k(x), X_k(0) = X_k(l) = 0)$,
$b_{j,k}(y) = (f_j, X_k), c_{i,k} = (T_i, X_k), i = 1;2, \delta_{k,m}$ is the Kroneker's symbol,

2) for BC (8.4) -
$u_j(x,y) = \sum_{k=-\infty}^{\infty} a_{j,k}(y) X_k(x), f_j(x,y) = \sum_{k=-\infty}^{\infty} b_{j,k}(y) X_k(x)$, $T_1(x) = \sum_{k=-\infty}^{\infty} c_{1,k} X_k(x)$
$T_2(x) = \sum_{k=-\infty}^{\infty} c_{2,k} X_k(x)$,
where $X_k(x) = \sqrt{\frac{1}{l}} \exp(2\pi i k x / l)$,
$X_k^*(x) = \sqrt{\frac{1}{l}} \exp(-2\pi i k x / l), i = \sqrt{-1}$ are the biorthonormed complex eigenvectors
$(X_k, X_m^*) = \int_0^l X_k(x) X_m^*(x) dx = \delta_{k,m}$ with eigenvalues
$\lambda_k = (\frac{2k\pi}{l})^2, (-\frac{d^2 X_k(x)}{dx^2} = \lambda_k X_k(x), X_k(0) = X_k(l), X_k'(0) = X_k'(l))$,
$b_{j,k}(y) = (f_j, X_k^*), c_{1,k} = (T_1, X_k^*), c_{2,k} = (T_2, X_k^*)$.

For the Fourier coefficients $a_{j,k}(y)$ we have following boundary value problem for the system of ODEs:

$$\begin{cases} -k_j \lambda_k a_{j,k}(y) + k_j a''_{j,k}(y) = -b_{j,k}(y), y \in (y_{j-1}, y_j), j = \overline{1,M}, \\ a_{j,k}(y_j) = a_{j+1,k}(y_j), k_j a'_{j,k}(y_j) = k_{j+1} a'_{j+1,k}(y_j), j = \overline{1,M-1}, \\ \gamma_1 k_1 a'_{1,k}(0) - \alpha_1 (a_{1,k}(0) - c_{1,k}) = 0, \\ \gamma_2 k_M a'_{M,k}(L) + \alpha_2 (a_{M,k}(L) - c_{2,k}) = 0, \end{cases} \tag{8.6}$$

where $a''_{j,k}(y) = \frac{d^2 a_{j,k}(y)}{d^2 x}$.
We have following solution of the ODEs system (8.6)

$$a_{j,k}(y) = C_{j,k}\sinh(\sqrt{\lambda_k}(y-y_{j-1})) + B_{j,k}\cosh(\sqrt{\lambda_k}(y-y_{j-1})) - \frac{1}{k_j\sqrt{\lambda_k}}\int_{y_{j-1}}^{y}\sinh(\sqrt{\lambda_k}(y-t))b_{j,k}(t)dt, \tag{8.7}$$

where the constants $C_{j,k}, B_{j,k}$ can be determined from following system of algebraic equations: $k_{j+1}C_{j+1,k} = k_j(C_{j,k}ch_{j,k} + B_{j,k}sh_{j,k}) - \frac{1}{\sqrt{\lambda_k}}IC_{j,k}$,
$B_{j+1,k} = C_{j,k}sh_{j,k} + B_{j,k}ch_{j,k} - \frac{1}{k_j\sqrt{\lambda_k}}IS_{j,k}, j = \overline{1,M-1}$,
$\gamma_1 k_1 C_{1,k}\sqrt{\lambda_k} - \alpha_1(B_{1,k} - c_{1,k}) = 0$,
$\gamma_2(k_M\sqrt{\lambda_k}(C_{M,k}ch_{M,k} + B_{M,k}sh_{M,k}) - IC_{M,k}) +$
$\alpha_2(C_{M,k}sh_{M,k} + B_{M,k}ch_{M,k} - \frac{1}{k_M\sqrt{\lambda_k}}IS_{M,k}) - c_{2,k}) = 0.$
Here $sh_{j,k} = \sinh(\sqrt{\lambda_k}h_j), ch_{j,k} = \cosh(\sqrt{\lambda_k}h_j)$,
$IS_{j,k} = \int_{y_{j-1}}^{y_j}\sinh(\sqrt{\lambda_k}(y_j-t))b_{j,k}(t)dt$,
$IC_{j,k} = \int_{y_{j-1}}^{y_j}\cosh(\sqrt{\lambda_k}(y_j-t))b_{j,k}(t)dt$.

8.4 The AV–method with quadratic splines

The equation of (8.1) are averaged along the heights h_j of layers Ω_j and quadratic integral splines along y coordinate in following form one used [9]

$$u_j(x,y) = U_j(x) + m_j(x)(y-\overline{y_j}) + e_j(x)G_j((y-\overline{y_j})^2/h_j^2 - 1/12), \tag{8.8}$$

where $G_j = h_j/k_j, \overline{y_j} = (y_{j-1}+y_j)/2, y \in [y_{j-1}, y_j]$,
m_j, e_j, U_j are the unknown coefficients of the spline-function,
$U_j(x) = h_j^{-1}\int_{y_{j-1}}^{y_j}u_j(x,y)dy$ are the average values of u_j, $j = \overline{1,M}$.

After averaging the system (8.1) along every layer Ω_j, we obtain M system of ODEs

$$k_j\frac{d^2U_j(x)}{dx^2} + 2h_j^{-1}e_j(x) + F_j(x) = 0, \tag{8.9}$$

where $F_j = h_j^{-1}\int_{y_{j-1}}^{y_j}f_j(x,y)dy$ are the average values of f_j, $j = \overline{1,M}$.

From the BCs (8.3) follows

$$\begin{aligned}&\gamma_1(k_1m_1-e_1)-\alpha_1(U_1-m_1h_1/2+e_1G_1/6-T_1)=0,\\&\gamma_2(k_Mm_M-e_M)+\alpha_2(U_M+m_Mh_M/2+e_MG_M/6-T_2)=0,\end{aligned}\tag{8.10}$$

From the condition (8.2) follows

$$\begin{aligned}&6U_j+3h_jm_j+e_jG_j=6U_{j+1}-3h_{j+1}m_{j+1}+e_{j+1}G_{j+1},\\&k_jm_j+e_j=k_{j+1}m_{j+1}-e_{j+1},\ j=\overline{1,M-1}.\end{aligned}\tag{8.11}$$

From (1.12) excluding m_{j+1} we obtain

$$3m_jk_j(G_j+G_{j+1})+e_j(G_j+3G_{j+1})+2e_{j+1}G_{j+1}=6(U_{j+1}-U_j),\tag{8.12}$$

where $j=\overline{1,M-1}$. Decreasing index j and excluding m_{j-1} follows

$$3m_jk_j(G_j+G_{j-1})-e_j(G_j+3G_{j-1})-2e_{j-1}G_{j-1}=6(U_j-U_{j-1}),\tag{8.13}$$

where $j=\overline{2,M}$. Excluding m_j from (8.12,8.13) we obtain for determined e_j following system of $M-2$ algebraic equations

$$\begin{aligned}&2e_{j-1}G_{j-1}(G_j+G_{j+1})+e_j((G_j+3G_{j-1})(G_j+G_{j+1})+\\&(G_j+3G_{j+1})(G_j+G_{j-1}))+2e_{j+1}G_{j+1}(G_j+G_{j-1})=\\&6(U_{j+1}-U_j)(G_j+G_{j-1})-6(U_j-U_{j-1})(G_j+G_{j+1}),\end{aligned}\tag{8.14}$$

where $j=\overline{2,M-1}$.
From (1.11), excluding m_1,m_N we get

$$\begin{aligned}&e_1[3(G_1+G_2)(\gamma_1+\alpha_1G_1/6)+(G_1+3G_2)(\gamma_1+\alpha_1G_1/2)]+\\&2e_2G_2(\gamma_1+\alpha_1G_1/2)=6(U_2-U_1)(\gamma_1+\alpha_1G_1/2)-\\&3\alpha_1(U_1-T_1)(G_1+G_2),\\&2e_{M-1}G_{M-1}(\gamma_2+\alpha_2G_M/2)+e_M[3(G_M+G_{M-1})(\gamma_2+\alpha_2G_M/6)+\\&(G_M+3G_{M-1})(\gamma_2+\alpha_2G_M/2)]=\\&-6(U_M-U_{M-1})(\gamma_2+\alpha_2G_M/2)-3\alpha_2(U_M-T_2)(G_M+G_{M-1}).\end{aligned}\tag{8.15}$$

Therefore we have following system of linear algebraic equations

$$A_je_{j-1}+(A_j+B_j+1)e_j+B_je_{j+1}=a_j(U_{j+1}-U_j)-b_j(U_j-U_{j-1}),\tag{8.16}$$

where $e_0=e_{M+1}=0, A_j=G_{j-1}/(G_j+G_{j-1}), b_j=3/(G_j+G_{j-1})=a_{j-1}, j=\overline{2,M}$,
$B_j=G_{j+1}/(G_j+G_{j+1}), a_j=3/(G_j+G_{j+1}), j=\overline{1,M-1}$,
$a_M=1.5\alpha_2/(\gamma_2+\alpha_2G_M/2), b_1=1.5\alpha_1/(\gamma_1+\alpha_1G_1/2)$,

$A_1 = \gamma_1/(\gamma_1 + \alpha_1 G_1/2), B_M = \gamma_2/(\gamma_2 + \alpha_2 G_M/2),$
$U_0 = T_1, U_{M+1} = T_2.$

8.5 The finite difference approximation

For solving 2-D problems we consider an uniform grid in the x-direction $x_k = kh, Nh = l, k = \overline{0,N}$, We can the PDEs (8.1) rewriten in following vector form:

$$-k_j A v_j(y) + k_j v_j''(y) + g_j(y) = 0, j = \overline{1,M}, \quad (8.17)$$

where $v_j(y), g_j(y)$ are the vectors-column with elements
$v_{j,k} \approx u_j(x_k, y), g_{j,k} = f_j(x_k, y), k = \overline{1,K}$,
matrix A is the 3-diagonal of K order in following forms:
1) for **BCs of first kind** , K=N-1,

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 & 0 \\ -1 & 2 & -1 & ... & 0 & 0 & 0 \\ .. & .. & .. & ... & .. & .. & \\ 0 & 0 & 0 & ... & -1 & 2 & -1 \\ 0 & 0 & 0 & ... & 0 & -1 & 2 \end{pmatrix}$$

2) for **periodical BCs**, K=N,

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & ... & 0 & 0 & -1 \\ -1 & 2 & -1 & ... & 0 & 0 & 0 \\ .. & .. & .. & ... & .. & .. & \\ 0 & 0 & 0 & ... & -1 & 2 & -1 \\ -1 & 0 & 0 & ... & 0 & -1 & 2 \end{pmatrix}$$

- this circulant matrix can determined wth the first row
$A = \frac{1}{h^2}[2\ -1\ 0\ ...\ 0\ 0\ -1]$.
From conditions (1.3, 1.4) follows

$$\begin{aligned} & v_j(y_j) = v_{j+1}(y_j), \\ & k_j v_j'(y_j) = k_{j+1} v_{j+1}(y_j), \\ & \gamma_1 k_1 v_1(0) - \alpha_1 (v_1(0) - \mathbf{T}_1) = 0, \\ & \gamma_2 k_M v_M(L) + alpha_2 (v_M(L) - \mathbf{T}_2) = 0, \end{aligned} \quad (8.18)$$

where $\mathbf{T}_1, \mathbf{T}_1$ are the vectors -column of K order with elements $T_1(x_k), T_2(x_k), k = \overline{1,K}$.
The calculation of circulant matrix (matrix invertion and multiplication) can be carried out using simple formulae for obtaining the first N elements of matrix.

8.6 Analytical solution for finite difference schemes

The solution of the corresponding discrete spectral problem $Aw^k = \mu_k w^k$ is
1)for **BCs of first kind** $k = \overline{1,N-1}$,orthonormed eigenvectors $(w^k, w^m) = \sum_{j=1}^{N-1} w_j^k w_j^m = \delta_{k,m}$,
$w_j^k = \sqrt{\frac{2}{N}} \sin \frac{\pi jk}{N},\ j,k = \overline{1,N-1}$ (elements of the symmetrical matrix W),
eigenvalues $\mu_k = \frac{4}{h^2} \sin^2 \frac{k\pi}{2N}$ (elements of the diagonal matrix D),
2) for **periodical BCs** $k = \overline{1,N}$,biorthonormed eigenvectors $(w^k, w_*^m) = \sum_{j=1}^{N} w_j^k w_{*j}^m = \delta_{k,m}$,
$w_j^k = \sqrt{\frac{1}{N}} \exp(2\pi ikj/N), w_{*j}^k = \sqrt{\frac{1}{N}} \exp(-2\pi ikj/N), i = \sqrt{-1}$ (elements of the complex matrices W, W_*),
eigenvalues $\mu_k = \frac{4}{h^2} \sin^2 \frac{k\pi}{N}$ (elements of the diagonal matrix D).
For periodical BCs we can consider the finite difference approximation for second order derivative $u''(x_k)$ in the uniform grid with $p+1$ points stencil
$(x_{k-p/2}, \cdots, x_{k-1}, x_k, x_{k+1}, \cdots, x_{k-p/2})$. We consider the approximation of the $O(h^p)$ order in following form:

$$u''(x_k) = \frac{1}{h^2} \sum_{m=-p/2}^{p/2} C_m u(x_{k-m}) + E_p \frac{h^p u^{(p+2)}(\xi)}{(p+2)!}, x_{k-p/2} < \xi < x_{k+p/2}.$$

For $m = 0$ follows the equation $C_0 = -2\sum_{m=1}^{p/2} C_m$. The others coefficients $C_m, (m > 0)$ we can determined from the linear system. Solution of this system give following coefficients:
1)$p = 2 : C_1 = 1, C_0 = -2, E_2 = -2$,
2)$p = 4 : C_1 = \frac{4}{3}, C_2 = -\frac{1}{12}, C_0 = -\frac{5}{2}, E_4 == 8$,
3)$p = 6 : C_1 = \frac{3}{2}, C_2 = -\frac{3}{20}, C_3 = \frac{1}{90}, C_0 = -\frac{49}{18}, E_4 = -72$,

4)$p=8: C_1=\frac{8}{5}, C_2=-\frac{1}{5}, C_3=\frac{8}{315}, C_4=-\frac{1}{560}, C_0=-\frac{205}{72}$,
$E_8=1152$.
In this case the matrix A of FDS is circulant with the form

$$A=\frac{1}{h^2}[C_0,C_1,\cdots,C_{p/2},0,\cdots,0,C_{p/2},C_{p/2-1},\cdots,C_2,C_1]$$

and matrix A has following eigenvalues ($k=\overline{1,N}$):
1) $p=2: \mu_k=\frac{4}{h^2}\sin^2(\pi k/N)$,
2)$p=4: \mu_k=\frac{4}{h^2}(\sin^2(\pi k/N)+\frac{1}{3}\sin^4(\pi k/N))$,
3)$p=6: \mu_k=\frac{4}{h^2}(\sin^2(\pi k/N)+\frac{1}{3}\sin^4(\pi k/N)+\frac{8}{45}\sin^6(\pi k/N))$,
4) $p=8: \mu_k=\frac{4}{h^2}(\sin^2(\pi k/N)+\frac{1}{3}\sin^4(\pi k/N)+\frac{8}{45}\sin^6(\pi k/N)+\frac{4}{35}\sin^8(\pi k/N))$.

In the matrix form we get (for BCs of first kind $W_*=W$)

$$AW=WD, WW_*=E, W^{-1}=W_*, A=WDW_*,$$

where the elements of the diagonal matrix D is $d_k=\mu_k$.

Using in (8.17)the transformation $\bar{v}_j(y)=W^T v_j(y)$ ($W^T=W_*$ for periodical BCs, $W^T=W$ for BCs of first kind) follows the seperate system of ODEs

$$-k_j D\bar{v}_j(y)+k_j\bar{v}_j''(y)+\bar{g}_j(y)=0, j=\overline{1,M}, \tag{8.19}$$

or

$$-k_j d_k\bar{v}_{j,k}(y)+k_j\bar{v}_{j,k}''(y)+\bar{g}_{j,k}(y)=0, j=\overline{1,M}, k=\overline{1,K}, \tag{8.20}$$

where $\bar{g}_j=W^T g_j, d_k$ are the elemnts of matrix D, $\bar{v}_{j,k}, \bar{g}_{j,k}$ are the elemnts of vectors $\bar{v}_j, \bar{g}_j$.

For (8.19) we have following conditions:

$$\begin{cases} \bar{v}_{j,k}(y_j)=\bar{v}_{j+1,k}(y_j), k_j\bar{v}_{j,k}'(y_j)=k_{j+1}\bar{v}_{j+1,k}'(y_j), j=\overline{1,M-1}, \\ \gamma_1 k_1\bar{v}_{1,k}'(0)-\alpha_1(\bar{v}_{1,k}(0)-\bar{c}_{1,k})=0, \\ \gamma_2 k_M\bar{v}_{M,k}'(L)+\alpha_2(\bar{v}_{M,k}(L)-\bar{c}_{2,k})=0, k=\overline{1,K}, \end{cases} \tag{8.21}$$

where $\bar{c}_{1,k}, \bar{c}_{2,k}$ are the elements of vectors $W^T\mathbf{T}_1, W^T\mathbf{T}_2$.
We have solution of (1.22, 8.21) in the form (8.7), where $a_{j,k}, \lambda_k, c_{1,k}, c_{2,k}$ are replaced with $\bar{v}_{j,k}, d_k, , \bar{c}_{1,k}, \bar{c}_{2,k}$.

For the **finite difference scheme with the exact spectrum** (FDSES) the matrix A is represented in the form form $A = WDW^T$ and the diagonal matrix D contains the first K eigenvalues $d_k = \lambda_k^2$, $k=\overline{1,K}$ from the differential operator $(-\frac{\partial^2}{\partial x^2})$ correspondly (the eigenvectors remained).

For the FDSES with the periodical BCs the elements of matrix D are replaced in following way:
1)$d_k = \lambda_k^2$ for $k = \overline{1,N_2}$, where $N_2 = N/2$.
2)$d_k = \lambda_{N-k}^2$ for $k = \overline{N_2,N-1}, d_N = 0$.

8.7 Solving of the problem in 2 layers

For 2 layers (M=2) we consider following parameters: $N = 10, y_0 = 0, y_1 = 1, y_2 = 3, k_1 = 10, k_2 = 1, T_2(x) = 0, f_1(x,y) = 0, \gamma_1 = \gamma_2 = 0, \alpha_1 = \alpha_2 = 1$,
$f_2(x,y) = P\delta(y-2)\sin(2\pi x)$, or $f_2(x,y) = P\sin(2\pi x)$
$T_1(x) = \sin(\pi x)$- for BCs of first kind, $T_1(x) = \sin(2\pi x)$- for periodical BCs ,
where $P = 10, \delta(y-2)$- is the delta Dirax function.
In this case the analytical soutions are only depending on two eigenvectors X_1, X_2 and w^1, w_*^1.
The numerical results in y-direction are obtained on uniform grid with 6 and 30 mesh points.

8.7.1 The exact solution

The exact solution for **BCs of first kind** using the 2 eigenvectors $X_1, X_2, (\sin(\pi x) = \sqrt{l/2}X_1(x), \sin(2\pi x) = \sqrt{l/2}X_2(x))$ in the form

$$\begin{cases} u_1(x,y) = a_{1,1}(y)\ sin(\pi x) + a_{1,2}(y)\ sin(2\pi x), \\ u_2(x,y) = a_{2,1}(y)\ sin(\pi x) + (a_{2,2}(y) - P\frac{p_1(y)}{2\pi k_2})\sin(2\pi x), \end{cases} \tag{8.22}$$

w here

$$\begin{cases} a_{1,1}(y) = C_{1,1}\sinh(\pi y) + B_{1,1}\cosh(\pi y), \\ a_{1,2}(y) = C_{1,2}\sinh(2\pi y) + B_{1,2}\cosh(2\pi y), \\ a_{2,1}(y) = C_{2,1}\sinh(\pi(y-1)) + B_{2,1}\cosh(\pi(y-1)), \\ a_{2,2}(y) = C_{2,1}\sinh(2\pi(y-1)) + B_{2,1}\cosh(2\pi(y-1)), \end{cases} \tag{8.23}$$

$B_{1,1} = 1, C_{1,1} = -\frac{\cosh(\pi)+\kappa_1\sinh(\pi)\tanh(2\pi)}{\sinh(\pi)+\kappa_1\cosh(\pi)\tanh(2\pi)}$,
$\kappa_1 = \frac{k_1}{k_2}, B_{2,1} = C_{1,1}\sinh(\pi) + B_{1,1}\cosh(\pi), C_{2,1} = \kappa_1(C_{1,1}\cosh(\pi) + B_{1,1}\sinh(\pi)$,
$B_{1,2} = 0, C_{1,2} = \frac{Pp_1(3)}{2\pi k_2(\sinh(2\pi)+\kappa_1\cosh(2\pi)\sinh(4\pi))}$,
$C_{2,2} = \kappa_1 C_{1,2}\cosh(2\pi), B_{2,2} = C_{1,2}\sinh(2\pi)$,
$p_1(y) = [0, y \in [1,2]; \sinh(2\pi(y-2)), y \in [2,3]]$ for $f_2(x,y) = P\delta(y-2)\sin(2\pi x)$
$p_1(y) = (\cosh(2\pi(y-1)) - 1)/(2\pi)$ for $f_2(x,y) = P\sin(2\pi x)$.
For **periodical BCs** using the eigenvectors $X_1, X_1^*, (\sin(2\pi x) = \frac{\sqrt{l}}{2i}(X_1(x) - X_1^*(x), X_1^*(x) = X_{-1}(x))$ in the form

$$u_1(x,y) = a_{1,1}(y)\ sin(2\pi x), u_2(x,y) = a_{2,1}(y)\ sin(2\pi x), \tag{8.24}$$

w here

$$\begin{cases} a_{1,1}(y) = C_{1,1}\sinh(2\pi y) + B_{1,1}\cosh(2\pi y), \\ a_{2,1}(y) = C_{2,1}\sinh(2\pi(y-1)) + B_{2,1}\cosh(2\pi(y-1)) - P\frac{p_1(y)}{2/pik_2}, \end{cases} \tag{8.25}$$

$B_{1,1} = 1, C_{1,1} = -B_{1,1}\frac{\cosh(2\pi)+\kappa_1\sinh(2\pi)\tanh(4\pi)-Pp_1(3)/(2\pi k_2\cosh(4\pi))}{\sinh(2\pi)+\kappa_1\cosh(2\pi)\tanh(4\pi)}$,
$B_{2,1} = C_{1,1}\sinh(2\pi) + B_{1,1}\cosh(2\pi), C_{2,1} = \kappa_1(C_{1,1}\cosh(2\pi) + B_{1,1}\sinh(2\pi)$,

8.7.2 The averaging solution

For the averaging solution $F_1 = 0, F_2 = P\sin(2\pi x)$ for $f_2 = P\sin(2\pi x)$, $F_2 = 0.5P\sin(2\pi x)$ for $f_2 = P\delta(y-2)\sin(2\pi x)$ we have two from (8.16) linear algebraic equations

$$\begin{cases} (B_1+1)e_1 + B_1 e_2 = a_1(U_2 - U_1) - b_1(U_1 - T_1), \\ A_2 e_1 + (A_2+1)e_2 = a_2(T_2 - U_2) - b_2(U_2 - U_1), \end{cases} \tag{8.26}$$

where $B_1 = \frac{G_2}{G_1+G_2} A_2 = \frac{G_1}{G_1+G_2}, a_1 = \frac{3}{G_1+G_2}, b_1 = \frac{3}{G_1}$,
$b_2 = \frac{3}{G_1+G_2}, a_2 = \frac{3}{G_2}$.

The solutions of (8.26) is in the form

$$\begin{cases} e_1 = c_{1,1}U_1(x) + c_{1,2}U_2(x) + c_{1,0}T_1(x), \\ e_2 = c_{2,1}U_1(x) + c_{2,2}U_2(x) + c_{2,0}T_1(x), \end{cases} \tag{8.27}$$

where $c_{1,1} = -\frac{b_{0,3}+b_1(A_2+1)}{b_{0,1}}, c_{1,2} = \frac{b_{0,3}+B_1a_2}{b_{0,1}}, c_{1,0} = \frac{b_1(A_2+1)}{b_{0,1}}$,
$c_{2,1} = \frac{b_{0,2}+b_1A_2}{b_{0,1}}, c_{2,2} = -\frac{b_{0,2}+(B_1+1)a_2}{b_{0,1}}, c_{2,0} = -\frac{b_1A_2}{b_{0,1}}$,
$b_{0,1} = (B_1+1)(A_2+1), b_{0,2} = A_2a_1 + (B_1+1)b_2, b_{0,3} = (A_2+1)a_1 + B_1b_2$. The solution of the 2 ODEs (8.9) for the **BCs of first kind** we can obtain with using two orthonormed eigenvectors $X_1(x), X_2(x)$ in following form:

$$U_1(x) = a_1^1 \sin(\pi x) + a_2^1 \sin(2\pi x), U_2(x) = a_1^2 \sin(\pi x) + a_2^2 \sin(2\pi x),$$

where the coeficients a_j^k satisfy folowing system of algebraic equations

$$\begin{cases} -k_1\lambda_1 a_1^1 + 2(c_{1,1}a_1^1 + c_{1,2}a_1^2 + c_{1,0}) = 0, \\ -k_2\lambda_1 a_1^2 + (c_{2,1}a_1^1 + c_{2,2}a_1^2 + c_{2,0}) = 0, \\ -k_1\lambda_2 a_2^1 + 2(c_{1,1}a_2^1 + c_{1,2}a_2^2) = 0, \\ -k_2\lambda_2 a_2^2 + (c_{2,1}a_2^1 + c_{2,2}a_2^2) + \beta P = 0, \end{cases} \tag{8.28}$$

where $\beta = 1$ for $f_2 = P\sin(2\pi x), \beta = 0.5$ for $f_2 = P\delta(y-2)\sin(2\pi x)$.
For **the periodical BCs** the solution we can obtain using the biorthonormed eigenvectors $X_1(x), X_1^*(x)$ in following form:

$$U_1(x) = a_1^1 \sin(2\pi x), U_2(x) = a_1^2 \sin(2\pi x),$$

where the coeficients a_1^1, a_1^2 can be obtained from folowing system of 2 algebraic equations

$$\begin{cases} -k_1\lambda_1 a_1^1 + 2(c_{1,1}a_1^1 + c_{1,2}a_1^2 + c_{1,0}) = 0, \\ -k_2\lambda_1 a_1^2 + c_{2,1}a_1^1 + c_{2,2}a_1^2 + c_{2,0} + \beta P = 0. \end{cases} \tag{8.29}$$

We have following solution of (8.28,8.29)
$a_1^1 = (2c_{1,2}(c_{2,0} + p_2\beta P) - 2c_{1,0}(c_{2,2} - k_2\lambda_1))/det_1$,
$a_1^2 = (2c_{2,1}c_{1,0} - (c_{2,0} + p_2\beta P)(2c_{1,1} - k_1\lambda_1))/det_1$,
$det_1 = (2c_{1,1} - k_1\lambda_1)(c_{2,2} - k_2\lambda_1) - 2c_{1,2}c_{2,1}$,
$a_2^1 = 2c_{1,2}\beta P/det_2, a_2^2 = -\beta P(2c_{1,1} - k_1\lambda_2)/det_2$,

$det_2 = (2c_{1,1} - k_1\lambda_2)(c_{2,2} - k_2\lambda_2) - 2c_{1,2}c_{2,1}$, where $p_2 = 1$ for periodical BCs, $p_2 = 0$ for BCs of first kind.
From (8.8) we can determined $u_1(x,y), u_2(x,y)$ because

$$\begin{cases} m_1(x) = \frac{1}{3k_1(G_1+G_2)}(6(U_2(x) - U_1(x)) - e_1(x)(G_1 + 3G_2) - 2e_2(x)G_2), \\ m_2(x) = \frac{1}{3k_2(G_1+G_2)}(6(U_2(x) - U_1(x)) + e_2(x)(G_2 + 3G_1) + 2e_1(x)G_1). \end{cases} \tag{8.30}$$

8.7.3 The solutions of the FDS and FDSES

The analytical **solution of FDS** from (8.19 - 8.21) we can obtained in following form:

$$\begin{cases} \bar{v}_{1,1}(y) = C_{1,1}\sinh(\sqrt{d_1}y) + B_{1,1}\cosh(\sqrt{d_1}y), y \in [0,1], \\ \bar{v}_{1,2}(y) = C_{1,2}\sinh(\sqrt{d_2}y) + B_{1,2}\cosh(\sqrt{d_2}y), y \in [0,1], \\ \bar{v}_{2,1}(y) = C_{2,1}\sinh(\sqrt{d_1}(y-1)) + B_{2,1}\cosh(\sqrt{d_1}(y-1)), y \in [1,3], \\ \bar{v}_{2,2}(y) = C_{2,2}\sinh(\sqrt{d_2}(y-1)) + B_{2,2}\cosh(\sqrt{d_2}(y-1)), y \in [1,3], \end{cases} \tag{8.31}$$

where the coefficients $C_{j,k}, B_{j,k}$ can be obtained from (1.25) similarly for $a_{j,k}$, where the value $\pi, 2\pi$ are replaced with $\sqrt{d_1}, \sqrt{d_2}$, and $B_{1,1} = \sqrt{N/2}$. For FDS $d_1 = \mu_1, d_2 = \mu_2$, but for FDSES $d_1 = \lambda_1, d_1 = \lambda_2$.
For **the periodical BCs** using the eigenvectors $w^1, w_*^1, (\sin(2\pi x_j) = \frac{\sqrt{N}}{2i}(w^1(x_j) - w_*^1(x_j), w_*^1(x_j) = w^{N-1}(x_j), \mu_{N-1} = \mu_1)$ we get

$$\begin{cases} \bar{v}_{1,1}(y) = C_{1,1}\sinh(\sqrt{d_1}y) + B_{1,1}\cosh(\sqrt{d_1}y), y \in [0,1], \\ \bar{v}_{2,1}(y) = C_{2,1}\sinh(\sqrt{d_1}(y-1)) + B_{2,1}\cosh(\sqrt{d_1}(y-1)) - \\ P\frac{p_1(3)}{k_2\sqrt{d_1}}, y \in [1,3], \end{cases} \tag{8.32}$$

where the coefficients $C_{j,k}, B_{j,k}$ can be obtained from (8.25) similarly as $a_{j,k}$, where the value 2π are replaced with $\sqrt{d_1}$. For FDS $d_1 = \mu_1$, but for FDSES $d_1 = \lambda_1$ (we have the exact solution).

8.8 Some numerical results

For **the BCc of first kind** the numerical results can be represented in following way:
1) $f_2(x,y) = 10\sin(2\pi)$- exact solution (Fig. 8.1), FDSES solution (Fig. 8.2), FDS solution (Fig. 8.3), averaged solution (Fig. 8.4),
2) $f_2(x,y) = 10\sin(2\pi)\delta(y-2)$- exact solution (Fig. 8.5), FDSES solution (Fig. 8.6), FDS solution (Fig. 8.7), averaged solution (Fig. 8.8).
For **periodical BCc** we have following representations:
1) $f_2(x,y) = 10\sin(2\pi)$- exact and FDSES solution (Fig. 8.10), FDS solution (Fig. 8.11), averaged solution (Fig. 8.12),
2) $f_2(x,y) = 10\sin(2\pi)\delta(y-2)$- exact and FDSES solution (Fig. 8.9), FDS solution (Fig. 8.13), averaged solution (Fig. 8.14).

8.9 MATLAB programs

1) BDCs of first kind:

```
function  Divi_slani(M,N)%x,y plakne,N=6
l1=1;l2=2.0;k1=10;k2=1;G1=l1/k1; G2=l2/k2;kap=k1/k2;
P=10;l=1;L=3.0;M1=M+1;x=linspace(0,l,M1);h=l/M;M2=M-1;
N1=N+1;y=linspace(0,L,N1)'; N11=N/L+1; N12 =N1-N11;N20=N11+1;
lk=4/(h^2)*(sin(0.5*(1:M2)'*h*pi/l)).^2;CK1=sqrt(2/M);%FDS
lk0=((1:M2)'*pi/l).^2; %FDSES
lk=lk0;
W=CK1*sin(pi*(1:M2)'*x(2:M)/l)';
%Analitiskais atrisinajums
C11=-(cosh(pi)+kap*sinh(pi)*tanh(2*pi)). . .
/(sinh(pi)+kap*cosh(pi)*tanh(2*pi));
B110=sqrt(M/2); KM1=sqrt(lk(1)); KM2=sqrt(lk(2));
C110=-B110*(cosh(KM1)+kap*sinh(KM1)*tanh(2*KM1)). . .
/(sinh(KM1)+kap*cosh(KM1)*tanh(2*KM1));
B21=C11*sinh(pi)+cosh(pi);C21=kap*(C11*cosh(pi)+sinh(pi));
B210=C110*sinh(KM1)+B110*cosh(KM1);
C210=kap*(C110*cosh(KM1)+B110*sinh(KM1));
%C10=P*sinh(2*pi)/( 2*pi*k2); %∆ funkcija
%C100=P*sinh(KM2)/( KM2*k2*CK1); %∆ funkcija
C10=P*(cosh(4*pi)-1)/(4*pi^2*k2);%nep. avots
C100=P*(cosh(2*KM2)-1)/(lk(2)*k2*CK1);%nep. avots
B11=1; C12= C10/( sinh(2*pi)+...
kap*cosh(2*pi)*tanh(4*pi))/cosh(4*pi);
C120= C100/( sinh(KM2)+kap*cosh(KM2)*tanh(2*KM2))/cosh(2*KM2);
B22=C12*sinh(2*pi);C22=kap*C12*cosh(2*pi);B12=0;
```

```
B220=C120*sinh(KM2);C220=kap*C120*cosh(KM2);B120=0;
Y11=C11*sinh(pi*y(1:N11))+B11*cosh(pi*y(1:N11));
Y110=C110*sinh(KM1*y(1:N11))+B110*cosh(KM1*y(1:N11));
Y12=C12*sinh(2*pi*y(1:N11))+B12*cosh(2^pi*y(1:N11));
Y120=C120*sinh(KM2*y(1:N11))+B120*cosh(KM2*y(1:N11));
Y21=C21*sinh(pi*(y(N20:N1)-l1))+B21*cosh(pi*(y(N20:N1)-l1));
Y210=C210*sinh(KM1*(y(N20:N1)-l1))+...
B210*cosh(KM1*(y(N20:N1)-l1));
Y22=C22*sinh(2*pi*(y(N20:N1)-l1))+...
B22*cosh(2*pi*(y(N20:N1)-l1));
Y220=C220*sinh(KM2*(y(N20:N1)-l1))+...
B220*cosh(KM2*(y(N20:N1)-l1)). . .
-P*(cosh(KM2*(y(N20:N1)-1))-1)/KM2^2/k2/CK1;%nep,FDS
%for i=N20:N1
 %  if y(i)≤ 2,
%Y220(i-N11)=C220*sinh(KM2*(y(i)-l1))+...
%B220*cosh(KM2*(y(i)-l1)); else
%Y220(i-N11)=C220*sinh(KM2*(y(i)-l1))+B220*cosh(KM2*(y(i)-l1))
%-P*sinh(KM2*(y(i)-2))/KM2/k2/CK1;
 %   end
%end % Δ, FDS
v1=zeros(M2,N1);v1(1,1:N11)=Y110;v1(2,1:N11)=. . .
Y120;v1(1,N20:N1)=Y210;v1(2,N20:N1)=Y220;
v2=zeros(M2,N1);v=zeros(M1,N1);v2=W*v1;v(2:M,:)=v2;
u=zeros(N1,M1);u1=zeros(N1,M1);
for i=1:N1
if y(i)≤ l1, u(i,:)= sin(pi*x(:))*Y11(i)+sin(2*pi*x(:))*Y12(i);
%elseif (l1 < y(i)&& y(i)≤  2) %Δ,prec
%u(i,:)=sin(pi*x(:))*Y21(i-N11)+sin(2*pi*x(:))*Y22(i-N11);
%Δ,prec
  else
%u(i,:)=sin(pi*x(:))*Y21(i-N11)+sin(2*pi*x(:))*. . .
(Y22(i-N11)-P*0.5*sinh(2*pi*(y(i)-2))/pi/k2);%Δ,prec
u(i,:)=sin(pi*x(:))*Y21(i-N11)+sin(2*pi*x(:))*(Y22(i-N11)-. . .
P*0.25*(cosh(2* pi*(y(i)-1))-1)/pi^2/k2);%nep
  end
end
Mu1=max(u(N11,:));mu1 =min(u(N11,:));
Mu=max(max(u(N11:N1,:)));mu =min(min(u(N11:N1,:)));
Mv1=max(v(:,N11));mv1 =min(v(:,N11));
Mv=max(max(v(:,N11:N1)));mv =min(min(v(:,N11:N1)));
figure
[C,h]=contour(x,y,v',20);
clabel(C);
title(sprintf('LevFDSES-nep , Maxv=%6.4f, Minv=%6.4f',. . .
Mv,mv)),xlabel('x'),ylabel('y')
%title(sprintf('LevFDSES-del , Maxv=%6.4f,. . .
 Minv=%6.4f',Mv,mv)),xlabel('x'),ylabel('y')
%title(sprintf('LevFDS-nep , Maxv=%6.4f,. . .
 Minv=%6.4f',Mv,mv)),xlabel('x'),ylabel('y')
%title(sprintf('LevFDS-del , Maxv=%6.4f,. . .
 Minv=%6.4f',Mv,mv)),xlabel('x'),ylabel('y')
figure,plot(x,v(:,N11))
%title(sprintf('FDSES-del u(x,1), Maxu(l1)=%6.4f,. . .
```

```
 Minu(l1)=%6.4f',Mv1,mv1)),xlabel('x'),ylabel('u(l1)')
title(sprintf('FDSES-nep u(x,1), Maxu(l1)=%6.4f,. . .
 Minu(l1)=%6.4f',Mv1,mv1)),xlabel('x'),ylabel('u(l1)')
%title(sprintf('FDS-del u(x,1), Maxu(l1)=%6.4f,. . .
 Minu(l1)=%6.4f',Mv1,mv1)),xlabel('x'),ylabel('u(l1)')
%title(sprintf('FDS-nep u(x,1), Maxu(l1)=%6.4f,. . .
 Minu(l1)=%6.4f',Mv1,mv1)),xlabel('x'),ylabel('u(l1)')
figure,plot(x,u(N11,:))
%title(sprintf('Prec-del u(x,1), Maxu(l1)=%6.4f,. . .
 Minu(l1)=%6.4f',Mu1,mu1)),xlabel('x'),ylabel('u(l1)')
title(sprintf('Prec-nep u(x,1), Maxu(l1)=%6.4f, . . .
Minu(l1)=%6.4f',Mu1,mu1)),xlabel('x'),ylabel('u(l1)')
figure
[C,h]=contour(x,y,u,20);
clabel(C);
%title(sprintf('Prec-del u, Maxu =%6.4f,. . .
 Minu=%6.4f',Mu,mu)),xlabel('x'),ylabel('u(l1)')
title(sprintf('Prec-nep u, Maxu=%6.4f,. . .
 Minu=%6.4f',Mu,mu)),xlabel('x'),ylabel('u(l1)')
X1=ones(N1,1)*x;Y1=y*ones(1,M1);
figure, surfc(X1,Y1,u),colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('Virsma, Maxu=%6.4f, Minu=%6.4f',Mu,mu))
%Viduvesana
B1=G2/(G1+G2);a1=3/(G1+G2); b1=3/G1;
A2=G1/(G1+G2);a2=3/G2;b2=a1;
b03=(A2+1)*a1+B1*b2;b02=A2*a1+(B1+1)*b2;
b01=(A2+1)*(B1+1)-B1*A2;
c12=(b03+B1*a2)/b01;
c11=-(b03+b1*(A2+1))/b01;c10=(A2+1)*b1/b01;
c22=-(b02 +(B1+1)*a2)/b01;
c21=(b02+b1*A2)/b01;c20=-b1*A2/b01;
det1=(2*c11-k1*pi^2)*(c22-k2*pi^2)-2*c12*c21;
det2=(2*c11-k1*4*pi^2)*(c22-k2*4*pi^2)-2*c12*c21;
a12=(-(2*c11-k1*pi^2)*c20+2*c10*c21)/det1;
a11=(-(c22-k2*pi^2)*2*c10+2*c20*c12)/det1;
a22=-P*(2*c11-k1*4*pi^2)/det2; a21=2*P*c12/det2;% nep. gad.
%a22=-0.5*P*(2*c11-k1*4*pi^2)/det2; a21=P*c12/det2;% ∆
U1=a11*sin(pi*x)+a21*sin(2*pi*x);
U2=a12*sin(pi*x)+a22*sin(2*pi*x);
E1=c12*U2+c11*U1+c10*sin(pi*x);
E2=c22*U2+c21*U1+c20*sin(pi*x);
M11=(6*(U2-U1)-E1*(G1+3*G2)-2*E2*G2)/(3*k1*(G1+G2));
M2=(6*(U2-U1)+E2*(G2+3*G1)+2*E1*G1)/(3*k2*(G1+G2));
for i=1:N1
if y(i)≤ l1, u1(i,:)= U1(:)+M11(:)*(y(i)-0.5). . .
+E1(:)*G1*((y(i)-0.5)^2-1/12);
else
u1(i,:)= U2(:)+M2(:)*(y(i)-2)+E2(:)*G2*(0.25*(y(i)-2)^2-1/12);
end
end
Mu2=max(u1(N11,:));mu2 =min(u1(N11,:));MU1=max(U1);. . .
mU1=min(U1);MU2=max(U2);mU2=min(U2);
Mu=max(max(u1(N11:N1,:)));mu =min(min(u1(N11:N1,:)));
```

```
figure
[c,h]=contour(x,y,u1,20);
clabel(c);
%title(sprintf('LevAv-del u1, Maxu =%6.4f,. . .
 Minu=%6.4f',Mu,mu)),xlabel('x'),ylabel('y')
title(sprintf('LevAv-nep u1, Maxu =%6.4f, . . .
Minu=%6.4f',Mu,mu)),xlabel('x'),ylabel('y')
figure,plot(x,u1(N11,:))
%title(sprintf('FuncAv-del u(l1), Maxv(l1)=%6.4f,. . .
 Minv(l1)=%6.4f',Mu2,mu2)),xlabel('x'),ylabel('u(l1)')
title(sprintf('FuncAv-nep u(l1), Maxv(l1)=%6.4f,. . .
 Minv(l1)=%6.4f',Mu2,mu2)),xlabel('x'),ylabel('u(l1)')
figure,plot(x,U1)
%title(sprintf('V1Av-del , Max=%6.4f, Min=%6.4f',. . .
MU1,mU1)),xlabel('x'),ylabel('U1')
title(sprintf('V1Av-nep , Max=%6.4f, Min=%6.4f',. . .
MU1,mU1)),xlabel('x'),ylabel('U1')
figure,plot(x,U2)
%title(sprintf('V2Av-del, Maxv=%6.4f, Min=%6.4f',. . .
MU2,mU2)),xlabel('x'),ylabel('U2')
title(sprintf('V2Av-nep, Maxv=%6.4f, Min=%6.4f',. . .
MU2,mU2)),xlabel('x'),ylabel('U2')
X2=ones(N1,1)*x;Y2=y*ones(1,M1);
figure, surfc(X2,Y2,u1),colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('VirsmaAv, Maxv=%6.4f, Minv=%6.4f',Mu,mu))
```

Using the operator **Divi-slani(10,6)**we have following results for maximal Mu and minimal value mu of the solution $u_2(x,y)$
(for the exact and FDSES solution-$Mu = 0.2446, mu = -0.2381, f_2(x,y) = 10\sin(2\pi)$; $Mu = 0.7596, mu = -0.7548, f_2(x,y) = 10\sin(2\pi)\delta(y-2))$:
1) for FDS and $f_2(x,y) = 10\sin(2\pi), Mu = 0.2521, mu = -0.2460$,
2) for FDS and $f_2(x,y) = 10\sin(2\pi)\delta(y-2), Mu0.7722, mu = -0.7674$.

1) Periodical BDCs:

```
function  Divi_slaniP(M,N)%x,y plakne,N=6, M-para sk.
l1=1;l2=2.0;k1=10;k2=1;G1=l1/k1; G2=l2/k2;kap=k1/k2;
P=10;l=1;L=3.0;M1=M+1;x=linspace(0,l,M1);h=l/M;M2=M-1;NH=M/2;
h1=L/N; N1=N+1;y=linspace(0,L,N1)'; N11=N/L+1;N20=N11+1;
NT=(1:M)'/l;d=zeros(M,1);
lk=4/h^2*(sin(pi*h*NT)).^2; %2.order FDS
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4);
%4.order FDS
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4. . .
+8/45*(sin(pi*h*NT)).^6);%6.order FDS
%lk=4/h^2*((sin(pi*h*NT)).^2+1/3*(sin(pi*h*NT)).^4+. . .
8/45*(sin(pi*h*NT)).^6+4/35*(sin(pi*h*NT)).^8);
```

```
%8.order FDS
Ck=sqrt(h/L);
lk0=(2*(1:M)'*pi/l).^2;
d=lk; %FDS
d(1:NH)=lk0(1:NH);
d(NH:M2)=lk0(NH:-1:1);%FDSES
W=Ck*exp(2*pi*i*(1:M)'*x/l)';
W1=Ck*exp(-2*pi*i*(1:M)'*x/l)';
pi1=sqrt(d(1));
%Analitiskais atrisinajums
C10=P*sinh(pi1)/( pi1*k2);beta=0.5; %Δ funkcija
%C10=P*(cosh(2*pi1)-1)/(pi1^2*k2);beta=1;%nep. avots
C11=-(cosh(pi1)+kap*sinh(pi1)*tanh(2*pi1)-C10/cosh(2*pi1)). . .
/(sinh(pi1)+kap*cosh(pi1)*tanh(2*pi1));
B21=C11*sinh(pi1)+cosh(pi1);
C21=kap*(C11*cosh(pi1)+sinh(pi1));
B11=1;
Y11=C11*sinh(pi1*y(1:N11))+B11*cosh(pi1*y(1:N11));
Y21=C21*sinh(pi1*(y(N20:N1)-l1))+...
 B21*cosh(pi1*(y(N20:N1)-l1));
u=zeros(N1,M1);u1=zeros(N1,M1);
for i1=1:N1
    if y(i1)≤ l1, u(i1,:)= sin(2*pi*x(:))*Y11(i1);
     elseif (l1 < y(i1)&& y(i1)≤  2) %Δ,prec
      u(i1,:)=sin(2*pi*x(:))*Y21(i1-N11);%Δ,prec
    else
u(i1,:)=sin(2*pi*x(:))*(Y21(i1-N11)-...
P*sinh(pi1*(y(i1)-2))/pi1/k2);%Δ
%u(i1,:)=sin(2*pi*x(:))*(Y21(i1-N11)-...
%P*(cosh(pi1*(y(i1)-1))-1)/pi1^2/k2);%nep
    end
end
%C21*sinh(2*pi1)+B21*cosh(2*pi1) -C10, u(N1,:)'
%Y21
Mu=max(max(u(N11:N1,:)));mu =min(min(u(N11:N1,:)));
Mu1=max(u(N11,:));mu1 =min(u(N11,:));
figure,plot(x,u(N11,:))
%title(sprintf('Exact-cont u(x,1), Max u=%6.4f, . . .
Min u=%6.4f',Mu1,mu1)),xlabel('x'),ylabel('u(l1)')
%title(sprintf('FDS-cont u(x,1), Max u=%6.4f,. . .
 Min u=%6.4f',Mu1,mu1)),xlabel('x'),ylabel('u(l1)')
%title(sprintf('FDS-Δ u(x,1), Max u=%6.4f,. . .
 Min u=%6.4f',Mu1,mu1)),xlabel('x'),ylabel('u(l1)')
title(sprintf('Exact-Δ u(x,1), Max u=%6.4f,. . .
 Min u=%6.4f',Mu1,mu1)),xlabel('x'),ylabel('u(l1)')
figure
[C,h]=contour(x,y,u,20);
clabel(C);
%title(sprintf('Exact-cont u , Max u=%6.4f,. . .
 Min u=%6.4f',Mu,mu)),xlabel('x'),ylabel('y')
%title(sprintf('FDS-cont u , Max u=%6.4f,. . .
 Min u=%6.4f',Mu,mu)),xlabel('x'),ylabel('y')
%title(sprintf('FDS-Δ u , Max u=%6.4f,. . .
 Min u=%6.4f',Mu,mu)),xlabel('x'),ylabel('y')
```

```
title(sprintf('Exact-Δ u , Max u=%6.4f,. . .
 Min u=%6.4f',Mu,mu)),xlabel('x'),ylabel('y')
X1=ones(N1,1)*x;Y1=y*ones(1,M1);
figure, surfc(X1,Y1,u),colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('Virsma, Max u=...
%6.4f, Min u=%6.4f',Mu,mu))
%Viduvesana
pi1=2*pi;
B1=G2/(G1+G2);a1=3/(G1+G2); b1=3/G1;
A2=G1/(G1+G2);a2=3/G2;b2=a1;
b03=(A2+1)*a1+B1*b2;b02=A2*a1+(B1+1)*b2;
 b01=(A2+1)*(B1+1)-B1*A2;
c12=(b03+B1*a2)/b01;
 c11=-(b03+b1*(A2+1))/b01;c10=(A2+1)*b1/b01;
c22=-(b02 +(B1+1)*a2)/b01;
c21=(b02+b1*A2)/b01;
c20=-b1*A2/b01;
det1=(2*c11-k1*pi1^2)*(c22-k2*pi1^2)-2*c12*c21;
a12=(-(2*c11-k1*pi1^2)*(c20+beta*P)+...
2*c10*c21)/det1;
a11=(-(c22-k2*pi1^2)*2*c10+2*c12*(c20+...
beta*P))/det1;
U1=a11*sin(2*pi*x);
U2=a12*sin(2*pi*x);E1=c12*U2+c11*U1+...
c10*sin(2*pi*x);
E2=c22*U2+c21*U1+c20*sin(2*pi*x);
M11=(6*(U2-U1)-E1*(G1+3*G2)-...
2*E2*G2)/(3*k1*(G1+G2));
M2=(6*(U2-U1)+E2*(G2+3*G1)+...
2*E1*G1)/(3*k2*(G1+G2));
for i1=1:N1
if y(i1)≤ l1, u1(i1,:)= U1(:)+M11(:)*(y(i1)-0.5). . .
 +E1(:)*G1*((y(i1)-0.5)^2-1/12);
    else
u1(i1,:)= U2(:)+M2(:)*(y(i1)-2)+E2(:)*G2*(0.25*...
(y(i1)-2)^2-1/12);
    end
end
Mu2=max(u1(N11,:)); mu2=min(u1(N11,:));
Mu1=max(max(u1(N11:N1,:)));
mu1 =min(min(u1(N11:N1,:)));
MU1=max(U1);mU1=min(U1);
MU2=max(U2);mU2=min(U2);
figure
[c,h]=contour(x,y,u1,20);
clabel(c);
%title(sprintf('Av-cont u, Max v=%6.4f, Min v=%6.4f',. . .
Mu1,mu1)),xlabel('x'),ylabel('y')
title(sprintf('Av-Δ, Max v=%6.4f, . . .
Min v=%6.4f',Mu1,mu1)),xlabel('x'),ylabel('y')
figure,plot(x,u1(N11,:))
%title(sprintf('Av-cont u(x,1), Maxv(l1)=%6.4f,. . .
 Minv(l1)=%6.4f',Mu2,mu2)),xlabel('x'),ylabel('u(l1)')
```

```
title(sprintf('Av-Δ u(x,1), Maxv(l1)=%6.4f,. . .
 Minv(l1)=%6.4f',Mu2,mu2)),xlabel('x'),ylabel('u(l1)')
figure,plot(x,U1)
%title(sprintf('V1Av-cont , MaxU1=%6.4f,. . .
 MinU1=%6.4f',MU1,mU1)),xlabel('x'),ylabel('U1')
title(sprintf('V1Av-Δ , MaxU1=%6.4f,. . .
 MinU1=%6.4f',MU1,mU1)),xlabel('x'),ylabel('U1')
figure,plot(x,U2)
%title(sprintf('V2Av-cont , MaxU2=%6.4f,. . .
 MinU2=%6.4f',MU2,mU2)),xlabel('x'),ylabel('U2')
title(sprintf('V2Av-Δ , MaxU2=%6.4f,. . .
 MinU2=%6.4f',MU2,mU2)),xlabel('x'),ylabel('U2')
X2=ones(N1,1)*x;Y2=y*ones(1,M1);
figure, surfc(X2,Y2,u1),colorbar
xlabel('x'), ylabel('y'), zlabel('u')
title(sprintf('VirsmaAv, Max v=...
%6.4f, Min v=%6.4f',Mu1,mu1))
```

Using the operator **DiviP-slani(10,6)** and using the diferent finite difference aproximations of p order $(p = 2;4;6;8)$ we have following results for maximal and minimal value mu of the solution $u_2(x,y)$ (for the exact and FDSES solution-$mu = \pm 0.2401, f_2(x,y) = 10\sin(2\pi)$; $mu = \pm 0.7568, f_2(x,y) = 10\sin(2\pi)\delta(y-2))$:
1) for $f_2(x,y) = 10\sin(2\pi), mu = \pm 0.2480(p=2); \pm 0.2405(p=4)$; $\pm 0.2401(p=6); \pm 0.2401(p=8)$,
2) for $f_2(x,y) = 10\sin(2\pi)\delta(y-2), mu = \pm 0.7694(p=2)$; $\pm 0.7575(p=4)$; $\pm 0.7569(p=6); \pm 0.7568(p=8)$.
We can see, that the FDSES method is exact.

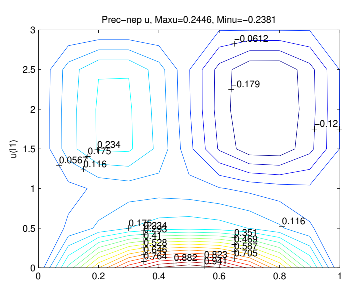


Fig. 8.1 Exact solution -cont

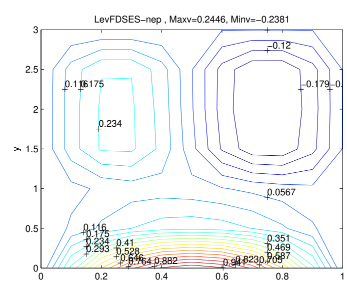


Fig. 8.2 FDSES solution -cont

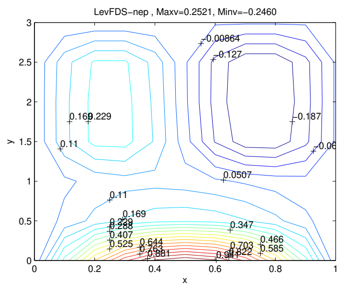


Fig. 8.3 FDS solution -cont

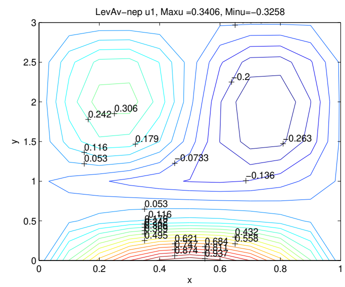


Fig. 8.4 Averaged solution-cont

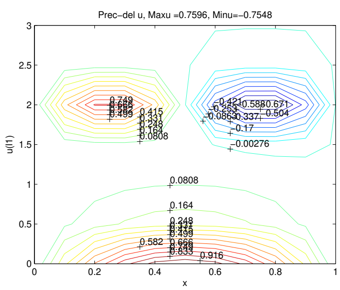


Fig. 8.5 Exact solution -delta

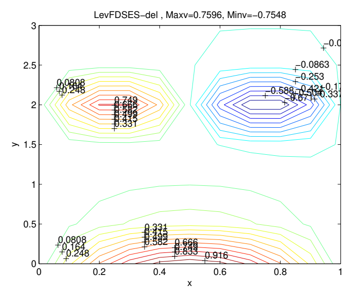


Fig. 8.6 FDSES solution -delta

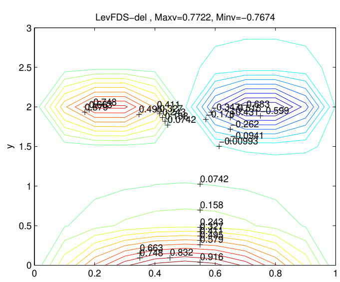


Fig. 8.7 FDS solution -delta

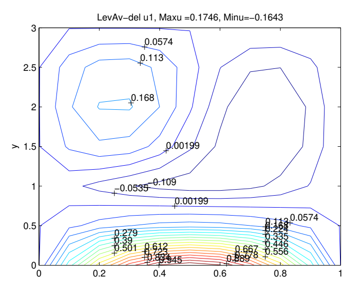


Fig. 8.8 Averaged solution-delta

8.10 Formulation of the 3-D diffusion-convection problem

The process of diffusion and convection in z-direction is considered in 3-D parallelepiped

$$\Omega = \{(x,y,z) : 0 \le x \le L_x, 0 \le y \le L_y, 0 \le z \le L_z\}.$$

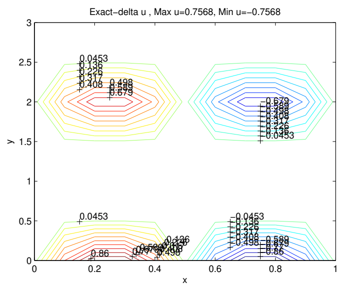


Fig. 8.9 FDSES solution -delta

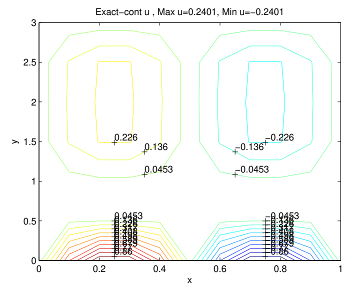


Fig. 8.10 FDSES solution -cont

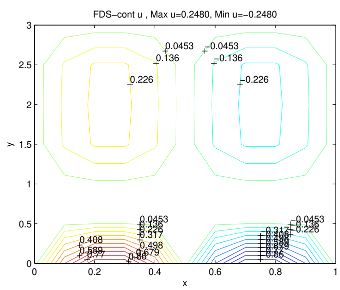


Fig. 8.11 FDS solution -cont

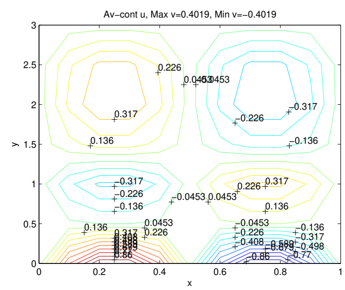


Fig. 8.12 Averaged solution-cont

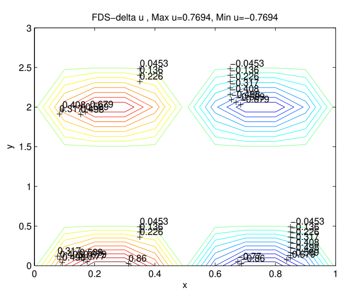


Fig. 8.13 FDS solution -delta

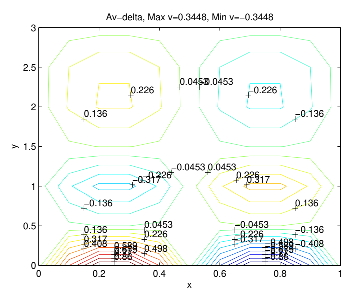


Fig. 8.14 Averaged solution-delta

We will consider the nonstationary 3-D problem of the linear diffusion theory for multilayered piece-wise homogenous materials of N layers in the z-direction

$$\Omega_i = \{(x,y,z) : x \in (0,L_x), y \in (0,L_y), z \in (z_{i-1},z_i)\}, i = \overline{1,N},$$

where $H_i = z_i - z_{i-1}$ is the height of layer $\Omega_i, z_0 = 0, z_N = L_z$.

We will find the distribution of concentrations $c_i = c_i(x,y,z,t)$ in every layer at the point $(x,y,z) \in \Omega_i$ and at the time t by solving the following 3-D initial-boundary value problem for partial differential equations (PDEs):

$$\begin{cases} \frac{\partial c_i}{\partial t} = L_{xy}(c_i) + \frac{\partial}{\partial z}(D_{iz}\frac{\partial c_i(x,y,z,t)}{\partial z}) + r_{iz}\frac{\partial c_i}{\partial z}, \\ x \in (0,L_x), y \in (0,L_y), z \in (z_{i-1},z_i), t \in (0,t_f), i = \overline{1,N}, \\ \frac{\partial c_i(0,y,z,t)}{\partial x} = \frac{\partial c_i(x,0,z,t)}{\partial y} = 0, \\ D_{1z}\frac{\partial c_1(x,y,0,t)}{\partial z} - \beta_z(c_1(x,y,0,t) - c_{oz}) = 0, \\ D_{ix}\frac{\partial c_i(L_x,y,z,t)}{\partial x} + \alpha_{ix}(c_i(L_x,y,z,t) - c_{iax}) = 0, i = \overline{1,N}, \\ D_{iy}\frac{\partial c_i(x,L_y,z,t)}{\partial y} + \alpha_{iy}(c_i(x,L_y,z,t) - c_{iay}) = 0, i = \overline{1,N}, \\ D_{Nz}\frac{\partial c_N(x,y,L_z,t)}{\partial z} + \alpha_z(c_N(x,y,L_z,t) - c_{az}) = 0, \\ r_{iz}c_i(x,y,z_i,t) = r_{i+1,z}c_{i+1}(x,y,z_i,t), \\ D_{iz}\frac{\partial c_i(x,y,z_i,t)}{\partial z} = D_{i+1,z}\frac{\partial c_{i+1}(x,y,z_i,t)}{\partial z}, i = \overline{1,N-1} \\ c_i(x,y,z,0) = c_{i0}, i = \overline{1,N}, \end{cases} \tag{8.33}$$

where $L_{xy}(c_i) = \frac{\partial}{\partial x}(D_{ix}\frac{\partial c_i}{\partial x}) + \frac{\partial}{\partial y}(D_{iy}\frac{\partial c_i}{\partial y}) - a_0^2 c_i$ are linear diferential expressions depending on x,y
$D_{ix}, D_{iy}, D_{iz}, r_{iz}, a_0$ are the constant coefficients, $\alpha_{ix}, \alpha_{iy}, \alpha_z, \beta_z, i = \overline{1,N}$ are the constant mass transfer coefficients in the 3 kind boundary conditions, $c_{az}, c_{iay}, c_{iax}, c_{oz}$ are the given concentration on the boundary, t_f is the final time, c_{i0} are the given initial concentration. We have following boundary conditions:
1) the homogenous 3-kind conditions by $x = L_x, y = L_y; z = L_z, z = 0$,
2) the symetrical conditions by $x = 0; y = 0$,
3) the continues conditions for values $r_{iz}c_i$ and the flux functions $D_{iz}\frac{\partial c_i}{\partial z}$ on the contact lines between the layers, $i = \overline{1,N-1}$
(for differen r_{iz} the functions c_i are discontinuos on the contact lines $z_i = H_i$.)

8.11 CAM in z-direction using integral hyperbolic type splines for 1-D problem

First we will consider following 1-D initial-boundary value problem in z-direction

$$\begin{cases} \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial z}(D_{iz}\frac{\partial u_i}{\partial z}) + r_{iz}\frac{\partial u_i}{\partial z}, \\ z \in (z_{i-1}, z_i), t \in (0, t_f), i = \overline{1,N}, \\ D_{1z}\frac{\partial u_1(0,t)}{\partial z} - \beta_z(u_1(0,t) - u_{oz}) = 0, \\ D_{Nz}\frac{\partial u_N(L_z,t)}{\partial z} + \alpha_z(u_N(L_z,t) - u_{az}) = 0, t = \overline{0,t_f} \\ r_{iz}u_i(z_i,t) = r_{i+1,z}u_{i+1}(z_i,t), \\ D_{iz}\frac{\partial u_i(z_i,t)}{\partial z} = D_{i+1,z}\frac{\partial u_{i+1}(z_i,t)}{\partial z}, i = \overline{1,N-1} \\ u_i(z,0) = u_{i0}, i = \overline{1,N}, \end{cases} \tag{8.34}$$

where $u_i = u_i(z,t)$ are the unknown functions in every layer, D_{iz}, r_{iz} are the constant coefficients, $\alpha_z, \beta_z, i = \overline{1,N}$ are the constant mass transfer coefficients.
Using the transformation $u_i(z,t) = \exp(-r_{iv}z)v_i(z,t), r_{iv} = \frac{r_{iz}}{2D_{iz}}$ we can reduced the problem (8.34) to the problem without the convective term:

$$\begin{cases} \frac{\partial v_i}{\partial t} = (D_{iz}(\frac{\partial^2 v_i}{\partial z^2} - r_{iv}^2 u_i), z \in (z_{i-1}, z_i), t \in (0, t_f), i = \overline{1,N}, \\ D_{1z}\frac{\partial v_1(0,t)}{\partial z} - r_{1v}v_1(0,t) - \beta_z(v_1(0,t) - c_{oz}) = 0, \\ D_{Nz}\frac{\partial v_N(L_z,t)}{\partial z} - r_{Nv}v_N(L_z,t) + \alpha_z(v_N(L_z,t) - c_{az}exp(r_{Nv}L_z)) = 0, \\ r_{iz}v_i(z_i,t) = r_{i+1,z}v_{i+1}(z_i,t)\exp((r_{iv} - r_{i+1,v})z_i), \\ D_{iz}\frac{\partial v_i(z_i,t)}{\partial z} = D_{i+1,z}\frac{\partial v_{i+1}(z_i,t)}{\partial z}\exp((r_{iv} - r_{i+1,v})z_i), i = \overline{1,N-1} \\ v_i(z,0) = u_{i0}exp(r_{iv}z), i = \overline{1,N}, t = \overline{0,t_f} \end{cases} \tag{8.35}$$

In this case we can easy determine the integral spline parameters for the conservative averaging method (CAM).

8.11.1 Stationary problem

For stationary problem we can obtain analytical solution at every layer in following form:
$v_i(z) = m_i \sinh(r_{iv}(z - z_i*)) + e_i \cosh(r_{iv}(z - z_i*))$,
where $z_i* = (z_{i-1} + z_i)/2$.
From BCs and the continuous conditions can be obtain N-linear algebraic equations for determine the unknown coefficients e_i, m_i. Using integral hyperbolic type splines:

$$v_i(z) = v_{iz} + m_{iz}\frac{0.5\ H_i \sinh(r_{iv}(z - z_i*))}{\sinh(0.5 r_{iv} H_i)} + e_{iz}\frac{\cosh(r_{iv}(z - z_i*)) - A_i}{8\sinh^2(0.25 r_{iv} H_i)}, \tag{8.36}$$

where $A_i = \frac{\sinh(0.5r_{iv}H_i)}{0.5r_{iv}H_i}, v_{iz} = \frac{1}{H_i}\int_{z_{i-1}}^{z_i} v_i(z)dz$ are the mean integral values of $v_i(z)$.
Similarly from BCs and the continuous conditions we can determine the unknown coefficients e_{iz}, m_{iz} depends on the mean integral values v_{iz} . By using the mean integral values of differential equation we obtain for determine v_{iz} linear algebraic equation in following form:
$0.5 e_{iz} r_{iv} \coth(0.25 r_{iv} H_i)/H_i - r_{iv}^2 v_{iz} = 0.$
We can see, that the hyperbolic type splines gives exact solution for the previous boundary-value problem with
$m_i = \frac{0.5 m_{iz} H_i}{\sinh(0.5 r_{iv} H_i)}, e_i = 0.125\frac{e_{iz}}{\sinh^2(0.25 r_{iv} H_i)}, v_{iz} = e_i A_i.$

Example 1.

For 2 layers (N=2) we have 4 algebraic equations for determine the unknown spline parameters m_{iz}, e_{iz} depending on $v_{1z}, v_{2z}, i = 1; 2$:
$z = 0 : d_{1z}m_{1z} - k_{1z}e_{1z} - a_1(v_{1z} - 0.5m_{1z}H_1 + b_{1z}e_{1z}) + \beta^* u_{0z} = 0,$
$z = L_z : d_{2z}m_{2z} + k_{2z}e_{2z} + a_2(v_{2z} + 0.5m_{2z}H_2 + b_{2z}e_{2z}) -$
$\alpha^* u_{az} exp(r_{2v}L_z) = 0,$
$z = H_1 : v_{1z} + 0.5m_{1z}H_1 + b_{1z}e_{1z} = a_{44}(v_{2z} - 0.5m_{2z}H_2 + b_{2z}e_{2z}),$
$z = H_1 : d_{1z}m_{1z} + k_{1z}e_{1z} = a_{33}(d_{2z}m_{2z} - k_{2z}e_{2z}),$
where $\alpha^* = \alpha/D_{2z}, \beta^* = \beta/D_{1z}, a_1 = \beta^* + r_{1v}, a_2 = \alpha^* - r_{2v}, D_{12} = D_{2z}/D_{1z},$
$d_{iz} = 0.5 r_{iv} H_i \coth(0.5 r_{iv} H_i), k_{iz} = 0.25 r_{iv} \coth(0.25 r_{iv} H_i),$
$b_{iz} = 0.125\frac{\cosh(0.5 r_{iv} H_i) - A_i}{\sinh^2(0.25 r_{iv} H_i)}, a_{33} = D_{12} exp((r_{1v} - r_{2v})H_1),$
$a_{44} = \frac{r_{2z}}{r_{1z}} exp((r_{1v} - r_{2v})H_1).$
For two first equations follows $m_{1z} = (b_{22}e_{1z} + a_1 v_{1z} - b_{55})/b_{11}$, $m_{2z} = (-b_{44}e_{2z} - a_2 v_{2z} - b_{66})/b_{33}$, where $b_{11} = d_{1z} + 0.5H_1 a_1, b_{22} = k_{1z} + b_{1z}a_1, b_{33} = d_{2z} + 0.5H_2 a_2, b_{44} = k_{2z} + b_{2z}a_2, b_{55} = \beta * u_{0z}, b_{66} = \alpha * u_{az} exp(r_{2v}L_z).$
Using other BCs we obtain two linear algebraic equations for determine the unknown parameters e_{1z}, e_{2z} depending on v_{1z}, v_{2z} :

$$\begin{cases} c_1 e_{1z} + c_2 e_{2z} = f_1 v_{1z} + f_2 v_{2z} + g_{11}, \\ c_3 e_{1z} - c_4 e_{2z} = f_3 v_{1z} + f_4 v_2 + g_{22}, \end{cases} \tag{8.37}$$

where $c_1 = d_{1z}b_{22}/b_{11} + k_{1z}, c_2 = a_{33}(d_{2z}b_{44}/b_{33} + k_{2z}),$
$c_3 = 0.5H_1 b_{22}/b_{11} + b_{1z}, c_4 = a_{44}(0.5H_2 b_{44}/b_{33} + b_{2z},$

$f_1 = -d_{1z}a_1/b_{11}, f_2 = -a_{33}a_2d_{2z}/b_{33}$,
$f_3 = -0.5H_1a_1/b_{11}, f_4 = a_{44}0.5H_2a_2/b_{33}$,
$g_{11} = d_{1z}b_{55}/b_{11} + a_{33}b_{66}d_{2z}/b_{33}, g_{22} = 0.5H_1b_{55}/b_{11} - a_{44}b_{66}0.5H_2/b_{33}$.
We have following solutions:

$$\begin{cases} e_{1z} = e_{11}v_{1z} + e_{12}v_{2z} + e_{10}, \\ e_{2z} = e_{21}v_{1z} + e_{22}v_{2z} + e_{20}, \end{cases} \tag{8.38}$$

where $e_{11} = (c_4f_1 + c_2f_3)/det, e_{12} = (c_4f_2 + c_2f_4)/det, e_{21} = (c_3f_1 - c_1f_3)/det, e_{22} = (c_3f_2 - c_1f_4)/det$,
$e_{10} = (c_4g_{11} + c_2g_{22})/det, e_{20} = (c_3g_{11} - c_1g_{22})/det, det = c_4c_1 + c_2c_3$.
By using the mean integral values of differential equation we obtain for determine v_{iz} linear algebraic equation in following form:

$$\begin{cases} (G_1e_{11} - r_{1v}^2)v_{1z} + G_1e_{12}v_{2z} = G_{11}, \\ (G_2e_{21} + (G_2e_{22} - r_{2v}^2)v_{2z} = G_{22}, \end{cases} \tag{8.39}$$

where $G_1 = 2k_{1z}/H_1, G_2 = 2k_{2z}/H_2, G_{11} = -G_1e_{10}, G_{22} = -G_2e_{20}$.
The solution is

$$\begin{cases} v_{1z} = (G_{11}(G_2e_{22} - r_{2v}^2) - G_{22}G_1e_{12})/det_1, \\ v_{2z} = (G_{22}(G_1e_{11} - r_{1v}^2) - G_{11}G_2e_{21})/det_1, \end{cases} \tag{8.40}$$

where $det_1 = (G_2e_{22} - r_{2v}^2)(G_1e_{11} - r_{1v}^2) - G_2e_{21}G_1e_{12}$.
In special case for N=2,$L_z = 3, H_1 = 1.8, H_2 = 1.2, r_{1z} = 0.4, r_{2z} = 0.1, \alpha = 100, \beta = 1000$, $u_{0z} = 1, u_{az} = 10, D_{1z} = 0.2, D_{2z} = 0.1$
we have discontinuous solutions with
$u_1(H_1) - u_2(H_1) = -7.413, v_1(H_1) - v_2(H_1) = -9.362, v_{1z} = 6.347, v_{2z} = 33.560$(see Fig. 8.15)
If $r_{1z} = 0.1, r_{2z} = 0.1, D_{1z} = 0.5, D_{2z}0.1$ then we have u-continuous solutions, v-discontinuous solutions with derivatives and discontinuous u-derivatives with $u_1(H_1) - u_2(H_1) = 0, , v_1(H_1) - v_2(H_1) = -5.605, v_{1z} = 3.150, v_{2z} = 27.036$ (see Fig. 8.16).

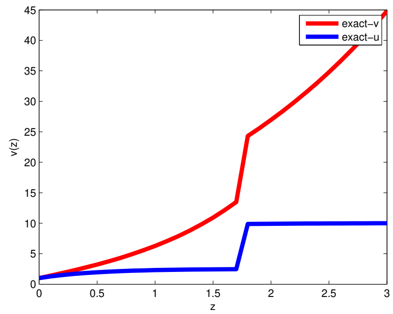


Fig. 8.15 Exact and spline v,u-solutions for $r_{1z} = 0.4, r_{2z} = 0.1$

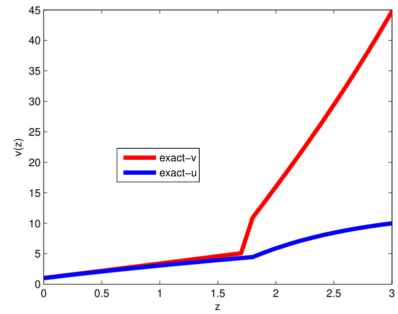


Fig. 8.16 Exact and spline v,u- solutions for $r_{1z} = 0.1, r_{2z} = 0.1$

8.11.2 Nonstationary problem

For nonstationary problem (8.35) using integral hyperbolic type splines (8.36) with $v_i(z,t), v_{iz}(t), m_{iz}(t), e_{iz}(t)$ from BCs and the continuous conditions we can determine the unknown functions $e_{iz}(t), m_{iz}(t)$ depends on the mean integral values $v_{iz}(t)$.
By using the mean integral values of PDEs we obtain for determine $v_{iz}(t)$ linear system of N ODEs in form:
$0.5e_{iz}(t)r_{iv}\coth(0.25r_{iv}H_i)/H_i - r_{iv}^2 v_{iz}(t) = \frac{1}{D_{iz}}\frac{\partial v_{iz}(t)}{\partial t}$.

Example 2.
For nonstationary problem in two layers we have the functions $m_{1z}, m_{2z}, e_{1z}, e_{2z}, v_{1z}, v_{2z}$ depending on t and using the equations (8.37, 8.38) we obtain the two ODEs from (8.39) in the following form:

$$\begin{cases} (G_1e_{11} - r_{1v}^2)v_{1z}(t) + G_1e_{12}v_{2z}(t) = \frac{1}{D_{1z}}\frac{dv_{1z}(t)}{dt} - G_1e_{10}, \\ (G_2e_{21}v_{1z}(t) + (G_2e_{22} - r_{2v}^2)v_{2z}(t) = \frac{1}{D_{2z}}\frac{dv_{2z}(t)}{dt} - G_2e_{20}. \end{cases} \quad (8.41)$$

or

$$\begin{cases} v_{1z}(t) - q_{10} = q_{11}(\frac{dv_{1z}(t)}{dt} - q_{12}\frac{dv_{2z}(t)}{dt}, \\ v_{2z}(t) - q_{20} = -q_{21}(\frac{dv_{1z}(t)}{dt} + q_{22}\frac{dv_{2z}(t)}{dt}, \end{cases} \quad (8.42)$$

where $q_{10} = (-G_1e_{10}(G_2e_{22} - r_{2v}^2) + G_2e_{20}G_1e_{12})/det$,
$q_{20} = (-G_2e_{20}(G_1e_{11} - r_{1v}^2) + G_2e_{21}G_1e_{10})/det$, $q_{11} = \frac{G_2e_{22}-r_{2v}^2}{detD_{1z}}$,
$q_{22} = \frac{G_1e_{11}-r_{1v}^2}{detD_{2z}}$, $q_{12} = \frac{G_1e_{12}}{detD_{2z}}$, $q_{21} = \frac{G_2e_{21}}{detD_{12z}}$,
$det = (G_2e_{22} - r_{2v}^2)(G_1e_{11} - r_{1v}^2) - G_2e_{21}G_1e_{12}$.

The normal system of 2 ODEs with first order is following form:

$$\begin{cases} \frac{dv_{1z}(t)}{dt} = d_{11}v_{1z}(t) + d_{12}v_{2z}(t) + d_{10}, \\ \frac{dv_{2z}(t)}{dt} = d_{21}v_{1z}(t) + d_{22}v_{2z}(t) + d_{20}, \end{cases} \tag{8.43}$$

where $d_{11} = \frac{q_{22}}{det_2}, d_{22} = \frac{q_{11}}{det_2}, d_{12} = \frac{q_{12}}{det_2}, d_{21} = \frac{q_{21}}{det_2}$,
$d_{10} = -(q_{22}q_{10} + q_{12}q_{20})/det_2, d_{20} = -(q_{21}q_{10} + q_{11}q_{20})/det_2$.
In the vector form we have ODEs $\frac{dv(t)}{dt} = Dv + d_0$, with vector-column $v = [v_{1z}, v_{2z}], d_0 = [d_{10}, d_{20}]$ and matrix $D = [d_{11}, d_{12}; d_{21}, d_{22}]$. The solution with homogenous initial conditions is $v(t) = (\exp(Dt) - E)D^{-1}d_0$,
E is unit matrix of second oprder.

In special case for N=2,
$v_{1z}(0) = v_{2z}(0) = 0, D_{1z} = 0.2, D_{2z} = 0.1, t_f = 30$ we have discontinuous solutions with
$v_{1z}(t_f) = 6.344, v_{2z}(t_f) = 33.555, u_1(H_1,t_f) = 2.47, u_2(H_1,t_f) = 9.88$, $v_1(H_1,t_f) = 14.94, v_2(H_1,t_f) = 24.30$ (see Fig. 8.17, Fig. 8.18)
For comparision we use Matlab routine "pdepe" with $u_1(H_1,t_f) = 2.67, u_2(H_1,t_f) =$ '10.01. For this routine the continuous conditions are approximated with first order differences with space step h=0.1.

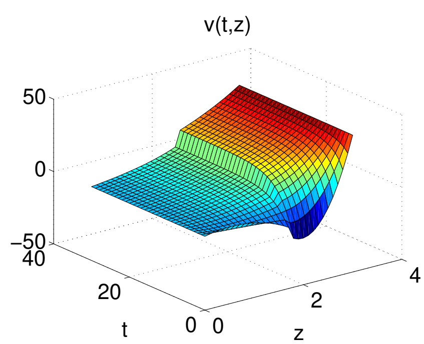


Fig. 8.17 Nonstationary spline v-solution for $r_{1z} = 0.4, r_{2z} = 0.1$

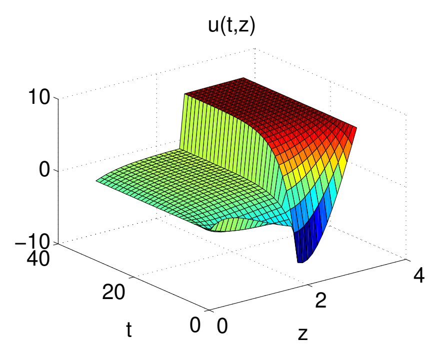


Fig. 8.18 Nonstationary spline u-solution for $r_{1z} = 0.4, r_{2z} = 0.1$

8.12 Reducing 3-D problems to 2-D initial-boundary value problems

Using transformation $c_i = \exp(-r_{ivz}z)v_i), r_{ivz} = \frac{r_{iz}}{2D_{iz}}$ we obtain from (8.33) following problem:

$$\begin{cases} \frac{\partial v_i}{\partial t} = L_{xy}(v_i) + D_{iz}(\frac{\partial^2 v_i}{\partial z^2} - r_{ivz}^2 v_i), \\ \frac{\partial v_i(0,y,z,t)}{\partial x} = \frac{\partial v_i(x,0,z,t)}{\partial y} = 0, \\ \frac{\partial v_1(x,y,0,t)}{\partial z} - (\beta_z^* + r_{1vz})(v_1(x,y,0,t) + \beta_z^* c_{oz} = 0, \\ \frac{\partial v_N(x,y,L_z,t)}{\partial z} + (\alpha_z^* - r_{Nvz})v_N(x,y,L_z,t) - \alpha_z^* c_{az}\exp(r_{Nvz}L_z) = 0, \\ \frac{\partial v_i(L_x,y,z,t)}{\partial x} + \alpha_{ix}^*(v_i(L_x,y,z,t) - c_{iax}\exp(r_{ivz}z)) = 0, i = \overline{1,N}, \\ \frac{\partial v_i(x,L_y,z,t)}{\partial y} + \alpha_{iy}^*(v_i(x,L_y,z,t) - c_{iay}\exp(r_{ivz}z)) = 0, i = \overline{1,N}, \\ r_{iz}v_i(x,y,z_i,t) = r_{i+1,z}v_{i+1}(x,y,z_i,t)\exp((r_{ivz} - r_{i+1,vz})z_i), \\ D_{iz}\frac{\partial v_i(x,y,z_i,t)}{\partial z} = D_{i+1,z}\frac{\partial v_{i+1}(x,y,z_i,t)}{\partial z}\exp((r_{ivz} - r_{i+1,vz})z_i), i = \overline{1,N-1} \\ v_i(x,y,z,0) = c_{i0}\exp(r_{ivz}z), i = \overline{1,N}, \end{cases} \tag{8.44}$$

where $\beta_z^* = \frac{\beta}{D_{1iz}}, \alpha_z^* = \frac{\alpha}{D_{Niz}}, \alpha_{ix}^* = \frac{\alpha_{ix}}{D_{ix}}, \alpha_{iy}^* = \frac{\alpha_{iy}}{D_{iy}}$.
Using for $v_i(x,y,z.t), r_{iv} = \sqrt{D_{iz}}r_{ivz}$ CAM with integral hyperbolic type splines (8.36) for unknown functions v_{iz}, m_{iz}, e_{iz} depends on (x,y,t) from BCs and the continuous conditions we can determine the unknown functions e_{iz}, m_{iz} depends on the mean integral values functions $v_{iz}(x,y,t)$. From 3-D PDEs (8.44) we obtain for determine $v_{iz}(x,y,t)$ the system of 2-D initial-value problem in following form:

$$\begin{cases} \frac{\partial v_{iz}}{\partial t} = L_{xy}(v_{iz}) + D_{iz}(0.5e_{iz}r_{ivz}\coth(0.25r_{ivz}H_i)/H_i - r_{ivz}^2 v_{iz}), \\ \frac{\partial v_{iz}(0,y,t)}{\partial x} = \frac{\partial v_{iz}(x,0,t)}{\partial y} = 0, \\ \frac{\partial v_{iz}(L_x,y,t)}{\partial x} + \alpha_{ix}^*(v_{iz}(L_x,y,t) - c_{iax}Q_i = 0, \\ \frac{\partial v_{iz}(x,L_y,t)}{\partial y} + \alpha_{iy}^*(v_{iz}(x,L_y,t) - c_{iay}Q_i = 0, \\ v_{iz}(x,y,0) = c_{i0}Q_i, i = \overline{1,N}, \end{cases} \tag{8.45}$$

where $Q_i = \frac{\exp(r_{ivz}z_i) - \exp(r_{ivz}z_{i-1}}{H_i r_{ivz}}$ is the intergral averaging values of $\exp(r_{ivz}z)$.
Similarly ([33], [35], [36], [34]), we can reduced 2-D problems to 1-D initial-boundary values problems.

Chapter 9
Exact FDS: H. Kalis, S. Rogovs, 2011 [74]

9.1 A mathematical model in the different coordinates

The finite-difference method is used only for solving the obtained 1-D problems for Poisson's equations in 3 way coordinates: decart and curved (cylindrical and spherical with radial symmetry) . The process of diffusion is considered in 1-D domain $\Omega = \{(x) : x_0 \leq x \leq l\}, x_0 = 0$ or $\Omega = \{r : r_0 \leq r \leq R\}, r_0 > 0$ in the curved coordinates.
The domain Ω consist of multilauer medium. We will consider the stationary 1-D problem of the linear diffusion theory for multilayered piece-wise homogenous materials of N layers in the form $\Omega_i = \{x : x \in (x_{i-1}, x_i)\}$ or $\Omega_i = \{r : r \in (r_{i-1}, r_i)\}, i = \overline{1,N}$,
where $h_i = x_i - x_{i-1}$ or $h_i = r_i - r_{i-1}$ is the height of layer Ω_i,$x_N = l, r_N = R$.
We will find the distribution of concentrations $u_i = u_i(x)$ or $u_i = u_i(r)$ in every layer Ω_i at the point $x, r \in \Omega_i$ by solving the following differential equations (ODE):

$$p_i \frac{d^2 u_i(x)}{dx^2} + f_i(x) = 0, \quad p_i \frac{1}{r^\alpha} \frac{d}{dr}\big(r^\alpha \frac{du_i(r)}{dr}\big) + f_i(r) = 0, \qquad (9.1)$$

where p_i are constant diffusions cefficients, $u_i = u_i(x), u_i = u_i(r)$ – the concentrations functions,
$f_i(x), f_i(r)$ - the fixed sours functions in every layer,$\alpha = 1, \alpha = 2$ are the coresponding parameters for cylindrical and spherical coordinates.
The values u_i and the flux functions $p_i \frac{du_i}{dx}$ or $p_i \frac{du_i}{dr}$ must be continues on the contact lines between the layers $x = x_i, r = r_i, i = \overline{1,N-1}$:

$$u_i|_{x_i} = u_{i+1}|_{x_i}, u_i|_{r_i} = u_{i+1}|_{r_i}$$
$$p_i \frac{du_i}{dx}|_{x_i} = p_{i+1}\frac{du_{i+1}}{dx}|_{x_i}, p_i\frac{du_i}{dr}|_{r_i} = p_{i+1}\frac{du_{i+1}}{dr}|_{r_i} \quad (9.2)$$

where $i = \overline{1, N-1}$.

We assume that the layered material is bounded above and below with the surfaces $x = 0, x = l$ or $r = r_0, r = R$ with fixed boundary conditions in following form:

$$u_1(0) = u_1(r_0) = U_0, \quad u_N(l) = u_N(R) = U_a \quad (9.3)$$

where U_0, U_a are given concentration-functions.

9.2 The analytical solutions of the 1-D continuous problems for Poissan's equations

The itegration of ODEs (9.1) for every system of coordinates give

$$p_iu_i(x) = -\int_{x_{i-1}}^{x}(x-t)f_i(t)dt + C_i(x-x_{i-1}) + B_i, x \in (x_{i-1}, x_i)$$
$$p_iu_i(r) = -\int_{r_{i-1}}^{r} t\ln\frac{r}{t} f_i(t)dt + C_i\ln\frac{r}{r_{i-1}} + B_i, r \in (r_{i-1}, r_i),$$
$$p_iu_i(r) = -\frac{1}{r}\int_{r_{i-1}}^{r} t(r-t)f_i(t)dt + C_i(\frac{1}{r_{i-1}} - \frac{1}{r}) + B_i, r \in (r_{i-1}, r_i) \quad (9.4)$$

where the unknown constants $C_i, B_i, i = \overline{1, N}$ can be deternined from conditions (1.3) and (1.4).

We have following systems of the linear algebraical equations:

1) for Decart coordinates

$$C_{i+1} = C_i - \int_{x_{i-1}}^{x_i} f_i(x)dx, i = \overline{1, N-1},$$
$$\frac{B_{i+1}}{p_{i+1}} - \frac{B_i}{p_i} = \frac{1}{p_i}(-\int_{x_{i-1}}^{x_i}(x_i - x)f_i(x)dx + C_ih_i), i = \overline{1, N-1}, \quad (9.5)$$
$$U_0 = \frac{B_1}{p_1}, U_a = \frac{1}{p_N}(-\int_{x_{N-1}}^{x_N}(x_N - x)f_N(x)dx + C_Nh_N + B_N),$$

2) for cylindrical coordinates

$$C_{i+1} = C_i - \int_{r_{i-1}}^{r_i} rf_i(r)dr, i = \overline{1, N-1},$$
$$\frac{B_{i+1}}{p_{i+1}} - \frac{B_i}{p_i} = \frac{1}{p_i}(-\int_{r_{i-1}}^{r_i} r\ln\frac{r_i}{r} f_i(r)dr + C_i\ln\frac{r_i}{r_{i-1}}), i = \overline{1, N-1}, \quad (9.6)$$
$$U_0 = \frac{B_1}{p_1}, U_a = \frac{1}{p_N}(-\int_{r_{N-1}}^{r_N} r\ln\frac{r_N}{r} f_N(r)dr + C_N\ln\frac{r_N}{r_{N-1}} + B_N),$$

3) for spherical coordinates

$C_{i+1} = C_i - \int_{r_{i-1}}^{r_i} r^2 f_i(r)dr, i = \overline{1,N-1},$
$\frac{B_{i+1}}{p_{i+1}} - \frac{B_i}{p_i} = \frac{1}{p_i}(-\frac{1}{r_i}\int_{r_{i-1}}^{r_i} r(r_i - r)f_i(r)dr + C_i(\frac{1}{r_{i-1}} - \frac{1}{r_i})), i = \overline{1,N-1},$
$U_0 = \frac{B_1}{p_1}, U_a = \frac{1}{p_N}(-\frac{1}{r_N}\int_{r_{N-1}}^{r_N} r(r_N - r)f_N(r)dr + C_N(\frac{1}{r_{N-1}} - \frac{1}{r_N}) + B_N).$ (9.7)

On the contact lines between the layers $u_i(r_i) = u_{i+1}(r_i) = \frac{B_{i+1}}{p_{i+1}}, i = \overline{1,N-1}$.
We sonsider the **example** in two layers ($N = 2$) with following data: $U_0 = U_a = 0, p_1 = 1, p_2 = 2, x_1 = 2, x_2 = l = 4, r_0 = 1, r_1 = 2, r_2 = R = 4, f_1(x) = f_2(x) = 5\delta(x - x_1), f_1(r) = f_2(r) = 5\delta(r - r_1)$.
Here $\delta(x - x_*)$ is the Dirax delta function concentrated in the point x_* or the

$$\lim_{\varepsilon \to 0} g(x, x_*, \varepsilon) = \delta(x - x_*),$$

where the function $g(x, x_*, \varepsilon) = \frac{1}{2\varepsilon}$ in the segment $[x_* - \varepsilon, x_* + \varepsilon]$ and equal to zero outside this segment. It is obviously that the integral $I(x_*) = \int_a^b f(x)\delta(x - x_*)dx$ is equal to $I(x_*) = f(x_*)$ for $x_* \in [a,b]$, $I(x_*) = 0.5f(x_*)$ for $x_* = a$ or $x_* = b$ and $I(x_*) = 0$ for x_* outside the segment $[a,b]$.
We have following results:
1) for decart coordinates $B_1 = 0; C_1 = \frac{5}{3}; C_2 = -\frac{5}{6}; B_2 = \frac{20}{3}; u_1(x_1) = u_2(x_1) = \frac{10}{3}$; $u_1(x) = \frac{5}{3}x, x \in [0,2]; u_2(x) = \frac{5}{3}(4 - x), x \in [2,4]$,
2) for cylindrica coordinates $B_1 = 0; C_1 = \frac{10}{3}; C_2 = -\frac{5}{3}; B_2 = \frac{20}{3}\ln(2)$; $u_1(r_1) = u_2(r_1) = \frac{10}{3}\ln(2)$; $u_1(r) = \frac{10}{3}\ln(r), r \in [1,2]$; $u_2(r) = \frac{10}{3}\ln(\frac{4}{r}), r \in [2,4]$,
3) for spherical coordinates $B_1 = 0; C_1 = 4; C_2 = -6; B_2 = 4; u_1(r_1) = u_2(r_1) = 2$; $u_1(r) = 4(1 - \frac{1}{r}), r \in [1,2]; u_2(r) = 2(\frac{4}{r} - 1), r \in [2,4]$.

9.3 The analytical solutios of the continuous problems for 1-D general equations

Similarly we can consider the ODEs with other terms

$$p_i \frac{d^2 u_i(x)}{dx^2} - 2a_i \frac{du_i(x)}{dx} - q_i u_i(x) + f_i(x) = 0, \tag{9.8}$$

where a_i, q_i are constant parameters, We consider the boundary condition of symmetry $\frac{du_N(l)}{dx} = 0$.

The equation (9.8) with the transformation $u_i(x) = \exp(b_i(x - x_{i-1})v_i(x)$, $v_1(0) = U_0, \frac{dv_N(l)}{dx} + b_N v_N(l) = 0$ we can use reduced in the following form

$$\frac{d^2 v_i(x)}{dx^2} - \kappa_i^2 v_i(x) + \exp(-b_i x) f_i(x)/p_i = 0, \tag{9.9}$$

where $b_i = \frac{a_i}{p_i}, \kappa_i^2 = \lambda_i^2 + b_i^2, \lambda_i = \frac{q_i}{p_i}$.
Integrating ODEs (9.9) give

$$v_i(x) = -\frac{1}{p_i \kappa_i} \int_{x_{i-1}}^{x} \sinh(\kappa_i((x-t)) \exp(-b_i(t - x_{i-1}) f_i(t) dt + \\ \frac{C_i}{\kappa_i} sinh(\kappa_i(x - x_{i-1})) + B_i \cosh(\kappa_i(x - x_{i-1})), x \in (x_{i-1}, x_i) \tag{9.10}$$

where the unknown constants $C_i, B_i, i = \overline{1,N}$ can be deternined from conditions (1.3) and (1.4).
We have following systems of the linear algebraical equations:

$$p_{i+1}(C_{i+1} + b_{i+1}) = p_i(\exp(b_i h_i)(C_i \cosh(\kappa_i h_i) + \kappa_i B_i \sinh(\kappa_i h_i)) - \\ \int_{x_{i-1}}^{x_i} \cosh(\kappa_i(x_i - x)) \exp(-b_i(x - x_{i-1}) f_i(x) dx) + b_i B_{i+1}), i = \overline{1, N-1}, \\ B_{i+1}\kappa_i = (-\frac{1}{p_i} \int_{x_{i-1}}^{x_i} \sinh(\kappa_i(x_i - x)) \exp(-b_i(x - x_{i-1}) f_i(x) dx + \\ C_i \sinh(\kappa_i h_i) + \kappa_i B_i \cosh(\kappa_i h_i)) \exp(b_i h_i), i = \overline{1, N-1}, \\ U_0 = B_1, \frac{b_N}{\kappa_N}(-\frac{1}{p_N} \int_{x_{N-1}}^{x_N} \sinh(\kappa_N(x_N - x)) \exp(-b_N(x - x_{N-1}) f_N(x) dx + \\ C_N \sinh(\kappa_N h_N) + B_N \kappa_N \cosh(\kappa_N h_N)) + C_N \cosh(\kappa_N h_N) + B_N \kappa_N \sinh(\kappa_N h_N) - \\ -\frac{1}{p_N} \int_{x_{N-1}}^{x_N} \cosh(\kappa_N(x_N - x)) \exp(-b_N(x - x_{N-1}) f_N(x) dx = 0. \tag{9.11}$$

We have following results for $N = 2$ (preliminary example with $f_1(x) = 0, f_2(x) = 5\delta(x-3), p_1 = 1 (B_1 = 0)$:

$$u_1(x) = \exp(b_1 x) \frac{C_1}{\kappa_1} \sinh(\kappa_1 x), x \in [0,2], \\ u_2(x) = \exp(b_2(x-2))(\frac{C_2}{\kappa_2} \sinh(\kappa_2(x-2)) + B_2 \cosh(\kappa_2(x-2)), x \in [2,3], \\ u_2(x) = \exp(b_2(x-2))(\frac{C_2}{\kappa_2} \sinh(\kappa_2(x-2)) + B_2 \cosh(\kappa_2(x-2)), \\ -\frac{5}{p_2 \kappa_2} \sinh(\kappa_2(x-3)) \exp(-b_2)), x \in [3,4], \tag{9.12}$$

where $c1 = (\frac{\cosh(2\kappa_1)}{p_2} + \frac{b_1 - p_2 b_2}{p_2} \frac{\sinh(2\kappa_1)}{\kappa_1})(\cosh(2\kappa_2) +$
$b_2 \frac{\sinh(2\kappa_2)}{\kappa_2}) + \frac{\sinh(2\kappa_1)}{\kappa_1}(\kappa_2 \sinh(2\kappa_2) + b_2 \cosh(2\kappa_2))$,
$C_1 = (\exp(-2b_1 - b_2)) \frac{5}{p_2} (\cosh(\kappa_2) + \sinh(\kappa_2) \frac{b_2}{\kappa_2}))/c1$,

$C_2 = \frac{\exp(2b_1)C_1}{p_2}(\cosh(2\kappa_1) + (b_1 - p_2 b_2)\frac{\sinh(3\kappa_1)}{\kappa_1})$,
$B_2 = C_1 \frac{\sinh(2\kappa_1)}{\kappa_1}\exp(2b_1)$.
Using the preliminary example for $N = 2$ we consider following parameters $(p_1 = 1, p_2 = 10)$:
1)$\kappa_1 = \kappa_2 = 0-$ piecwise linear solution $u_2(2) = 10$,
2)$a_1 = -1.5, a_2 = 2, q_1 = 1, q_2 = 0.1, p_1 = 1, p_2 = 10, \kappa_1 = 1.803, \kappa_2 = 0.224, u_2(4) = 8.00$.
The solutions are represented in the Fig. 9.1.

9.4 The exact FDS-method

For solving 1-D problems we consider the grid points equal with the contact lines between the layers $x = x_i, r = r_i, i = \overline{1, N-1}$, two end points $x_0 = 0, x_N = l, r_0 = R$ and the steps h_i.
The exact finite difference scheme (FDS) in the Decart coordinates we can represented in following form [3]:

$$\begin{aligned}
&a_{i+1}(u_{i+1} - u_i) - a_i(u_i - u_{i-1}) = -(F_i^- + F_i^+), i = \overline{1, N-1},\\
&a_i = (\textstyle\int_{x_{i-1}}^{x_i} \frac{dx}{p_i(x)})^{-1}, i = \overline{1, N},\\
&F_i^- = \textstyle\int_{x_{i-1}}^{x_i}(1 - a_i \int_x^{x_i} \frac{dt}{p_i(t)}) f_i(x)dx,\\
&F_i^+ = \textstyle\int_{x_i}^{x_{i+1}}(1 - a_{i+1} \int_{x_i}^{x} \frac{dt}{p_{i+1}(t)}) f_{i+1}(x)dx.
\end{aligned} \tag{9.13}$$

For the curved coordinates the functions $f_i(r)$ are multiply with r^α. In the case of piecewise coefficients p_i we have following formulas:
1) for Decart coordinates

$$\begin{aligned}
&a_i = \frac{p_i}{x_i - x_{i-1}}, a_{i+1} = \frac{p_{i+1}}{x_{i+1} - x_i},\\
&F_i^- = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i}(x - x_{i-1}) f_i(x)dx,\\
&F_i^+ = \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}}(x_{i+1} - x) f_{i+1}(x)dx,
\end{aligned} \tag{9.14}$$

2) for cylindrical coordinates

$$\begin{aligned}
&a_i = \frac{p_i}{\ln(r_i/r_{i-1})}, a_{i+1} = \frac{p_{i+1}}{\ln(r_{i+1}/r_i)},\\
&F_i^- = \frac{1}{\ln(r_i/r_{i-1})} \int_{r_{i-1}}^{r_i} r \ln(r/r_{i-1}) f_i(r)dr,\\
&F_i^+ = \frac{1}{\ln(r_{i+1}/r_i)} \int_{r_i}^{r_{i+1}} r \ln(r_{i+1}/r) f_{i+1}(r)dr,
\end{aligned} \tag{9.15}$$

3) for spherical coordinates

$$\begin{aligned} &a_i = \frac{p_i r_i r_{i-1}}{(r_i - r_{i-1})}, a_{i+1} = \frac{p_{i+1} r_i r_{i+1}}{(r_{i+1} - r_i)}, \\ &F_i^- = \frac{r_i}{r_i - r_{i-1}} \int_{r_{i-1}}^{r_i} r(r - r_{i-1}) f_i(r) dr, \\ &F_i^+ = \frac{r_i}{r_{i+1} - r_i} \int_{r_i}^{r_{i+1}} r(r_{i+1} - r) f_{i+1}(r) dr. \end{aligned} \tag{9.16}$$

The FDS can be solved by Thomas algorithm ([15]). In the preliminary example for $N = 2$ we have following results:
1) for decart coordinates $a_1 = 0.5; a_2 = 1; F_1^- = F_1^+ = 2.5; u_1 = u(x_1) = 10/3$,
2) for cylindrica coordinates $a_1 = 1/\ln(2); a_2 = 2/\ln(2); F_1^- = F_1^+ = 5; u_1 = u(r_1) = 10\ln(2)/3$,
3) for spherical coordinates $a_1 = 2; a_2 = 8; F_1^- = F_1^+ = 10; u_1 = u(r_1) = 2$.
We can see, that the values on the contact lines are identy in the continuously and discrete case.

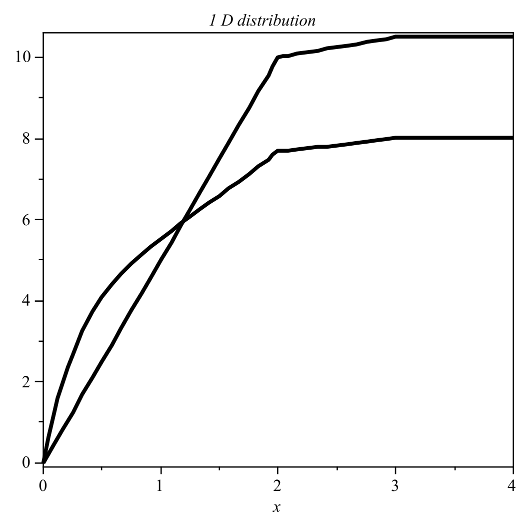


Fig. 9.1 2 solutions $u(x)$

Chapter 10
CAM: I. Kangro, H. Kalis, E. Teirumnieka, E. Teirumnieks, 2011 [31]

The task of sufficient accuracy numerical simulation of quick solution 3-D problems for mathematical physics in multi-layered media is important in the known areas of the applied sciences. With regard to the numerical analysis several numerical methods are known for solving 3-D problems: FEM, BEM, ADM, spectral methods, multigrids and others methods.
For simple engineering calculations two methods are applied [30], [12]: special finite difference scheme and conservative averaging method (CAM) by using special integral hyperbolic type splines with two parameters in every layer. We chose the CAM for engineering calculation and the solution of 3-D problem can be obtained analytically.
We consider C AM for solving the 3-D boundary-value problem in multilayer domain. A specific feature of these problems is that it is necessary to solve the 3-D initial-boundary-value problems of second order with piece-wise parameters in multilayer domain.
The special exponential and hyperbolic type integral splines, which interpolation middle integral values of piece-wise smooth functions, are considered. These functions contain the independent solutions of coresponding homogeneous linear ODEs with parameters-characteristic values. With the help of this splines is reduce the problems of mathematical physics in 3-D with piece-wise coefficients to respect one coordinate to problems for system of equations in 2-D. This procedure allows to reduce also the 2-D problem to a 1-D problems and the solution of the aproximated problem can be obtained anlytically.

The solution of corresponding averaged 2-D initial-boundary value problem is obtained also numerically, using for approach differential equations the discretization in space applying the differences. This method may be considered as a generalization of the method of finite volumes for the layered systems. In the case of constant piece-wise coefficients we obtain the exact discrete approximation of steady-state 1-D boundary-value problem. The approximation of the 2-D nonstationary problem is based on the implicid finite-difference and alternating direction (ADI) methods. The numerical solution is compared with the analytical solution.

10.1 Formulation of 3-D problem of the process of diffusion

The process of diffusion is consider in 3-D parallelepiped

$$\Omega = \{(x,y,z) : 0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq z \leq L_z\}.$$

The domain Ω consist of multilayer medium.

We will consider the nonstationary 3-D problem of the linear diffusion theory for multilayered piece-wise homogenous materials of N layers in the domain

$$\Omega_i = \{(x,y,z) : x \in (0,L_x), y \in (0,L_y), z \in (z_{i-1},z_i)\}, i = \overline{1,N},$$

where $H_i = z_i - z_{i-1}$ is the height of layer $\Omega_i, z_0 = 0, z_N = L_z$.

We will find the distribution of concentrations $c_i = c_i(x,y,z,t)$ in every layer Ω_i at the point $(x,y,z) \in \Omega_i$ and at the time t by solving the following 3-D initial-boundary value problem for partial differential equation (PDE):

$$\begin{cases} \frac{\partial c_i(x,y,z,t)}{\partial t} = \frac{\partial}{\partial x}(D_{ix}\frac{\partial c_i(x,y,z,t)}{\partial x})+ \\ \frac{\partial}{\partial y}(D_{iy}\frac{\partial c_i(x,y,z,t)}{\partial y}) + \frac{\partial}{\partial z}(D_{iz}\frac{\partial c_i(x,y,z,t)}{\partial z}) + f_i(x,y,z,t), \\ x \in (0,L_x), y \in (0,L_y), z \in (z_{i-1},z_i), t \in (0,t_f), i = \overline{1,N}, \\ \frac{\partial c_i(0,y,z,t)}{\partial x} = \frac{\partial c_i(x,0,z,t)}{\partial y} = 0, \\ D_{1z}\frac{\partial c_1(x,y,0,t)}{\partial z} - \beta_z(c_1(x,y,0,t) - c_{oz}(x,y)) = 0, \\ D_{ix}\frac{\partial c_i(L_x,y,z,t)}{\partial x} + \alpha_{ix}(c_i(L_x,y,z,t) - c_{iax}(y,z)) = 0, i = \overline{1,N}, \\ D_{iy}\frac{\partial c_i(x,L_y,z,t)}{\partial y} + \alpha_{iy}(c_i(x,L_y,z,t) - c_{iay}(x,z)) = 0, i = \overline{1,N}, \\ D_{Nz}\frac{\partial c_N(x,y,L_z,t)}{\partial z} + \alpha_z(c_N(x,y,L_z,t) - c_{az}(x,y)) = 0, \\ c_i(x,y,z_i,t) = c_{i+1}(x,y,z_i,t), \\ D_{iz}\frac{\partial c_i(x,y,z_i,t)}{\partial z} = D_{i+1,z}\frac{\partial c_{i+1}(x,y,z_1,t)}{\partial z}, i = \overline{1,N-1} \\ c_i(x,y,z,0) = c_{i0}(x,y,z), i = \overline{1,N}, \end{cases} \quad (10.1)$$

where $c_i = c_i(x,y,z,t)$ are the concentrations functions in every layer, $f_i(x,y,z,t)$ - the fixed sours functions, D_{ix}, D_{iy}, D_{iz} are the constant diffusion coefficients, $\alpha_{ix}, \alpha_{iy}, \alpha_z, \beta_z, i = \overline{1,N}$ are the constant mass transfer coefficients for the 3 kind boundary conditions, $c_{az}, c_{iay}, c_{iax}, c_{oz}$ are the given concentration on the boundary, t_f is the final time, $c_{i0}(x,y,z)$ are the given initial concentration.

We have following boundary conditions:

1) the homogenous 3-kind conditions by $x = L_x, y = L_y; z = L_z, z = 0$,

2) the symetrical conditions by $x = 0; y = 0$. The values c_i and the flux functions $D_{iz}\frac{\partial c_i}{\partial z}$ must be continues on the contact lines between the layers, $i = \overline{1,N-1}$.

10.1.1 The averaged method in z-direction

Using **averaged method with respect to** z or hiperbolic trigonometric functions

$c_i(x,y,z,t) = c_{iz}(x,y,t) + m_{iz}(x,y,t)\frac{0.5H_i \sinh(a_i(z-\overline{z_i}))}{\sinh(0.5a_iH_i)} +$

$e_{iz}G_{iz}(0.25\frac{\sinh^2(a0_i(z-\overline{z_i}))}{\sinh^2(0.5a0_iH_i} - A_{i0z}),$

where $c_{iz}(x,y,t) = \frac{1}{H_i}\int_{z_{i-1}}^{z_i} c_i(x,y,z,t)dz, G_{iz} = \frac{H_i}{D_{iz}}$,

$A_{i0z} = 0.25\frac{(sinh(a0_iH_i))/(a0_iH_i)-1}{cosh(a0_iH_i)-1} \in [0,1/12]$,

$\overline{z_i} = (z_{i-1} + z_i)/2, z \in [z_{i-1}, z_i], i = \overline{1,N}$.
We can see that the parameters $a_i > 0, a0_i > 0$ tend to zero then the limit is the integral parabolic spline (A. Buikis [9]), because $A_{i0z} \to \frac{1}{12}$:
$c_i(x,y,z,t) = c_{iz}(x,y,t) + m_{iz}(x,y,t)(z - \overline{z_i}) + e_{iz}(x,y,t)G_{iz}(\frac{(z-\overline{z_i})^2}{H_i^2} - \frac{1}{12})$.
The unknown functions $m_{iz}(x,y,t), e_{iz}(x,y,t)$, we can determined from boundary conditions (10.1) in z-direction:
1) for z=0 ,$d_{1z}D_{1z}m_{1z} - d0_{1z}e_{1z} - \beta_z(c_{1z} - 0.5m_{1z}H_1 + e_{1z}G_{1z}A_{11z} - c_{0z}) = 0$,
2) for z=L_z,$d_{Nz}D_{Nz}m_{Nz} + d0_{Nz}e_{Nz} + \alpha_z(c_{Nz} + 0.5m_{Nz}H_N + e_{Nz}G_{Nz}A_{N1z} - c_{az}) = 0$,
3)for z= z_i, $c_{iz} + 0.5m_{iz}H_i + e_{iz}G_{iz}A_{i1z} =$
$c_{i+1,z} - 0.5m_{i+1,z}H_{i+1} + e_{i+1,z}G_{i+1,z}A_{i+1,1z}$,
$d_{iz}D_{iz}m_{iz} + d0_{iz}e_{iz} = d_{i+1,z}D_{i+1,z}m_{i+1,z} - d0_{i+1,z}e_{i+1,z}, i = \overline{1,N-1}$,
where
$d_{iz} = 0.5H_ia_i\coth(0.5a_iH_i) \to 1$,$d0_{iz} = 0.5H_ia0_i\coth(0.5a0_iH_i) \to 1$,
$A_{i1z} = 0.25 - A_{i0z} \in [1/6, 1/4]$ and $A_{i1z} \to \frac{1}{6}$ if $a0_i \to 0$.
From conditions on the contact lines by $z = z_i, i = \overline{1,N-1}$ excluding $m_{i+1,z}$ follows:
$m_{iz}D_{iz}(G_{iz} + G_{i+1,z}\kappa_{iz}) + e_{iz}(2G_{iz}A_{i,1z} + \kappa0_{iz}G_{i+1,z}) +$
$e_{i+1,z}G_{i+1,z}(\frac{d0_{i+1,z}}{d_{i+1,z}} - 2A_{i+1,1z}) = 2(c_{i+1,z} - c_{iz})$,
where $\kappa_{iz} = \frac{d_{iz}}{d_{i+1,z}}$, $\kappa0_{iz} = \frac{d0_{iz}}{d_{i+1,z}}$.
From conditions on the contact lines by $z = z_{i-1}, i = \overline{2,N}$ excluding $m_{i-1,z}$ follows:
$m_{iz}D_{iz}(G_{iz} + G_{i-1,z}\kappa_{i-1,z}^{-1}) - e_{iz}(2G_{iz}A_{i,1z} + \kappa_{i-1,z}^{-1}G_{i-1,z}) -$
$e_{i-1,z}G_{i-1,z}(\frac{d0_{i-1,z}}{d_{i-1,z}} - 2A_{i-1,1z}) = 2(c_{i,z} - c_{i-1,z})$.
Excluding m_{iz} we obtain for determined e_{iz} following system of $N-2$ algebraic equations $a_{i,i}e_{iz} + a_{i,i+1}e_{i+1,z} + a_{i,i-1}e_{i-1,z} =$
$b_{i,i}c_{iz} + b_{i,i+1}c_{i+1,z} + b_{i,i-1}c_{i-1,z}, i = \overline{2,N-1}$,
where $a_{i,i} = (G_{iz} + G_{i-1,z}\kappa_{i-1,z}^{-1})(2G_{iz}A_{i,1z} +$
$1\kappa0_{iz}G_{i+1,z}) + (G_{iz} + G_{i+1,z}\kappa_{iz})(2G_{iz}A_{i,1z} + \kappa0_{i-1,z}^{-1}G_{i-1,z})$,
$a_{i,i+1} = G_{i+1,z}(\frac{d0_{i+1,z}}{d_{i+1,z}} - 2A_{i+1,1z})(G_{iz} + G_{i-1,z}\kappa_{i-1,z}^{-1})$,
$a_{i,i-1} = G_{i-1,z}(\frac{d0_{i-1,z}}{d_{i-1,z}} - 2A_{i-1,1z})(G_{iz} + G_{i+1,z}\kappa_{iz})$.
$b_{i,i} = -(b_{i,i+1} + b_{i,i-1}), b_{i,i+1} = 2(G_{iz} + G_{i-1,z}\kappa_{i-1,z}^{-1}), b_{i,i-1} = 2(G_{iz} +$

$G_{i+1,z}\kappa_{iz})$.
From boundary conditions by $z=0, z=L_z$ and previous expresions by $i=1$ and $i=N$ excluding m_{1z}, m_{Nz} follows:
$a_{1,1}e_{1z}+a_{1,2}e_{2z}=b_{1,1}c_{1z}+b_{1,2}c_{2z}+b_{01}c_{0z}$,
$a_{N,N}e_{Nz}+a_{N,N-1}e_{N-1,z}=b_{N,N}c_{Nz}+b_{N,N-1}c_{N-1,z}+b_{0N}c_{az}$,
where
$a_{1,1}=(2G_{1z}A_{1,1z}+\kappa_{1z}G_{2,z})(d_{1z}+0.5\beta_z G_{1z})+(G_{1z}+\kappa_{1z}G_{2,z})((d0_{1z}+\beta_z A_{i,1z}G_{1z})$,
$a_{1,2}=G_{2,z}(\frac{d0_{i+1,z}}{d_{i+1,z}}-2A_{i,1z})(d_{1z}+0.5\beta_z G_{1z})$,
$b_{1,1}=-(2(d_{1z}+0.5\beta_z G_{1z})+\beta_z(G_{1z}+\kappa_{1z}G_{2,z}))$,
$b_{1,2}=2(d_{1z}+0.5\beta_z G_{1z})$, $b_{01}=\beta_z(G_{1z}+\kappa_{1z}G_{2,z})$,
$a_{N,N}=(2G_{Nz}A_{N1z}+\kappa^{-1}_{N-1,z}G_{N-1,z})(d_{Nz}+0.5\alpha_z G_{Nz})+$
$(G_{Nz}+\kappa^{-1}_{N-1z}G_{N-1,z})((d0_{Nz}+\alpha_z G_{Nz}A_{N,1z})$,
$a_{N,N-1}=G_{N-1,z}(\frac{d0_{N-1,z}}{d_{N-1,z}}-2A_{N-1,1z})(d_{Nz}+0.5\alpha_z G_{Nz})$,
$b_{N,N}=-(2(d_{Nz}+0.5\alpha_z G_{Nz})+\alpha_z(G_{Nz}+\kappa^{-1}_{N-1,z}G_{N-1,z}))$,
$b_{N,N-1}=2(d_{Nz}+0.5\alpha_z G_{Nz})$, $b_{0N}=\alpha_z(G_{Nz}+\kappa^{-1}_{N-1,z}G_{N-1,z})$.

Using the obtained values $a_{i,j}, b_{i,j}, i,j=\overline{1,N}$ we can determine the 3-diagonal N-order matrices A_z, B_z with these elements, N-order diagonal matrix B_0 with the elements $[b_{01},0,0,...0,b_{0N}]$, the N-order vectors-column e_z, c_z with the elements e_{iz}, c_{iz} and the N-order vector-column c_0 with the elements $[c_{0z},0,0,...0,c_{az}]$. Then we have the system of N algebraic equations in following vector form

$$A_z e_z(x,y,t)=B_z c_z(x,y,t)+B_0 c_0(x,y).$$

The matrix A_z is diagonal dominant and we can the unique solution write in the form
$e_z(x,y,t)=B_{1z}c_z(x,y,t)+B_{2z}c_0(x,y)$, where $B_{1z}=A_z^{-1}B_z, B_{2z}=A_z^{-1}B_0$.

Now the initial-boundary value 2D problem is in the form

$$\begin{cases}\frac{\partial c_{iz}(x,y,t)}{\partial t}=\frac{\partial}{\partial x}(D_{ix}\frac{\partial c_{iz}(x,y,t)}{\partial x})+\frac{\partial}{\partial y}(D_{iy}\frac{\partial c_{iz}(x,y,t)}{\partial y})+dh_{iz}e_{iz}(x,y,t)+f_{iz}(x,y,t),\\ \frac{\partial c_{iz}(0,y,t)}{\partial x}=\frac{\partial c_{iz}(x,0,t)}{\partial y}=0,\\ D_{ix}\frac{\partial c_{iz}(L_x,y,t)}{\partial x}+\alpha_{ix}(c_{iz}(L_x,y,t)-c^v_{iax})(y)=0,\\ D_{iy}\frac{\partial c_{iz}(x,L_y,t)}{\partial y}+\alpha_{iy}(c_{iz}(x,L_y,t)-c^v_{iay}(x)=0,\\ c_{iz}(x,y,0)=c_{iz,0}(x,y), x\in(0,L_x), y\in(0,L_y), t\in(0,t_f), i=\overline{1,N},\end{cases} \tag{10.2}$$

where $dh_{iz} = \frac{2d0_{iz}}{H_i}$,
$f_{iz}(x,y,t) = \frac{1}{H_i}\int_{z_{i-1}}^{z_i} f_i(x,y,z,t)dz$, $c^v_{iax})(y) = \frac{1}{H_i}\int_{z_{i-1}}^{z_i} c_{iax}(y,z)dz$,
$c^v_{iay})(x) = \frac{1}{H_i}\int_{z_{i-1}}^{z_i} c_{iay}(x,z)dz$, $c_{iz,0}(x,y) = \frac{1}{H_i}\int_{z_{i-1}}^{z_i} c_{i0}(x,y,z)dz$.

We can determine the diagonal N-order matrices $D_x, D_y, \alpha_x, \alpha_y, Dh_z$ with the corresponding elements
$D_{ix}, D_{iy}, \alpha_{ix}, \alpha_{iy}, dh_{iz}$ and the N-order vectors-column
$f_z(x,y,t), c^v_{ay}(x), c^v_{ax})(y), c_{z,0}$
with the elements $f_{iz}(x,y,t), c^v_{iay}(x), c^v_{iax})(y), c_{iz,0}(x,y)$.
Then we have the initial-boundary value 2-D problem (10.2) in vector form:

$$\begin{aligned}
&\frac{\partial c_z(x,y,t)}{\partial t} = \frac{\partial}{\partial x}(D_x\frac{\partial c_z(x,y,t)}{\partial x}) + \frac{\partial}{\partial y}(D_y\frac{\partial c_z(x,y,t)}{\partial y})+\\
&Dh_z(B_{1z}c_z(x,y,t) + B_{2z}c_0(x,y)) + f_z(x,y,t), \frac{\partial c_z(0,y,t)}{\partial x} = \frac{\partial c_z(x,0,t)}{\partial y} = 0,\\
&D_x\frac{\partial c_z(L_x,y,t)}{\partial x} + \alpha_x(c_z(L_x,y,t) - c^v_{ax})(y) = 0,\\
&D_y\frac{\partial c_z(x,L_y,t)}{\partial y} + \alpha_y(c_z(x,L_y,t) - c^v_{ay}(x) = 0,\\
&c_z(x,y,0) = c_{z,0}(x,y), x \in (0,L_x), y \in (0,L_y), t \in (0,t_f).
\end{aligned} \tag{10.3}$$

10.1.2 The averaged method in y-direction

Using **averaged method with respect to** y
$c_{iy}(x,t) = \frac{1}{L_y}\int_0^{L_y} c_{iz}(x,y,t)dy$,
$c_{iz}(x,y,t) = c_{iy}(x,t) + m_{iy}(x,t)\frac{0.5L_y \sinh(a_i(y-0.5L_y))}{\sinh(0.5a_iL_y)}+$
$e_{iy}(x,t)G_{iy}(\frac{1}{4}\frac{\sinh^2(a0_i(y-0.5L_y))}{\sinh^2(0.5a0_iL_y} - A_{i0y})$,
with the unknown functions $m_{iy}(x,t), e_{iy}(x,t)$, we can determined these functions from boundary conditions (10.2) in following form:
$D_{iy}d_{iy}m_{iy}(x,t) = d0_{iy}e_{iy}(x,t), e_{iy} = -b_{iy}(c_{iy}(x,t) - c^v_{iay}(x))$,
where $b_{iy} = \alpha_{iy}/(\alpha_{iy}G_{iy}(0.5\frac{d0_{iy}}{d_{iy}} + A_{i,1y}) + 2d0_{iy}), A_{i1y} = 0.25 - A_{i0y})$,
$A_{i0y} = 0.25\frac{(sinh(a0_iL_y))/(a0_iL_y)-1}{cosh(a0_iL_y)-1}$,
$d_{iy} = 0.5L_ya_i\coth(0.5L_ya_i), G_{iy} = \frac{L_y}{D_{iy}}, d0_{iy} = 0.5L_ya0_i\coth(0.5L_ya0_i)$.

Then the initial-boundary value problem (10.3) is in following form

$$\begin{cases} \frac{\partial c_y(x,t)}{\partial t} = \frac{\partial}{\partial x}(D_x \frac{\partial c_y(x,t)}{\partial x}) - \\ Dh_y B_y (c_y(x,t) - c_{ay}^v(x)) + Dh_z(B_{1z} c_y(x,t) + B_{2z} c_0^v(x)) + f_y(x,t), \\ \frac{\partial c_y(0,t)}{\partial x} = 0, D_x \frac{\partial c_y(L_x,t)}{\partial x} + \alpha_x (c_y(L_x,t) - c_{ax}^{vv}) = 0, \\ c_y(x,0) = c_{y,0}(x), x \in (0,L_x), t \in (0,t_f), \end{cases} \tag{10.4}$$

where Dh_y, B_y are the N-order diagonal matrices with the elements $\frac{2d0_{iy}}{L_y}, b_{iy}$, c_y, f_y are the N-order vectors-column with the elements c_{iy}, f_{iy}, $c_0^v(x)$ is the N-order vector-column with 2 nonzeros elements $c_{oz}^v(x), c_{az}^v(x)$, $f_y(x,t) = \frac{1}{L_y}\int_0^{L_y} f_z(x,y,t)dy$,
$c_{az}^v(x) = \frac{1}{L_y}\int_0^{L_y} c_{az}(x,y)dy$, $c_{oz}^v(x) = \frac{1}{L_y}\int_0^{L_y} c_{oz}(x,y)dy$,
$c_{ax}^{vv} = \frac{1}{L_y}\int_0^{L_y} c_{ax}^v(y)dy$, $c_{y,0}(x) = \frac{1}{L_y}\int_0^{L_y} c_{z,0}(x,y)dy$.

10.1.3 The averaged method in x-direction

It is possible make the averaging also **with respect to** x
$c_{ix}(t) = \frac{1}{L_x}\int_0^{L_x} c_{iy}(x,t)dx$,
$c_{iy}(x,t) = c_{ix}(t) + m_{ix}(t)\frac{0.5L_x \sinh(a_i(x-0.5L_x))}{\sinh(0.5a_iL_x)} +$
$e_{ix}G_{ix}(\frac{1}{4}\frac{\sinh^2(a0_i(x-0.5L_x))}{\sinh^2(0.5a0_iL_x} - A_{i0x})$,
with the unknown functions $m_{ix}(t), e_{ix}(t)$. We can determined these functions from boundary conditions (10.4) in following form:
$D_{ix}d_{ix}m_{ix} = d0_{ix}e_{ix}, e_{ix} = -b_{ix}(c_{ix}(t) - c_{iax}^{vv})$,
where $b_{ix} = \alpha_{ix}/(\alpha_{ix}G_{ix}(0.5\frac{d0_{ix}}{d_{ix}} + A_{i,1x}) + 2d0_{ix}), A_{i,1x} = 0.25 - A_{i0x})$,
$A_{i0x} = 0.25\frac{(sinh(a0_iL_x))/(a0_iL_x)-1}{cosh(a0_iL_x)-1}$,
$d_{ix} = 0.5L_xa_i\coth(0.5L_xa_i), G_{ix} = \frac{L_x}{D_{ix}}, d0_{ix} = 0.5L_xa0_i\coth(0.5L_xa0_i)$.
Then the initial-boundary value problem (1.14) is in the following form:

$$\begin{cases} \frac{\partial c_x(t)}{\partial t} = -Dh_xB_x(c_x(t) - c_{ax}^{vv}) - Dh_yB_y(c_x(t) - c_{ay}^{vv}) + \\ Dh_z(B_{1z}c_x(t) + B_{2z}c_0^{vv}) + f_x(t), \\ c_x(0) = c_{x,0}, t \in (0,t_f), \end{cases} \tag{10.5}$$

where Dh_x, B_x are the N-order diagonal matrices with the elements $\frac{2d0_{ix}}{L_x}, b_{ix}$, c_x, f_x are the N-order vectors-column with the elements

c_{ix}, f_{ix}, c_0^{vv} is the N-order vector-column with 2 nonzeros elements c_{oz}^{vv}, c_{az}^{vv}, $f_x(t) = \frac{1}{L_x}\int_0^{L_x} f_y(x,t)dx$,
$c_{x,0} = \frac{1}{L_x}\int_0^{L_x} c_{y,0}(x)dx, c_{az}^{vv} = \frac{1}{L_x}\int_0^{L_x} c_{az}^{v}(x)dx$, $c_{0z}^{vv} = \frac{1}{L_x}\int_0^{L_x} c_{0z}^{v}(x)dx$,

Therefore we have from (10.5) the initial problem for the system of N ODEs of the first order. The solutions of this system can be with classical methods obtained . For the averaged stationary solution (f_x is constant vector) follows the analytical solution in the form

$$c_x = (Dh_xB_x + Dh_yB_y - Dh_zB_{1z})^{-1}(f_x + Dh_xB_xc_{ax}^{vv} + Dh_yB_yc_{ay}^{vv} + Dh_zB_{2z}c_0^{vv}).$$

10.1.4 Domain with homogenous material, one layer

For N=i=1 the unknown functions $m_z(x,y,t), e_z(x,y,t)$, are determined from boundary conditions by $z = 0, z = L_z$:
$d_zD_zm_z - d0_ze_z - \beta_z(c_z - 0.5m_zL_z + e_zG_zA_{1z} - c_{oz}) = 0$,
$d_zD_zm_z + d0_ze_z + \alpha_z(c_z + 0.5m_zL_z + e_zG_zA_{1z} - c_{az}) = 0$,
where $m_z(x,y,t) = (\beta_za_{22}(c_z(x,y,t) - c_{oz}(x,y,t)) + \alpha_za_{12}(-c_z(x,y,t) + c_{az}(x,y,t)))/det$,
$e_z(x,y,t) = -c_z(x,y,t)g_z + c_{az}(x.y)a_z + c_{oz}(x,y)b_z$,
$g_z = (a_{11}\alpha_z + a_{21}\beta_z)/det, a_z = \alpha_za_{11}/det, b_z = \beta_za_{21}/det$,
$det = a_{11}a_{22} + a_{12}a_{21}, a_{11} = d_zD_z + 0.5\beta_zL_z, a_{21} = d_zD_z + 0.5\alpha_zL_z$,
$a_{12} = d0_z + \beta_zG_zA_{1z}, a_{22} = d0_z + \alpha_zG_zA_{1z}, G_z = L_z/D_z$.

The initial-boundary value 2D problem is in following form

$$\begin{cases} \frac{\partial c_z(x,y,t)}{\partial t} = \frac{\partial}{\partial x}(D_x\frac{\partial c_z(x,y,t)}{\partial x}) + \\ \frac{\partial}{\partial y}(D_y\frac{\partial c_z(x,y,t)}{\partial y}) - B_zg_zc_z(x,y,t) + f_z(x,y,t) + B_z(a_zc_{az}(x,y) + b_zcoz(x.y)), \\ \frac{\partial c_z(0,y,t)}{\partial x} = \frac{\partial c_z(x,0,t)}{\partial y} = 0, \\ D_x\frac{\partial c_z(L_x,y,t)}{\partial x} + \alpha_x(c_z(L_x,y,t) - c_{ax}^{v})(y) = 0, \\ D_y\frac{\partial c_z(x,L_y,t)}{\partial y} + \alpha_y(c_z(x,L_y,t) - c_{ay}^{v}(x) = 0, \\ c_z(x,y,0) = c_{z,0}(x,y), \end{cases} \tag{10.6}$$

where $B_z = 2d0_z/L_z$, $c_{ax}^{v})(y) = \frac{1}{L_z}\int_0^{L_z} c_{ax}(y,z)dz$,
$c_{ay}^{v})(x) = \frac{1}{L_z}\int_0^{L_z} c_{ay}(x,z)dz$.

Using **averaged method with respect to** y
with the unknown functions $m_y(x,t), e_y(x,t)$, we can determine this

functions from boundary conditions
$D_y d_y m_y(x,t) = d0_y e_y(x,t), e_y = -b_y(c_y(x,t) - c^v_{ay}(x))$.
Then the initial-boundary value problem (10.6) is in following form

$$\begin{cases} \frac{\partial c_y(x,t)}{\partial t} = \frac{\partial}{\partial x}(D_x \frac{\partial c_y(x,t)}{\partial x}) - \\ (g_z B_z + b_y B_y)c_y(x,t) + f_y(x,t) + B_z(a_z c^v_{az}(x) + b_z c^v_{oz}(x)) + B_y b_y c^v_{ay} x, \\ \frac{\partial c_y(0,t)}{\partial x} = 0, D_x \frac{\partial c_y(L_x,t)}{\partial x} + \alpha_x(c_y(L_x,t) - c^{vv}_{ax}) = 0, \\ c_y(x,0) = c_{y,0}(x), \end{cases} \tag{10.7}$$

where
$B_y = 2d0_y/Ly, G_y = L_y/D_y$, $c^v_{az}(x) = \frac{1}{L_y}\int_0^{L_y} c_{az}(x,y)dy$,
$c^v_{oz}(x) = \frac{1}{L_y}\int_0^{L_y} c_{oz}(x,y)dy$, $c^{vv}_{ax} = \frac{1}{L_y}\int_0^{L_y} c^v_{ax}(y)dy$,
$c_{y,0}(x) = \frac{1}{L_y}\int_0^{L_y} c_{z,0}(x,y)dy$.

Make the averaging also **respect to** x
with the unknown functions $m_x(t), e_x(t)$, we can determined this functions from boundary conditions (10.7) $D_x d_x m_x(t) = d0_x e_x(t), e_x = -b_x(c_x(t) - c^{vv}_{ax})$.
Then the initial-boundary value problem (10.7) or the initial problem for ODEs of the first order is in following form

$$\begin{cases} \frac{\partial c_x(t)}{\partial t} = -(B_z g_z + B_y b_y + B_x b_x)c_x(t) + f_x(t) + \\ B_z(a_z c^{vv}_{az} + b_z c^{vv}_{oz}) + B_y b_y c^{vv}_{ay} + B_x b_x c^{vv}_{ax}, \\ c_x(0) = c_{x,0}, \end{cases} \tag{10.8}$$

where
$B_x = 2d0_x/L_x, G_x = L_x/D_x, c^{vv}_{az} = \frac{1}{L_x}\int_0^{L_x} c^v_{az}(x)dx$,
$c^{vv}_{0z} = \frac{1}{L_x}\int_0^{L_x} c^v_{0z}(x)dx$.
The solution of this problem is

$$c_x(t) = exp(-\gamma t)c_{x,0} + \int_0^t exp(-\gamma(t-\xi))\tilde{f}_x(\xi)d\xi, \tag{10.9}$$

where $\tilde{f}_x(t) = f_x(t) + B_z(a_z c^{vv}_{az} + b_z c^{vv}_{oz}) + B_y b_y c^{vv}_{ay} + B_x b_x c^{vv}_{ax}$,
$\gamma = B_z g_z + B_y b_y + B_x b_x$.

For the averaged stationary solution follows the formula

$$c_x = \frac{\tilde{f}_x}{\gamma}$$

and we have the analytical solution for $c(x,y,z)$.

We can obtain also the stationary solution $c(x) = c_y(x)$ from (10.7) by solving the boundary-value problem for ODEs of second order

$$c''(x) - a_1^2 c(x) + g(x) = 0, c'(0) = 0, D_x c'(L_x) + \alpha_x(c(L_x) - C_0)) = 0,$$

where $a_1^2 = (g_z B_z + b_y B_y)/D_x, g(x) = (f_y(x) + B_z(a_z c_{az}^v(x) + b_z c_{oz}^v(x)) + B_y b_y c_{ay}^v(x))/D_x, C_0 = c_{ax}^{vv}$.
Then $c(x) = C_1 \cosh(a_1 x) - \frac{1}{D_x a_1}\int_0^x \sinh(a_1(x-\tau)g(\tau)d\tau$,
$C_1 = (\frac{\alpha_x}{a_1}\int_0^{L_x} \sinh(a_1(L_x - \tau)g(\tau)d\tau + \alpha_x C_0)/ (D_x a_1 \sinh(a_1 L_x) + \alpha_x \cosh(a_1 L_x))$.
If $g(x) = const$, then $c(x) = C_1 \cosh(a_1 x) + \frac{g}{a_1^2}$,
$C_1 = \alpha_x(C_0 - \frac{g}{a_1^2})/(D_x a_1 \sinh(a_1 L_x) + \alpha_x \cosh(a_1 L_x))$.

10.1.5 Analytical model for estimating the parameters a, a0

We consider the special 1-D diffusion problem in the z -direction for $f = F0\cos(\frac{\pi x}{L_x})\cos(\frac{\pi y}{L_y})$,
$\alpha_x = \alpha_y = 0, c_{az}(x,y) = C_a \cos(\frac{\pi x}{L_x})\cos(\frac{\pi y}{L_y})$
$c_{0z}(x,y) = C_0 \cos(\frac{\pi x}{L_x})\cos(\frac{\pi y}{L_y})$. Then the stationary solution of (10.1) is in the form $c(x,y,z) = c(z)\cos(\frac{\pi x}{L_x})\cos(\frac{\pi y}{L_y})$, where the function $c(z)$ is solution for following boundary value problem:

$$\begin{cases} \frac{\partial}{\partial z}(\frac{\partial c(z)}{\partial z}) - b^2 c(z) + f0 = 0, z \in (0, L_z), \\ D_z \frac{\partial c(0)}{\partial z} - \beta_z(c(0) - C_0) = 0, \\ D_z \frac{\partial c(L_z)}{\partial z} + \alpha_z(c(L_z) - C_a) = 0, \end{cases} \tag{10.10}$$

where $b = \pi\sqrt{(\frac{D_x}{L_x^2} + \frac{D_y}{L_y^2})/D_z}, f0 = \frac{F0}{D_z}$.
Therefore the exact solution is
$c(z) = P_1 \sinh(bz) + P_2 \cosh(bz) + \frac{f0}{b^2}$,
where the constants $P_1 = \frac{\beta_z}{D_z b}(P_2 + \frac{f0}{b^2} - C_0)$,

$P_2 = \alpha_z(C_a - \frac{f0}{b^2}) + \beta(C_0 - \frac{f0}{b^2})(\cosh(bL_z) + \alpha_z \sinh(bL_z)/(bD_z))/$
$(\cosh(bL_z)(\alpha_z + \beta_z) + \sinh(bL_z)(bD_z + \alpha_z\beta_z/(bD_z)))$.
The averaged values are
$c^v = L_z^{-1}\int_0^{L_z} c(z)dz = \frac{1}{L_z b}(P_1(\cosh(bL_z) - 1) + P_2 \sinh(bL_z)) + \frac{f0}{b^2}$.
This form of solution remained also for discrete aproximation
$c(z_j), z_j = jh, h = \frac{L_z}{N}, j = \overline{0,N}$ by using exact finite difference scheme (FDS) from N. Bahvalov [30]. Then from first order of approximation the boundary conditions are
$P_1 = (-b_4 b_{11} + b_3 b_{21})/det, P_2 = (b_4 b_{12} - b_3 b_{22})/det, det = b_{11}b_{22} - b_{12}b_{21}$,
where $b_{11} = \cosh(bh) - b_1, b_{12} = \sinh(bh)$,
$b_{21} = \cosh(b(L_z - h)) - b_2 \cosh(bL_z)$,
$b_{22} = \sinh(b(L_z - h)) - b_2 \sinh(bL_z)$, $b_1 = 1 + h\beta_z/D_z$,
$b_2 = 1 + h\alpha_z/D_z, b_3 = h\beta_z/D_z(C_0 - \frac{f0}{b^2}), b_4 = h\alpha_z/D_z(Ca - \frac{f0}{b^2})$.

For comparision we use the averaged method respect to z with exponential spline. Then $c^v = (B_z(a_z C_a + b_z C_0) + F0)/(B_z g_z + D_z b^2)$,
$e_z = -c^v g_z + C_a a_z + C_0 b_z$,
$m_z = (\beta_z a_{22}(c^v - C_0) + \alpha_z a_{12}(-c^v + C_a))/det$.
We have following numerical results
$(F0 = 0, L_z = 1, L_x = 1, L_y = 1, C_0 = 0.3, C_a = 2.0, D_z = 10^{-3}, D_x = D_y = 3.10^{-4}, b = 2.4335)$ for maximal error and averaged values depending on $a, a0, \alpha_z, \beta_z$. The numerical results are given in the Table eftab:10.1 (a=a0=0 for parabolic spline). In Fig 10.1 is represented the solution $c(z)$ for 4 methods ($N = 20, \alpha_z = 20, \beta_z = 0, a = a0 = 2.3$,for FDS $\delta = 0.0222$, for a=b, a0=b/2 follows $\delta = 0.$)

For $L_z = 3, F0 = 0.1, b = 2.4335$ follows: $c^v_{ap} = 23.14$, $c^v_{an} = 16.87, \delta = 6.27$ (for parabolic spline), $c^v_{ap} = 16.87, c^v_{an} = 16.87, \delta = 0$ (for exponential type spline with a=b, a0=b/2.)

For exponential spline with 2 parameters $a = b, a0 = b/2$ we have exact solution for every values of the parameters
$L_z, F0, b, D_z, C_0, C_a, \alpha, \beta$.

10.1.6 The numerical approximations for the 2-D problem,one layer

We use uniform grid in the space $((M+1)\times(N+1))$:
$\{(y_i, x_j),\ y_i = (i-1)hy,\ x_j = (j-1)hx\}, i = \overline{1,M+1},\ j = \overline{1,N+1}$,

Table 10.1 The maximal error δ and averaged values depending on a for α_z, β_z, c^v_{an}-exact, c^v_{ap} -approx)

α_z	β_z	a	a0	δ	c^v_{ap}	c^v_{an}
2	1	0	0	0.092	0.769	0.792
-	-	0.5	0.5	0.084	0.802	-
-	-	1.0	1.0	0.062	0.785	-
-	-	1.4	1.4	0.050	0.791	-
-	-	1.5	1.5	0.050	0.802	-
-	-	b	b/2	0.0	0.792	0.792
2	0	0	0	0.339	0.672	0.808
-	-	1.3	1.3	0.243	0.727	-
-	-	2.0	2.0	0.134	0.791	-
-	-	2.5	2.5	0.088	0.843	-
-	-	2.3	2.3	0.081	0.822	-
-	-	b	b/2	0.0	0.808	0.808
20	0	0	0	0.339	0.672	0.809
-	-	2.3	2.3	0.081	0.823	-
-	-	b	b/2	0.0	0.809	0.809

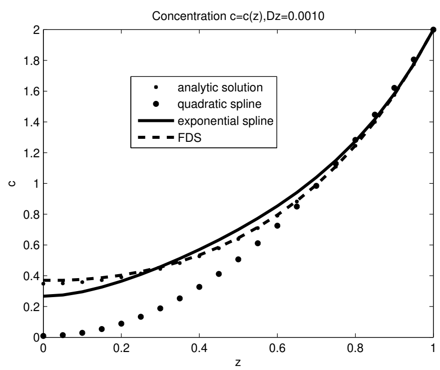


Fig. 10.1 Solution c(z)for $\alpha_z = 20, \beta_z = 0, a = 2.3$

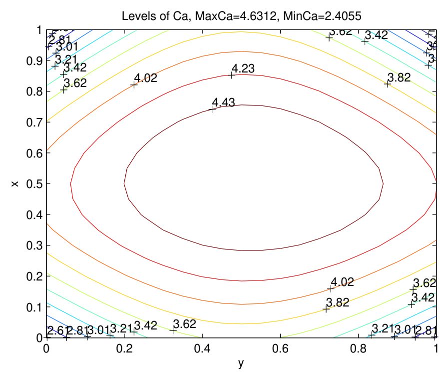


Fig. 10.2 Levels of concentration from interpolation by $z = L_z = 3$

$M\,hy = L_y, N\,hx = L_x$.
For the time t we use the moments $t_n = n\tau, n = 0, 1, \cdots$. Subscripts (i, j, n) refer to y, x, t indices with the mesh spacing and for approximation the function $c_z(x, y, t)$ we have the grid function with values $U^n_{i,j} \approx c_z(x_j y_i, t_n)$.

For solving 2D problem (1.13) we use the discrete approximation in the form
$(U^{n+1}_{i,j} - U^n_{i,j})/\tau = (\Lambda_x + \Lambda_y)U^{n+1}_{i,j} + f^n_{i,j}, n \geq 0, i = \overline{1, N+1}, j = \overline{1, M+1}$
and ADI method of Douglas and Rachford (1955)

$$(U_{i,j}^{n+0.5} - U_{i,j}^{n})/\tau = \Lambda_x U_{i,j}^{n+0.5} + \Lambda_y U_{i,j}^{n} + f_{i,j}^{n}, i = \overline{2,N}, j = \overline{2,M}$$

$$(U_{i,j}^{n+1} - U_{i,j}^{n+0.5})/\tau = \Lambda_y (U_{i,j}^{n+1} - U_{i,j}^{n}), i = \overline{2,N}, j = \overline{1,M+1}.$$

Elimate the half time step $t_{n+0.5}$ we obtain the previous discrete problem with approximation error $O(\tau^2)$. Here Λ_x, Λ_y are the discrete difference operators, approximated second order derivatives
$\frac{\partial}{\partial x}(D_x \frac{\partial c_z(x,y,t)}{\partial x}) - B_z g_z c_z(x,y,t)$,
$\frac{\partial}{\partial y}(D_y \frac{\partial c_z(x,y,t)}{\partial y})$ respect to x and y and boundary conditions with central differences, $f_{i,j}^{n} = f_z(x_j, y_i, t_n) + B_z(a_z c_{az}(x_j, y_i) + b_z coz(x_j, y_i))$.
We use following discrete operators:
$\Lambda_x U_{i,j} = D_x \delta_x^2 U_{i,j} - B_z g_z U_{i,j}$, $\Lambda_y U_{i,j} = D_y \delta_y^2 U_{i,j} i = \overline{2,N}, j = \overline{2,M}$.
where $\delta_x^2 U_{i,j} = \frac{1}{hx^2}(U_{i,j+1} - 2U_{i,j} + U_{i,j-1}), \delta_y^2 U_{i,j} = \frac{1}{hy^2}(U_{i+1,j} - 2U_{i,j} + U_{i-1,j})$.
For solving $U^{n+0.5}$ and U^{n+1} we use Tomas algorithm in x and y directions respectively. We can write the 3-point difference equation at every direction in following form:

$$\begin{cases} A_k U_{k-1} - C_k U_k + B_k U_{k+1} = F_k, k = \overline{2,K}, \\ -C_1 U_1 + B_1 U_2 = F_1, j = 1, \\ A_{K+1} U_K - C_{K+1} U_{K+1} = F_{K+1}, k = K+1, \end{cases} \tag{10.11}$$

where
$C_1 = B_1 = 1, F_1 = 0, C_{K+1} = 1 + \frac{h\alpha}{D}, AK + 1 = 1, F_{K+1} = \frac{h\alpha}{D} Ca$,
$k = i$ or j, $K = M$ or N, $h = hx$ or hy, $D = D_x$ or D_y, $\alpha = \alpha_x$ or α_y.
Using the Tomass algorithm we have:
$U_k = X_k U_{k+1} + Z_k, k = K(-1)1$,
where $X_k = \frac{B_k}{d_k}, Z_k = \frac{F_k + A_k Z_{k-1}}{d_k}, d_k = C_k - A_k X_{k-1}, k = \overline{2,K+1}$,
$X_1 = \frac{B_1}{C_1}, Z_1 = \frac{F_1}{C_1}, U_{K+1} = Z_{K+1}$.
We have following coefficients in (10.11):
$A_k = B_k = \frac{D}{h^2}$, $C_k = 1/\tau + 2\frac{D}{h^2} + \delta B_z g_z$, $F_k = U_k^n/\tau + \Lambda_y U_k^n + f_k^n$
or $F_k = U_k^{n+0.5}/\tau - \Lambda_y U_k^n$.

10.2 Some numerical results

We consider the metal concentration in the peat block. The layered peats block are modelled in [32], [31]. On the top of earth $(z = L_z)$ the

concentration $c[\frac{mg}{kg}]$ of metals is measured in following nine points in the (x,y) plane:
$c(0.1,0.2) = 3.69, c(0.5,0.2) = 4.43, c(0.9,0.2) = 3.72, c(0.1,0.5) = 4.00, c(0.5,0.5) = 4.63, c(0.9,0.5) = 4.11, c(0.1,0.8) = 3.71, c(0.5,0.8) = 4.50, c(0.9,0.8) = 3.73$.
This date are smoothing in matrix c_{az} by 2D interpolation with MATLAB operator, using the spline functions.
In Fig. 10.2 we can see the distribution of concentration c for Ca in the (x,y) plane by $z = L_z$. On the below of peat block $z = 0$ the elements of matrix c_{0z} are constant $1.30\frac{mg}{kg}$.

The numerical results are obtained for $z_m = m\,hz, m = \overline{0,10}, hz = \frac{L_z}{10}$, $(D_x = D_y = 310^{-4}, D_y = 10^{-3}, L_z = 3, L_x = L_y = 1, \alpha_z = 20, \beta_z = 10, \alpha_x = \alpha_y = 2.5, M = N = 20, a = 1.3$.
For the initial condition the averaged solutions $c_z(x,y)$ are choosed.
We have the stationary solution with $\tau = 1, t_f = 1000$ the maximal error 10^{-6}, the maximal value of $c_z(x,y)$ 2.6446 for averaged method 2.6892 for ADI method (following results can see in Figs. 10.3-10.13).
The spline functions and results are represented in Figs. 10.14-10.16).
Depending on the number of the grid points (N,M) we have following maximal values for ADI method:
$2.6974(M = N = 10), 2.6892(M = N = 28), 2.6859(M = N = 40)$,
$2.6841(M = N = 60)$. Depending on the parameter a by $M = N = 20$ are obtained following maximal values corresponding for averaged and ADI methods:
$2.6446; 2.6892(a = 1.3), 2.6172; 2.6406(a = 0.1), 2.6582; 2.7300(a = 2.0)$.

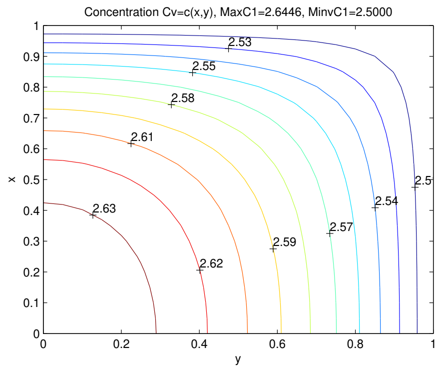


Fig. 10.3 Levels of averaged concentration $c_z(x,y)$

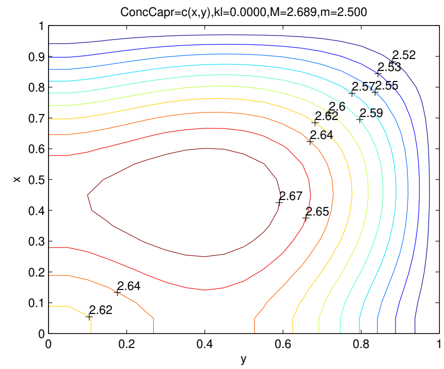


Fig. 10.4 Levels of numerical concentration $c_z(x,y)$

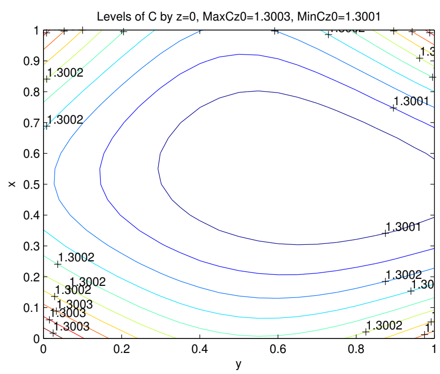


Fig. 10.5 Levels of averaged concentration $c_z(x,y)$ for $z=0$

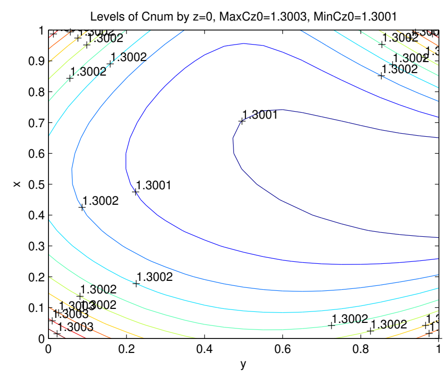


Fig. 10.6 Levels of numerical concentration $c_z(x,y)$ for $z=0$

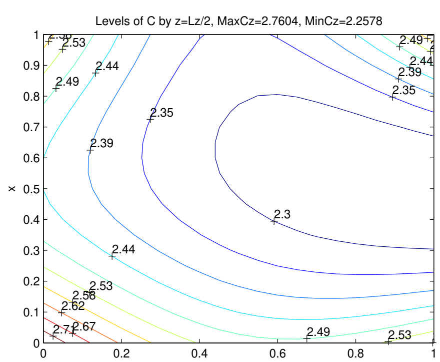


Fig. 10.7 Levels of averaged concentration $c_z(x,y)$ for $z=L_z/2$

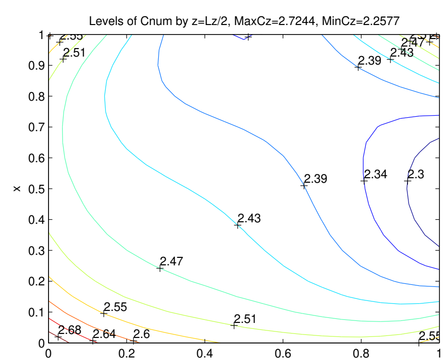


Fig. 10.8 Levels of numerical concentration $c_z(x,y)$ for $z=L_z/2$

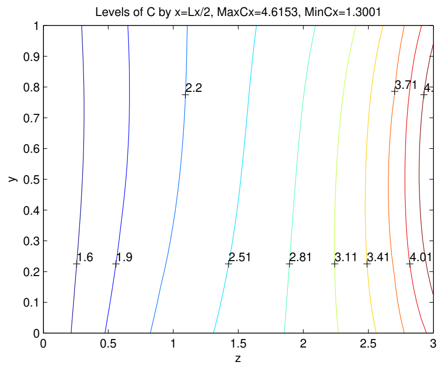


Fig. 10.9 Levels of averaged concentration $c_z(x,y)$ for $x = L_x/2$

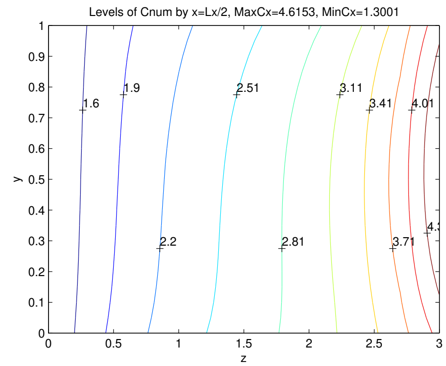


Fig. 10.10 Levels of numerical concentration $c_z(x,y)$ for $x = L_x/2$

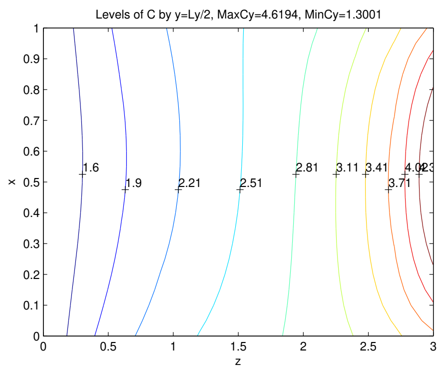


Fig. 10.11 Levels of averaged concentration $c_z(x,y)$ for $y = L_y/2$

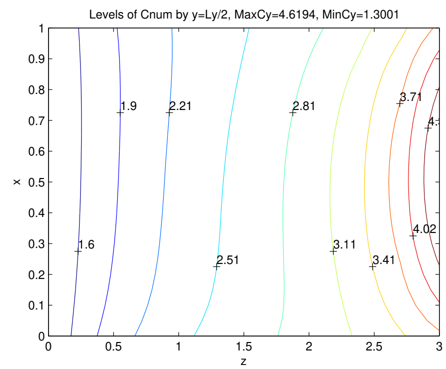


Fig. 10.12 Levels of numerical concentration $c_z(x,y)$ for $y = L_y/2$

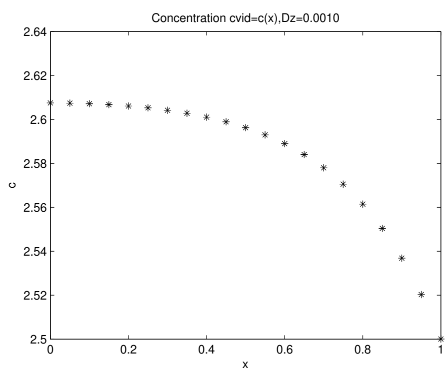


Fig. 10.13 Levels of averaged concentration $c_{z,y}(x)$

10.2.1 Matlab programm

We solve following boundary-value problem:

$$\begin{cases} u''(z) - k^2 u(z) = 0, z \in (0, L), \\ u'(0) = \beta(u(0) - c0), z = 0, \\ u'(L) - = -\alpha(u(L) - ca), z = L. \end{cases} \tag{10.12}$$

For $a = k, a0 = a/2$ in the exponential spline we have exact solution for every $c0, ca, L, k, \beta, \alpha$. The numerical results are obtaining with MATLAB using the following m. file **"EXPgRAF(20)"**:

```
% u''-k^2u=0, u'(0)=beta (u(0)-c0),u'(L)=-alfa(u(L)-ca)
%a=a1=k;a0=a2=k/2, then exp.spline are exact
%eksp.spline functions for L=1
 function EXPgRAF(N)
L=1;NP=N+1;
z=linspace(0,L,NP)';
a=0.0001:1:50.0001;y=0.5*sinh((z-L/2)*a)/diag(sinh(a/2));
plot (z,y),
title ('GRAF y=0.5*sinh(a*(z-L/2))/sinh(a/2),a=0(1)50')
figure, y=0.25*(sinh((z-L/2)*a)).^2/diag((sinh(a/2)).^2);
plot (z,y),
 title ('GRAF y=0.25*sinh^2(a*(z-L/2))/sinh^2(a/2),a=0(1)50')
L=5;ca=1;c0=1;beta=10;alfa=10;k=5;U=zeros(NP,1);
U1=zeros(NP,1);
U2=zeros(NP,1);a1=k;a2=k/2;z=linspace(0,L,NP)';h=L/N;
C2=(ca+c0*(beta/k*sinh(k*L)+beta/alfa*cosh(k*L)))/. . .
((beta/alfa +1)*cosh(k*L)+(k/alfa+beta/k)*sinh(k*L));
C1=beta/k*(C2-c0);
for j= 1:NP
    U(j)=C2*cosh(k*(z(j)))+C1*sinh(k*(z(j)));
end
v1=1/(L*k)*(C1*(cosh(k*L)-1)+C2*sinh(k*L));
%Exponent. spline
A0=0.25*(sinh(a2*L)/(a2*L)-1)/(cosh(a2*L)-1);
dz2=0.5*L*a2*coth (a2*L/2);dz1=0.5*L*a1*coth (a1*L/2);
A1=0.25-A0; sauc= beta*L^2*A1 +dz2*0.5*L +dz1*L*A1+. . .
((dz1+0.5*beta*L)*dz2+(dz2+beta*L*A1)*dz1)/alfa;
a22=((dz1+beta*L/2)*ca + beta*c0*(dz1/alfa +0.5*L))/sauc;
a11=(beta*L +dz1+beta/alfa*dz1)/sauc;
b11=dz2*(1-beta/alfa)/sauc;
b22=(ca*(dz2 +beta*L*A1)-beta*c0*(dz2/alfa+L*A1))/sauc;
uv=2*dz2*a22/(2*dz2*a11 +L*k^2); e=-a11*uv+a22;
m=-b11*uv+b22;
for i=1:NP
U1(i)=uv+0.5*m*L*sinh(a1*(z(i)-L/2))/sinh(a1*L/2)+. . .
e*L*(0.25*(sinh(a2*(z(i)-L/2)))^2/(sinh(a2*L/2))^2 -A0);
```

```
end
atv=(-U1(3)-3*U1(1)+4*U1(2))/(2*h)
%Paraboliskie splaini
dz1=1;dz2=1;A0=1/12;A1=1/6;
sauc= beta*L^2*A1 +dz2*0.5*L +dz1*L*A1+. . .
((dz1+0.5*beta*L)*dz2+(dz2+beta*L*A1)*dz1)/alfa;
a22=((dz1+beta*L/2)*ca + beta*c0*(dz1/alfa +0.5*L))/sauc;
a11=(beta*L +dz1+beta/alfa*dz1)/sauc;
b11=dz2*(1-beta/alfa)/sauc;
b22=(ca*(dz2 +beta*L*A1)-beta*c0*(dz2/alfa+L*A1))/sauc;
uv1=2*dz2*a22/(2*dz2*a11 +L*k^2); e=-a11*uv1+a22;
m=-b11*uv1+b22;
for i=1:NP
U2(i)=uv1+m*(z(i)-L/2)+e*L*((z(i)-L/2)^2/L^2 -1/12);
end
[max(U),max(U1), max(U2)],[v1,uv,uv1]
[max(abs(U-U1)),max(abs(U-U2))]
figure
plot(z,U,'k*',z,U1,'k-',z,U2,'ko','LineWidth',2.5)
title(sprintf('du/dz-k^2u=0, k=...
%4.2f, beta=%4.2f,L=%4.2f',k,beta,L))
legend('analytic solution','eksp. spline', 'par. spline')
```

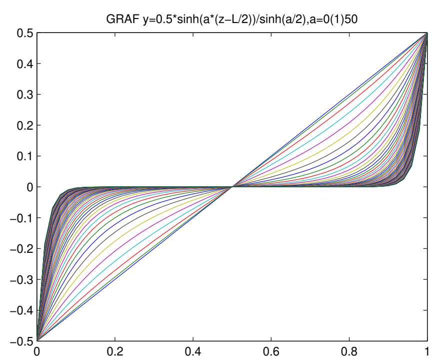


Fig. 10.14 Spline function-multiplier of m for $a \in [0.0001, 50], L = 1, N = 20$

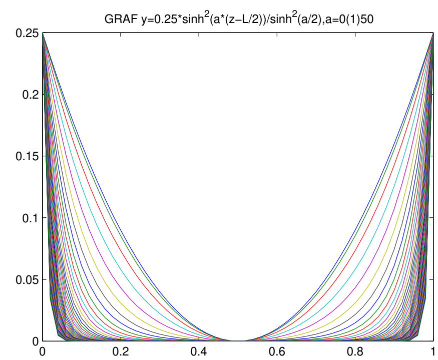


Fig. 10.15 Spline function-multiplier of e for $a \in [0.0001, 50], L = 1, N = 20$

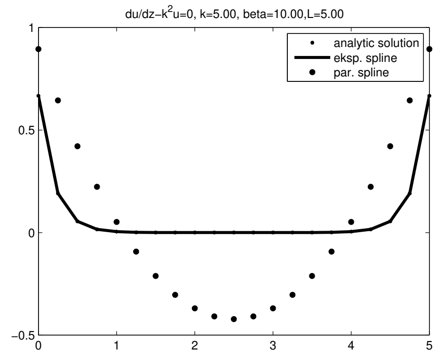


Fig. 10.16 Exact, exponential and parabolic values for$k = L = 5, \alpha = \beta = 10, c0 = ca = 1$

10.3 Formulation of special 3-D problem in Decart coordinates

The process of diffusion is consider in 3-D parallelepiped

$$\Omega = \{(x,y,z) : 0 \leq x \leq L_x, 0 \leq y \leq L_y, 0 \leq z \leq L_z\}.$$

We will consider the stationary 3-D problem of the linear diffusion theory We will find the distribution of concentrations $c = c(x,y,z)$ in Ω at the point (x,y,z) by solving the following special 3-D boundary value problem for partial differential equation (PDE):

$$\begin{cases} \frac{\partial}{\partial x}(D_x \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y}(D_y \frac{\partial c}{\partial y}) + \\ \frac{\partial}{\partial z}(D_z \frac{\partial c}{\partial z}) + f_0 \cos \frac{\pi x}{2L_x} \cos \frac{\pi y}{2L_y} = 0, \\ \frac{\partial c(0,y,z)}{\partial x} = \frac{\partial c(x,0,z)}{\partial y} = 0, c(L_x,y,z) = 0, c(x,L_y,z) = 0, \\ D_z \frac{\partial c(x,y,0)}{\partial z} - \beta_z (c(x,y,0) - c_o \cos \frac{\pi x}{2L_x} \cos \frac{\pi y}{2L_y}) = 0, \\ D_z \frac{\partial c(x,y,L_z)}{\partial z} + \alpha_z (c(x,y,L_z) - c_a \cos \frac{\pi x}{2L_x} \cos \frac{\pi y}{2L_y}) = 0, \end{cases} \quad (10.13)$$

where $f_0, c_o, c_a)$ - are the fixed constants,
D_x, D_y, D_z are the constant diffusion coefficients, α_z, β_z are the constant mass transfer coefficients in the 3 kind boundary conditions. We can obtained the analytical solution of (10.13)in following form:
$c(x,y,z) = g(z) \cos \frac{\pi x}{2L_x} \cos \frac{\pi y}{2L_y}$,
where the function$g(z)$ is solution of boundary value-problem for ODE

$$\begin{cases} g''(z) - a_0^2 g(z) + f_1 = 0, \\ g'(0) - \beta(g(0) - c_o) = 0, g'(L_z) + \alpha(g(L_z) - c_a) = 0, \end{cases} \tag{10.14}$$

where $f_1 = f_0/D_z, \beta = \beta_z/D_z, \alpha = \alpha_z/D_z, a_0^2 = \frac{\pi^2(D_y/L_y^2 + D_x/L_x^2)}{4D_z}$.
We have following solutions

$$g(z) = C_1 \sinh(a_0 z) + C_2 \cosh(a_0 z) + f_2,$$

where $C_1 = \beta/a_0(C_2 + f_2 - c_o), C_2 = \frac{(c_a - f_2)(\alpha/a_0 + \beta/a_0 c_3)}{\beta/a_0 c_3 + c_4}$,
$f_2 = f_1/a_0^2, c_3 = \cosh(a_0 L_z) + \alpha/a_0 \sinh(a_0 L_z)$,
$c_4 = \sinh(a_0 L_z) + \alpha/a_0 \cosh(a_0 L_z)$.

10.3.1 CAM in z-direction using integral hyperbolic type spline with two fixed parametrical functions

Using averaged method for (10.14) with fixed parametrical functions f_{z1}, f_{z2}
$g(z) = g_a + m f_{z1}(z - L_z/2) + e f_{z2}(z - L_z/2)$,
where $g_a = \frac{1}{L_z}\int_0^{L_z} g(z)dz$ is the averaged value, $\int_0^{L_z} f_{z1}dz = \int_0^{L_z} f_{z2}dz = 0$,
$f_{z1} = \frac{0.5L_z \sinh(a(z - L_z/2))}{\sinh(aL_z/2)} . f_{z2} = \frac{\cosh(a(z - L_z/2)) - A_0}{8\sinh^2(aL_z/4)}$,
$A_0 = \sinh(aL_z/2)/(aL_z/2), a = a_0$ is the optimal parameter.
We can see that the parameters a tend to zero then the limit is the integral parabolic spline (A.Buikis [9]), because $A_0 \to \frac{1}{12}$:
$f_{z1} \to (z - L_z/2), f_{z2} \to (\frac{(z - L_z/2)^2}{L_z^2} - \frac{1}{12})$.

The unknown coefficients m, e we can determine from boundary conditions (10.14):
1) for z=0 ,$md - ek - \beta(g_a - 0.5mL_z + eb - c_o)$,
2) for z=L_z,$md + ek + \alpha(g_a + 0.5mL_z + eb - c_a)$,
where $d = (aL_z/2)\coth(aL_z/2), k = (a/4)\coth(aL_z/4), b = \frac{\cosh(aL_z/2)) - A_0}{8\sinh^2(aL_z/4)}$.
Therefore $e = g_e g_a + a_e, m = g_a g_m + a_m, g_e = (a_{11}\alpha + a_{21}\beta)/det, g_m = (a_{22}\beta - a_{12}\alpha)/det$,
$a_e = (c_o a_{21}\beta + c_a a_{11}\alpha)/det, a_m = (c_a a_{12}\alpha - c_o a_{22}\beta)/det, det = a_{11}a_{22} + a_{12}a_{21}$,
$a_{11} = d + \beta L_z/2, a_{21} = d + \alpha L_z/2, a_{12} = k + b\beta, a_{22} = k + b\alpha$.

Now the-boundary value problem (10.14) is in the form

$$1/L_z(g'(L_z) - g'(0)) - a_0^2 g_a + f_1 = 0, g'(L_z) - g'(0) = 2ek \quad (10.15)$$

or $g_a = \frac{f_1 L_z + 2ka_e}{2kg_e + a_0^2 L_z}$.

Example 1.We consider following parameters:
$L_x = L_y = L_z = 1, f_0 = 0.1, \alpha_z = 0.3, \beta_z = 0.1, c_o = 5, c_a = 2, D_x = 10^{-2}, D_y = 10^{-3}, D_z = 10^{-4}, a_0 = 16.4747$.
We have following maximal errors: for hyperbolic spline $er_h = 210^{-9}$, for parabolic $er_p = 0.9653$ (see $g(z)$ in Fig. 10.17 and $c(x,0,z)$ in Fig. 10.18) If $D_y = 0, y = 0$, then $er_h = 810^{-8}, er_p = 1.0423$.

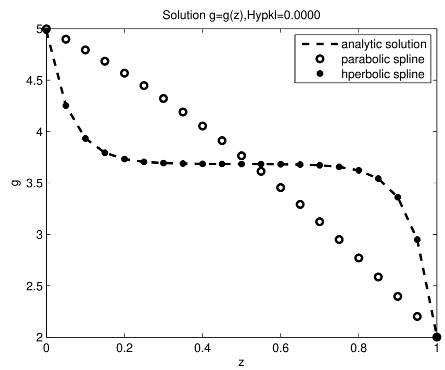


Fig. 10.17 Solution $g(z)$

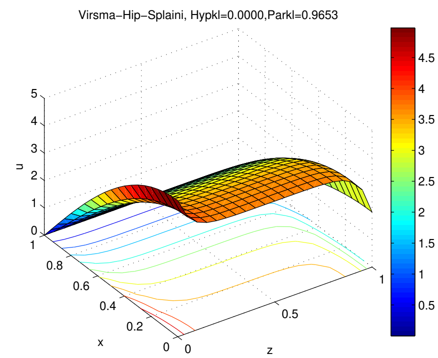


Fig. 10.18 Solution $c(x,0,z)$

10.3.2 The problem in cylindrical coordinates with axial symmetry

The process of diffusion is consider in 3-D cylinder

$$\Omega = \{(r, z.\phi) : 0 \le r \le R, 0 \le z \le L_z, 0 \le \phi \le 2\pi\}.$$

We will consider the stationary boundary-value problem with axial symmetry . We will find the distribution of concentrations $c = c(r,z)$ in Ω at the point (r,z) by solving the following special boundary value problem for partial differential equation (PDE):

$$\begin{cases} D_r \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial c}{\partial r}) + \frac{\partial}{\partial z}(D_z\frac{\partial c}{\partial z}) + f_0 \cos\frac{\pi z}{2L_z} = 0, \\ \frac{\partial c(r,0)}{\partial z} = \frac{\partial c(0,z)}{\partial r} = 0, c(r, L_z) = 0, \\ D_r \frac{\partial c(R,z)}{\partial r} + \alpha_r(c(R,z) - c_a \cos\frac{\pi z}{2L_z}) = 0, \end{cases} \tag{10.16}$$

where $f_0, c_a)$ - are the fixed constants,
D_r, D_z are the constant diffusion coefficients, α_r is the constant mass transfer coefficient in the 3 kind boundary conditions. We can obtain the analytical solution of (10.16)in following form: $c(r,z) = g(r)\cos\frac{\pi z}{2L_z}$,
where the function $g(r)$ is solution of boundary-value problem for ODE

$$\begin{cases} g''(r) + \frac{1}{r}g'(r) - a_0^2 g(r) + f_1 = 0, \\ g'(0) = 0, g'(R) + \alpha(g(R) - c_a) = 0, \end{cases} \tag{10.17}$$

where $f_1 = f_0/D_r, \alpha = \alpha_r/D_r, a_0^2 = \frac{\pi^2 D_z/L_z^2}{4D_r}$.
We have following solutions

$$g(r) = C_1 I_0(a_0 r) + f_2,$$

where $f_2 = \frac{f_1}{a_0^2}$, $C_1 = \frac{\alpha(c_a - f_2)}{a_0 I_1(a_0 R) + \alpha I_0(a_0 R)}$, I_0, I_1 are the modified Bessel functions.

10.3.3 The averaged method in r-direction using integral spline with two fixed parametrical functions

Using averaged method (10.17) with fixed parametrical functions f_{r1}, f_{r2}
$g(r) = g_a + m f_{r1}(r - R/2) + e f_{r2}(r - R/2)$,
where
$g_a = \frac{2}{R^2}\int_0^R r g(r) dr$ is the averaged value, $\int_0^R r f_{r1} dr = \int_0^R r f_{r2} dr = 0$,
$f_{r1} = \frac{R^2 a^2 \sinh(a(r-R/2))}{4\sinh(aR/2)(d-1)} - 1, f_{r2} = \frac{\cosh(a(r-R/2)) - A_0}{8\sinh^2(aR/4)}$,
$A_0 = \sinh(aR/2)/(aR/2), d = Ra/2\coth(aR/2), a = a_0 + kor, kor$ is the corection for optimal parameter.
We can see that the parameter a tend to zero then the limit is the integral parabolic spline.

The unknown coefficients m, e we can determine from boundary conditions (10.17):

1) for r=0 ,$md_r - ek = 0$,
2) for r=R,$md_r + ek + \alpha(g_a + mb_m + eb_e - c_a)$,
where
$k = (a/4)\coth(aR/4), d_r = 0.5 * Rda^2/(d-1)$,
$b_e = \frac{\cosh(aR/2))-A_0}{8\sinh^2(aR/4)}, b_m = \frac{R^2a^2}{4(d-1)} - 1$.
Therefore $e = (c_a - g_a)/g_1, m = ek/d_r, g_1 = 2k/\alpha + b_m k/d_r + b_e$,
Now the-boundary value problem (10.17) is in the form

$$2/R^2(Rg'(R)) - a_0^2 g_a + f_1 = 0, g'(R) = 2ek \tag{10.18}$$

or $g_a = \frac{f_1 g_1 R + 4kc_a}{4k + a_0^2 g_1 R}$.
Example 2.We consider following parameters $L_z = R = 1, f_0 = -0.1, \alpha_r = 10.01, c_a = 10, D_r = 10^{-2}, D_z = 10^{-2}, D_z = 10^{-4}, a_0 = 1.5708$.
We have following maximal errors:
for hyperbolic spline $er_h = 0.00084, kor = -0.202$ for parabolic $er_p = 0.7476$ (see $g(r)$ in Fig. 10.19 and $c(r,z)$ in Fig. 10.20) If $\alpha_r =$

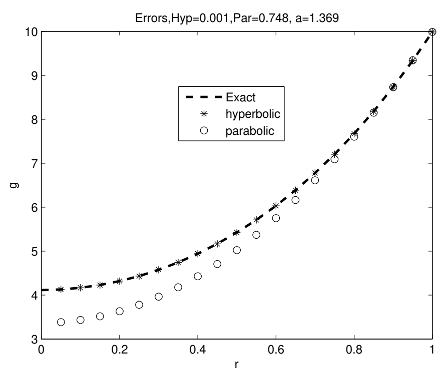


Fig. 10.19 Solution $g(r)$

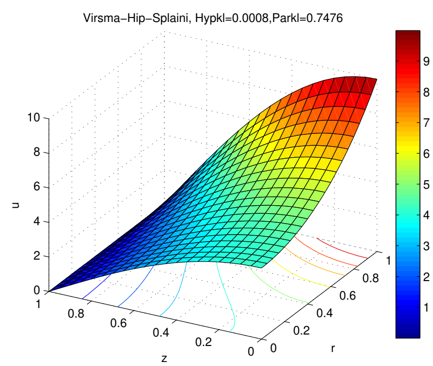


Fig. 10.20 Solution $c(r,z)$

0.01 then $er_h = 0.00032, er_p = 0.3412$, but for $f_0 = 1.01$: $er_h = 0.00079, er_p = 0.8412$.

10.3.4 The problem in sferical coordinates with axial symmetry

The process of diffusion is consider in half ball

$$\Omega = \{(r,\theta,\phi) : 0 \le r \le R, 0 \le \theta \le \pi/2, 0 \le \phi \le 2\pi\}.$$

We will consider the stationary boundary-value problem with axial symmetry. We will find the distribution of concentrations $c = c(r,\theta)$ in Ω at the point (r,θ) by solving the following special boundary-value problem for partial differential equation (PDE):

$$\begin{cases} D_r \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial c}{\partial r}) + \\ D_\theta \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}(\sin(\theta)\frac{\partial c}{\partial \theta}) + f_0\cos(\theta)/r^2 = 0, \\ \frac{\partial c(r,0)}{\partial \theta} = \frac{\partial c(0,\theta)}{\partial r} = 0, c(r,\pi/2) = 0, \\ D_r\frac{\partial c(R,\theta)}{\partial r} + \alpha_r(c(R,\theta) - c_a\cos(\theta) = 0, \end{cases} \qquad (10.19)$$

where f_0, c_a)-are the fixed constants,
D_r, D_θ are constant diffusion coefficients, α_r is the constant mass transfer coefficient.
We can obtain the analytical solution of (10.19) in following form: $c(r,\theta) = g(r)\cos(\theta)$,
where the function$g(r)$ is solution of boundary-value problem for ODE

$$\begin{cases} r^2g''(r) + 2rg'(r) - a_0^2 g(r) + f_1 = 0, \\ g'(0) = 0, g'(R) + \alpha(g(R) - c_a) = 0, \end{cases} \qquad (10.20)$$

where $f_1 = f_0/D_r, \alpha = \alpha_r/D_r, a_0^2 = \frac{2D_\theta}{D_r} > 2$.
We have following solutions

$$g(r) = C_1 r^\gamma + f_2,$$

where
$f_2 = \frac{f_1}{a_0^2}, \gamma = -0.5 + \sqrt{0.25 + a_0^2}\, C_1 = \frac{\alpha(c_a - f_2)}{R^{\gamma-1}(\gamma + \alpha R)}$.

10.3.5 CAM in r-direction with two fixed parametrical functions

Using averaged method (10.19) simirally Decart coordinates with functions f_{r1}, f_{r2}
$g(r) = g_a + m f_{r1}(r - R/2) + e f_{r2}(r - R/2)$,
where
$g_a = \frac{1}{R}\int_0^R g(r)dr$ is the averaged value, $\int_0^R f_{r1}dr = \int_0^R f_{r2}dr = 0$,
$f_{r1} = \frac{0.5R\sinh(a(r-R/2))}{\sinh(aR/2)}.f_{r2} = \frac{\cosh(a(r-R/2)) - A_0}{8\sinh^2(aR/4)}$,

$A_0 = \sinh(aR/2)/(aR/2), a = a_0 + kor, kor$ is the corection for optimal parameter.
We can see that the parameters a tend to zero then the limit is the integral parabolic spline (A. Buikis [9]).
The unknown coefficients m, e we can determine from boundary conditions (10.20):
1) for r=0 ,$md - ek = 0$,
2) for r=R,$md + ek + \alpha(g_a + mR/2 + eb - c_a)$,
where $k = (a/4)\coth(aR/4), d = 0.5 * Ra\coth(aR/2), b = \frac{\cosh(aR/2)) - A_0}{8\sinh^2(aR/4)}$.
Therefore $e = (c_a - g_a)/g_1, m = ek/d, g_1 = 2k/\alpha + b + 0.5kR/d$.
Now the-boundary value problem (10.20) is in the form

$$1/R(R^2 g'(R)) - a_0^2 g_a + f_1 = 0, g'(R) = 2ek \tag{10.21}$$

or $g_a = \frac{f_1 g_1 R + 2kRc_a}{2Rk + a_0^2 g_1}$.
Example 3. We consider following parameters:
$R = 2, f_0 = 0.1, \alpha_r = 0.003, c_a = 1, D_t = 10^{-2}, D_r = 10^{-4}, a_0 = 14.142$.
We have following maximal errors:
for hyperbolic spline $er_h = 0.0350, kor = -7.0$
for parabolic $er_p = 1.968$ (see $g(r)$ in Fig. 10.21 and $c(r,\theta)$ in Fig. 10.22) If $R = 1$ then $er_h = 0.0712, kor = -1.0, er_p = 1.966$.

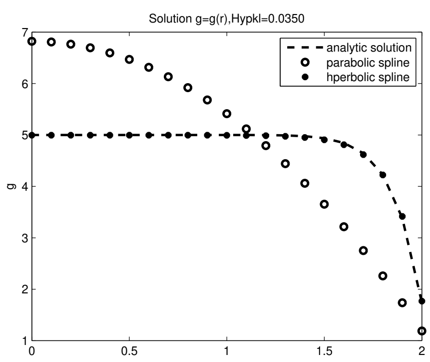


Fig. 10.21 Solution $g(r)$

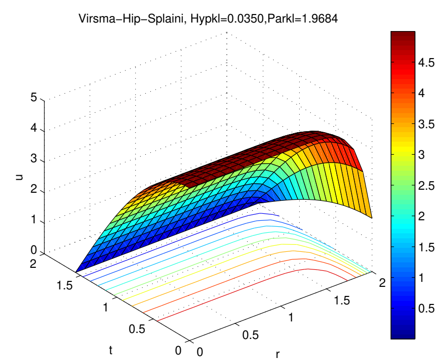


Fig. 10.22 Solution $c(r,\theta)$

10.4 The solution of 3-D two layer stationary diffusion problem

The process of diffusion is consider in 3-D parallelepiped

$$\Omega = \{(x,y,z) : 0 \le x \le L_x, 0 \le y \le L_y, 0 \le z \le L_z\}.$$

The domain Ω consist of 2 layer medium in the z direction

$$\Omega_i = \{(x,y,z) : x \in (0,L_x), y \in (0,L_y), z \in (z_{i-1}, z_i)\}, i = \overline{1,2},$$

where $H_i = z_i - z_{i-1}$ is the height of layer $\Omega_i, z_0 = 0, z_2 = L_z$.

We will find the distribution of concentrations $c_i = c_i(x,y,z)$ in every layer Ω_i at the point $(x,y,z) \in \Omega_i$ solving the following 3-D boundary-value problem for partial differential equations (PDEs):

$$\begin{cases} \frac{\partial}{\partial x}(D_x \frac{\partial c_i}{\partial x}) + \frac{\partial}{\partial y}(D_y \frac{\partial c_i}{\partial y}) + \frac{\partial}{\partial z}(D_{iz} \frac{\partial c_i}{\partial z}) - a_{i0}^2 c_i + F_i = 0, \\ x \in (0,L_x), y \in (0,L_y), z \in (z_{i-1}, z_i), \frac{\partial c_i(0,y,z)}{\partial x} = \frac{\partial c_i(x,0,z)}{\partial y} = 0, i = \overline{1,2}, \\ c_1(x,y,0) = 0, c_i(L_x,y,z) = c_i(x,L_y,z) = 0, i = \overline{1,2}, \frac{\partial c_2(x,y,L_z)}{\partial z} = 0, \\ c_1(x,y,z_1) = c_2(x,y,z_1), D_{1z} \frac{\partial c_1(x,y,z_1)}{\partial z} = D_{2z} \frac{\partial c_2(x,y,z_1)}{\partial z}, \end{cases} \tag{10.22}$$

where $c_i = c_i(x,y,z)$ are the concentrations functions in every layer, F_i - the fixed constants, $D_x, D_y, D_{1z}, D_{2z}, a_{i0}$ are the constant coefficients.

10.4.1 The CAM with the hyperbolic type integral spline approximation in z-direction

Using averaged method with respect to z with fixed parametrical functions $f_{iz1}, f_{iz2}, i = \overline{1,2}$
$c_i(x,y,z) = c_{iz}(x,y) + m_{iz}(x,y) f_{iz1}(z - \overline{z_i}) + e_{iz} f_{iz2}(z - \overline{z_i})$,
where $c_{iz}(x,y) = \frac{1}{H_i} \int_{z_{i-1}}^{z_i} c_i(x,y,z) dz$, are the averaged values,
$\int_{z_{i-1}}^{z_i} f_{iz1}(z) dz = \int_{z_{i-1}}^{z_i} f_{iz2}(z) dz = 0, \overline{z_i} = (z_{i-1} + z_i)/2, z \in [z_{i-1}, z_i]$,
$f_{iz1} = \frac{0.5 H_i \sinh(a_{iz}(z - \overline{z_i}))}{\sinh(0.5 a_{iz} H_i)}$, $f_{iz2} = \frac{\cosh(a_{iz}(z - \overline{z_i})) - A_{iz}}{8 \sinh^2(0.25 a_{iz} H_i)}$,
$A_{iz} = \frac{0.5 sinh(0.5 a_{iz} H_i)}{0.5 a_{iz} H_i}, i = \overline{1,2}$, and $a_{iz} > 0$ are fixed parameters (un-

known).
If $a_{i0} \neq 0$ then we can chooze $a_{iz} = |a_{i0}|\sqrt{1/D_{iz}}$. We can see that the parameters a_{iz} tend to zero then the limit is the integral parabolic spline from (A. Buikis [9]).

The unknown functions $m_{iz}(x,y), e_{iz}(x,y)$ we can determine from boundary conditions by $z = 0, z = L_z$:
$d_{2z}m_{2z} + k_{2z}e_{2z} = 0, m_{2z} = -p_{2z}e_{2z}, p_{2z} = k_{2z}/d_{2z}$,
$d_{iz} = 0.5a_{iz}H_i \coth(0.5a_{iz}H_i), k_{iz} = 0.25a_{iz} \coth(0.25a_{iz}H_i$,
$c_{1z} - 0.5m_{1z}H_1 + b_{1z}e_{1z} = 0, b_{iz} = \frac{\cosh(0.5a_{iz}H_i)-A_{iz}}{8\sinh^2(0.25a_{iz}H_i)}$,
$D_{1z}(d_{1z}m_{1z} + k_{1z}e_{1z}) = D_{2z}(d_{2z}m_{2z} - k_{2z}e_{2z})$,
$c_{1z}+0.5m_{1z}H_1+b_{1z}e_{1z} = c_{2z}-0.5m_{2z}H_2+b_{2z}e_{2z}, m_{1z} = 2(u_{1z}+b_{1z}e_{1z})/H_1$.
Therefore we have the system of 2 algebraic equations for $e_{iz}, i = \overline{1,2}$:
$b_{11}e_{1z} + b_{12}e_{2z} = -2c_{1z}d_{1z}/H_1, b_{21}e_{1z} + b_{22}e_{2z} = c_{2z} - 2c_{1z}$,
where $b_{11} = k_{1z}+2d_{1z}b_{1z}/H_1, b_{12} = 2D_{21}k_{2z}, b_{21} = 2b_{1z}, b_{22} = -(b_{2z}+ 0.5p_{2z}H_2)$.
The solution is: $e_{1z} = a_{11}c_{1z} + a_{12}c_{2z}, e_{2z} = a_{21}c_{1z} + a_{22}c_{2z}$,
where $a_{11} = (2b_{12} - 2b_{22}d_{1z}/H_1)/d, a_{12} = -b_{12}/d, a_{21} = (2b_{21}d_{1z}/H_1 - 2b_{11})/d, a_{22} = b_{11}/d,\ d = b_{11}b_{22} - b_{12}b_{21}$,

The boundary value 2-D problem is in following form

$$\begin{cases} \frac{\partial}{\partial x}(D_x\frac{\partial c_{1z}}{\partial x}) + \frac{\partial}{\partial y}(D_y\frac{\partial c_{1z}}{\partial y}) + a0_{1z}c_{1z} + b0_{1z}c_{2z} - a_{10}^2c_{1z} + F_1 = 0, \\ \frac{\partial}{\partial x}(D_x\frac{\partial c_{2z}}{\partial x}) + \frac{\partial}{\partial y}(D_y\frac{\partial c_{2z}}{\partial y}) + b0_{2z}c_{1z} + a0_{2z}c_{2z} - a_{20}^2c_{2z} + F_2 = 0, \\ \frac{\partial c_{iz}(0,y)}{\partial x} = \frac{\partial c_{iz}(x,0)}{\partial y} = 0, (c_{iz}(L_x,y) = c_{iz}(x,L_y) = 0, \end{cases} \tag{10.23}$$

where $a0_{iz} = \frac{2D_{iz}k_{iz}a_{i,i}}{H_i}, b0_{iz} = \frac{2D_{iz}k_{iz}a_{i,j}}{H_i}, i = \overline{1,2}, j = \overline{2,1}$.

10.4.2 The CAM in y-direction

Using averaged method with respect to y with fixed parametrical functions $f_{iy1}, f_{iy2}, i = \overline{1,2}$
$c_{iz}(x,y) = c_{iy}(x) + m_{iy}(x)f_{iy1}(y-L_y/2) + e_{iy}f_{iy2}(y-L_y/2)$,
where $c_{iy}(x) = \frac{1}{L_y}\int_0^{L_y} c_{iz}(x,y)dy$, are the averaged values,
$f_{iy1} = \frac{0.5L_y \sinh(a_{iy}(y-L_y/2))}{\sinh(0.5a_{iy}L_y)}$, $f_{iy2} = \frac{\cosh(a_{iy}(y-L_y/2))-A_{iy}}{8\sinh^2(0.25a_{iy}L_y)}$,
$A_{iy} = \frac{0.5sinh(0.5a_{iy}L_y)}{0.5a_{iy}L_y}, i = \overline{1,2}$, and for the parameter we choose $a_{iy} =$

$\sqrt{(-a0_{iz}+a_{i0}^2)/D_y}$.
Similarly, we determine the unknown functions $m_{iy}(x), e_{iy}(x)$,from boundary conditions by $y=0, y=L_y$
and $e_{iy}(x)=\frac{c_{iy}(x)}{g_{iy}}, g_{iy}=-b_{iy}-0.5p_{iy}L_y,\ m_{iy}(x)=p_{iy}e_{iy}(x), p_{iy}=k_{iy}/d_{iy}, d_{iy}=0.5a_{iy}L_y\coth(0.5a_{iy}L_y), k_{iy}=0.25a_{iy}\coth(0.25a_{iy}L_y)$,
$b_{iy}=\frac{\cosh(0.5a_{iy}L_y)-A_{iy}}{8\sinh^2(0.25a_{iy}L_y)}, A_{iy}=\frac{0.5sinh(0.5a_{iy}L_y)}{0.5a_{iy}L_y}$.
The boundary-value 1-D problem is in following form

$$\begin{cases} \frac{\partial}{\partial x}(D_x\frac{\partial c_{1y}(x)}{\partial x})+a01yc_{1y})+a0_{1z}c_{1y}+b0_{1z}c_{2y}-a_{10}^2c_{1y}+F_1=0,\\ \frac{\partial}{\partial x}(D_x\frac{\partial c_{2y}(x)}{\partial x})+a02yc_{2y})+b0_{2z}c_{1y}+a0_{2z}c_{2y}-a_{20}^2c_{2y}+F_2=0,\\ \frac{\partial c_{iy}(0)}{\partial x}=0, c_{iy}(L_x)=0, i=\overline{1,2}, \end{cases} \tag{10.24}$$

where $a0_{iy}=\frac{2D_yk_{iy}}{L_yg_{iy}}$.

10.4.3 The CAM in x-direction

Using averaged method with respect to x with fixed parametrical functions $f_{ix1}, f_{ix2}, i=\overline{1,2}$
$c_{iy}(x)=c_{ix}+m_{ix}f_{ix1}(x-L_x/2)+e_{ix}f_{ix2}(x-L_x/2)$,
where $c_{ix}=\frac{1}{L_x}\int_0^{L_x}c_{iy}(x)dx$, are the averaged values,
$f_{ix1}=\frac{0.5L_x\sinh(a_{ix}(x-L_x/2))}{\sinh(0.5a_{ix}L_x)}$, $f_{ix2}=\frac{\cosh(a_{ix}(x-L_x/2))-A_{ix}}{8\sinh^2(0.25a_{ix}L_x)}$,
$A_{ix}=\frac{0.5sinh(0.5a_{ix}L_x)}{0.5a_{ix}L_x}, i=\overline{1,2}$, and for the parameter we chooze $a_{ix}=\sqrt{(-a0_{iy}-a0_{iz}+a_{i0}^2)/D_x}$.
Similarly, we determine the unknown constants m_{ix}, e_{ix}
from boundary conditions by $x=0, x=L_x$ and $e_{ix}(x)=\frac{c_{ix}}{g_{ix}}, g_{ix}=-b_{ix}-0.5p_{ix}L_x$,
$m_{ix}=p_{ix}e_{ix}, p_{ix}=k_{ix}/d_{ix}, d_{ix}=0.5a_{ix}L_x\coth(0.5a_{ix}L_x), k_{ix}=0.25a_{ix}\coth(0.25a_{ix}L_x)$,
$b_{ix}=\frac{\cosh(0.5a_{ix}L_x)-A_{ix}}{8\sinh^2(0.25a_{ix}L_x)}, A_{ix}=\frac{0.5sinh(0.5a_{ix}L_x)}{0.5a_{ix}L_x}$.

From problem (10.24) follows 2 linear algebraic equations
$c_{11}c_{1x}+c_{12}c_{2x}+F_1=0, c_{21}c_{1x}+c_{22}c_{2x}+F_2=0$,
where $c_{11}=a0_{1x}+a0_{1y}+a0_{1z}-a_{10}^2, c_{12}=b0_{1z}, c_{21}=b0_{2z}, c_{22}=a0_{2x}+a02y+a0_{2z}-a_{20}^2$, $a0_{ix}=\frac{2D_xk_{ix}}{L_xg_{ix}}$.

The solution is $c_{1x} = (c_{12}F_2 - c_{22}F_1)/d1, c_{2x} = (c_{21}F_1 - c_{11}F_2)/d1, d1 = c_{11}c_{22} - c_{12}c_{21}$.

10.4.4 The CAM in y-direction and z- direction

If $a_{i0} = 0$ then for the parameters a_{iz} we determine the iteration process with using the CAM first in inverse directions
(y-direction and then z-direction). In y-directon $c_i(x,y,z) = c_{iy}(x,z) + m_{iy}(x,z)f_{iy1} + e_{iy}f_{iy2}$,
where $c_{iy}(x,z) = \frac{1}{L_y}\int_0^{L_y} c_i(x,y,z)dy$ is the averaged value and $a_{iy} = \sqrt{-a0_{iz}/D_y}$ is the previous value. In z-directon $c_{iy}(x,z) = c_{iz}(x) + m_{iz}(x)f_{iz1} + e_{iz}(x)f_{iz2}$,
where $c_{iz}(x) = \frac{1}{H_i}\int_{z_{i-1}}^{z_i} c_{iy}(x,z)dz$ and $a_{iz} = \sqrt{-a0_{iy}/D_z}$ is the new value for parameter a_{iz}. We can use the iteration process for obtain the parameters a_{iz}, a_{iy}, a_{ix}.

10.4.5 The 2-order Fourier method

For comparision we use Fourier series method in the domain $\Omega_{k1} = \{(x_1,y_1,z) : 0 \le x_1 \le 2L_x, 0 \le y_1 \le 2L_y, z \in (z_{k-1}, z_k)\}, k = \overline{1,2}$ and $x_1 = x + L_x \in [L_x, 2L_x], y_1 = y + L_y \in [Ly, 2L_y]$, if $x \in [0, Lx], y \in [0, L_y]$.
Then for the partial equation (10.21) we have following BCs:
$c_1(x_1,y_1,0) = c_k(x_1,2L_y,z) = c_k(2L_x,y_1,z) = c_k(x_1,0,z) = c_k(0,y_1,z) = \frac{\partial c_1(x_1,y_1,L_z)}{\partial z} = 0$,
$c_1(x_1,y_1,H_1) = c_2(x_1,y_1,H_1), D_{1z}\frac{\partial c_1(x_1,y_1,H_1)}{\partial z} = D_{2z}\frac{\partial c_2(x_1,y_1,H_1)}{\partial z}$,
and we can be obtain the solution in the series form
$c_k(x_1,y_1,z) = \sum_{i,j}^{\infty} A_{i,j}^k(z)W_{i,j}(x_1,y_1), F_k = \sum_{i,j}^{\infty} B_{i,j}^k W_{i,j}$,
where $B_{i,j}^k = F_k\int_0^{2L_x}\int_0^{2L_y} W_{i,j}(x_1,y_1)dx_1dy_1 = \frac{16F_k\sqrt{L_xL_y}}{\pi^2(2i-1)(2j-1)}$,
$W_{i,j}(x_1,y_1) = \sqrt{\frac{1}{L_xL_y}}\sin\frac{i\pi x_1}{2L_x}\sin\frac{j\pi y_1}{2L_y}$ are the orthonormed eigenfunctions $(i,j) = \overline{1,\infty}$ with the coresponding eigenvalues $\mu_{i,j} = (\frac{i\pi}{2L_x}\frac{j\pi}{2L_y})^2$.
Therefore for $A_{i,j}^k(z)$ we have following boundary-value problem of the system ODEs:

$$\begin{cases} \frac{d^2A^k_{i,j}(z)}{dz^2} - a^2_{i,j,k}A^k_{i,j}(z) + b^k_{i,j} = 0, k = \overline{1,2} \\ \frac{dA^2_{i,j}(L_z)}{dz} = 0, A^1_{i,j}(0) = 0, A^1_{i,j}(H_1) = A^2_{i,j}(H_1), \\ D_{1z}\frac{dA^1_{i,j}(H_1)}{dz} = D_{2z}\frac{dA^2_{i,j}(H_1)}{dz}, \end{cases} \tag{10.25}$$

where
$a^2_{i,j,k} = \frac{\pi^2}{4D_{kz}}(D_x((2i-1)/L_x)^2 + D_y((2j-1)/L_y)^2) + a^2_{k0}/D_{kz}, b^k_{i,j} = B^k_{i,j}/D_{kz}$.
The solution is $A^1_{i,j}(z) = C_1\sinh(a_{i,j,1}z) + C_2\cosh(a_{i,j,1}z) + g_{i,j,1} = 0$, $A^2_{i,j}(z) = C_3\sinh(a_{i,j,2}z) + C_4\cosh(a_{i,j,2}z) + g_{i,j,2} = 0$, where $C_2 = -g_{i,j,1}, C_3 = -C_4\tanh(a_{i,j,2}L_z), g_{i,j,k} = \frac{b^k_{i,j}}{a^2_{i,j,k}}$,
$C_4 = \frac{g_{i,j,2} - g_{i,j,1}(1 - cosh^{-1}(a_{i,j,1}H_1)}{D_{21}d_{i,j}\tanh(a_{i,j,1}H_1)b_{i,j,3}b_{i,j,4}}, D_{21} = \frac{D_{2z}}{D_{1z}}$,
$d_{i,j} = \frac{a_{i,j,2}}{a_{i,j,1}}, b_{i,j,3} = \sinh(a_{i,j,2}H_1) - \tanh(a_{i,j,2}L_z)\cosh(a_{i,j,2}H_1)$,
$b_{i,j,4} = -\cosh(a_{i,j,2}H_1) + \tanh(a_{i,j,2}L_z)\sinh(a_{i,j,2}H_1)$,
$C_1 = D_{21}d_{i,j}b_{i,j,3}/\cosh(a_{i,j,1}H_1) + g_{i,j,1}\tanh(a_{i,j,1}H_1)$.
For maximal value depends on z $M_k(z) = c_k(Lx, Ly, z) = \sum_{i,j}^{\infty}(-1)^{i+j}A^k_{i,j}(z)$.
For averaging in z-direction
$c_{1a} = \frac{1}{H_1}\int_0^{H_1} c_1(L_x, L_y, z)dz$, $c_{2a} = \frac{1}{H_2}\int_{H_1}^{L_z} c_2(L_x, L_y, z)dz$
For averaging in z-direction by $x = L_x, y = L_y$ $c_z = \frac{1}{L_z}\int_{L_z}^{2L_z} c(L_x, L_y, z_1)dz_1$ follows
$c_{1a} = \sum_{i,j}^{\infty}(-1)^{i+j}(C_1(\cosh(a_{i,j,1}H_1) - 1)/(a_{i,j,1}H_1) + g_{i,j,1}(1 - \sinh(a_{i,j,1}H_1)/(a_{i,j,1}H_1))$,
$c_{2a} = \sum_{i,j}^{\infty}(-1)^{i+j}(C_3(\cosh(a_{i,j,2}L_z) - \cosh(a_{i,j,2}H_1))/(a_{i,j,2}H_2)) + C_4(\sinh(a_{i,j,2}L_z) - \sinh(a_{i,j,2}H_1)/(a_{i,j,1}H_1)) + g_{i,j,2})$,
For maximal value by $z = L_z, M = \sum_{i,j}^{\infty}(-1)^{i+j}(C_4/\cosh(a_{i,j,2}L_z) + g_{i,j,2})$.

Example. We consider $F_1 = 2, F_2 = 1, L_z = 1, H_1 = 0.6, H_2 = 0.4, L_x = 2, L_y = 3, D_{1z} = 10^{-2}, D_{2z} = 10^{-3}, D_x = 3.10^{-3}, D_y = 3.10^{-4}$.
For initial value $a_{1z} = 1, a_{2z} = 3$ we obtain with the 5 iterations
$a_{1z} = 0.44, a_{2z} = 0.89, a_{1y} = 19.07, a_{2y} = 7.52$,
$a_{1x} = 6.08, a_{2x} = 2.43$ with maximal error 10^{-6}.
We obtain the maximal values M in 2 layers: $M_F = [134.5, 59.2]$ for Fourier method with 10 series summ, $M_H = [121.6, 50.3$ for hyperbolic approximation, $M_P = [259.8, 119.9]$ for parabolic approximation with $a_{ix} = a_{iy} = a_{iz} = 0.0001, (i = 1; 2)$;

and the averaged values $A : A_F = [108.0, 35.3]$ $A_H = [75.5, 27.1,$ $A_P = [94.7, 32.4.$
In Figs. 10.23-10.28 are represented the solutions.

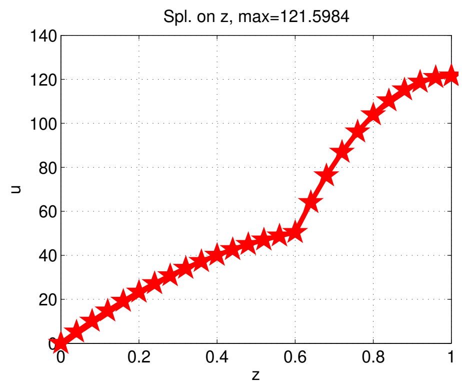


Fig. 10.23 Maximal solution with hiperbolic CAM depending on z

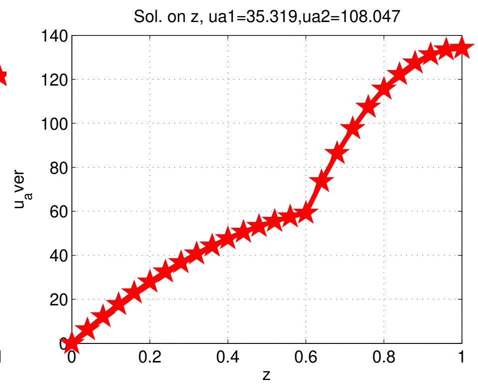


Fig. 10.24 Maximal solution with Fourier method depending on z

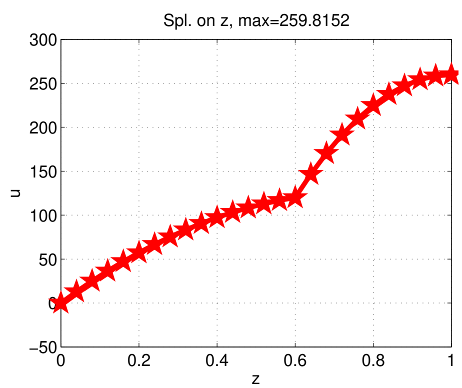


Fig. 10.25 Maximal solution with parabolic CAM depending on z

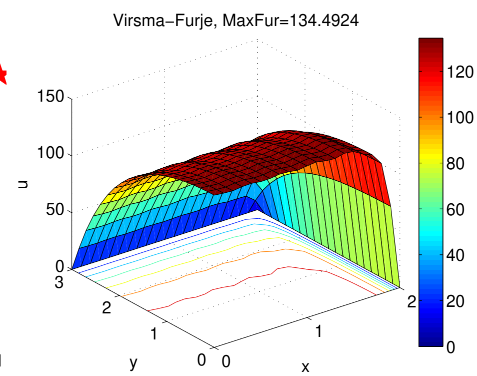


Fig. 10.26 Solution c_2 with with Forier method by $z = L_z$ depends on (x,y)

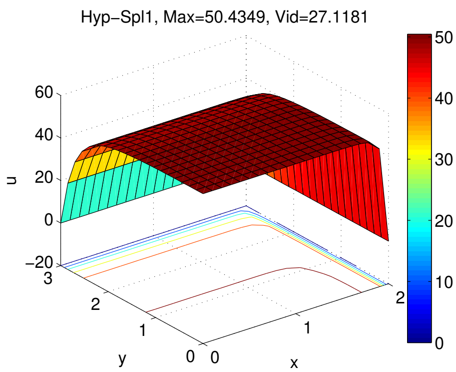


Fig. 10.27 Solution c_1 with with hiperbolic CAM by $z = H_1$ depends on (x,y)

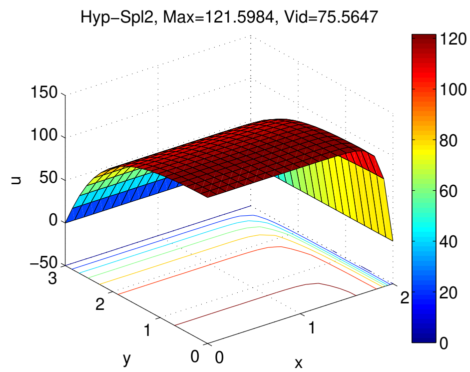


Fig. 10.28 Solution c_2 with with hiperbolic CAM by $z = L_z$ depends on (x,y)

10.5 Special hyperbolic type spline for 3-D nonstationary diffusion problem in peat block

The process of diffusion is consider in 3-D parallelepiped

$$\Omega = \{(x,y,z) : 0 \le x \le L_x, 0 \le y \le L_y, 0 \le z \le L_z\}.$$

We will find the distribution of concentrations of metals in the peat block $c = c(x,y,z,t)$ by solving the following 3-D initial-boundary value problem for partial differential equation (PDE):

$$\begin{cases} \frac{\partial c}{\partial t} = \frac{\partial}{\partial x}(D_x \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y}(D_y \frac{\partial c}{\partial y}) + \\ \frac{\partial}{\partial z}(D_z \frac{\partial c}{\partial z}) + f_0, x \in (0,L_x), y \in (0,L_y), z \in (0,L_z), \\ \frac{\partial c(0,y,z,t)}{\partial x} = \frac{\partial c(x,0,z,t)}{\partial y} = 0, c(x,y,z,0) = c_0(x,y,z), \\ D_x \frac{\partial c(L_x,y,z,t)}{\partial x} + \alpha_x (c(L_x,y,z,t) - c_{ax}) = 0, \\ D_y \frac{\partial c(x,L_y,z,t)}{\partial y} + \alpha_y (c(x,L_y,z,t) - c_{ay}) = 0, \\ D_z \frac{\partial c(x,y,L_z,t)}{\partial z} + \alpha_z (c(x,y,L_z,t) - c_{az}) = 0, \\ D_z \frac{\partial c(x,y,0,t)}{\partial z} - \beta_z (c(x,y,0,t) - c_{0z}) = 0, \end{cases} \tag{10.26}$$

where f_0 is the fixed constant, $c_0(x,y,z)$ is the concentration of the metals by t=0,
D_x, D_y, D_z are the constant coefficients,
$\alpha_x, \alpha_y, \alpha_z, \beta_z$ are the constant mass transfer coefficients,
$c_{az}, c_{0z}, c_{ay}, c_{ax}$ are the given concentration on the boundary.

10.5.1 The CAM with the hyperbolic type integral spline approximation in z-direction

For solving (10.26) we consider with respect to z-direction the CAM in following form:
$c(x,y,z,t) = c_z(x,y,t) + m_z(x,y,t)f_{z1} + e_z(x,y,t)f_{z2}$,
with following two fixed hyperbolic type functions and parameter a_z:
$f_{z1} = \frac{0.5L_z \sinh(a_z(z-0.5L_z))}{\sinh(0.5a_zL_z)}$, $f_{z2} = \frac{\cosh(a_z(z-0.5L_z))-A_{0z}}{8\sinh^2(0.25a_zL_z)}$,
where $A_{0z} = \frac{sinh(0.5a_zL_z)}{0.5a_zL_z}, c_z(x,y,t) = \frac{1}{L_z}\int_0^{L_z} c(x,y,z,t)dz$ is the averaged value and $a_z > 0$ is fixed initial parameter (unknown !).
We can see that the parameters a_z tend to zero then the limit is the integral parabolic spline from (A. Buikis [9]).

The unknown functions $m_z(x,y,t), e_z(x,y,t)$, are determine from boundary conditions by $z = 0, z = L_z$:
$D_z(d_zm_z - k_ze_z) - \beta_z(c_z - 0.5m_zL_z + b_ze_z - c_{0z}) = 0$,
$d_z = 0.5a_zL_z \coth(0.5a_zL_z), k_z = 0.25a_z \coth(0.25a_zL_z)$,
$D_z(d_zm_z + k_ze_z) + \alpha_z(c_z + 0.5m_zL_z + b_ze_z - c_{az}) = 0$,
where $b_z = \frac{\cosh(0.5a_zL_z)-A_{0z}}{8\sinh^2(0.25a_zL_z)}$.
Therefore $e_z(x,y) = c_{az}g_{az} + c_{0z}g_{0z} - c_z(x,y)g_z, g_z = g_{az} + g_{0z}$,
$g_{0z} = \beta_z(d_z + 0.5\alpha_zL_z/D_z)/S_z, g_{az} = \alpha_z(d_z + 0.5\beta_zL_z/D_z)/S_z$,
$S_z = 2D_zk_zd_z + b_zL_z\alpha_z\beta_z/D_z + (\alpha_z + \beta_z)(d_zb_z + 0.5k_zL_z)$,
$m_z(x,y,t) = c_{az}h_{az} + c_{0z}h_{0z} + c_z(x,y,t)h_z, h_z = h_{az} + h_{0z}$,
$h_{0z} = -\beta_z(k_z + \alpha_zb_z/D_z)/S_z, h_{az} = \alpha_z(k_z + \beta_zb_z/D_z)/S_z$.
The boundary-value 2-D problem is in following form

$$\begin{cases} \frac{\partial c_z}{\partial t} = \frac{\partial}{\partial x}(D_x\frac{\partial c_z}{\partial x}) + \frac{\partial}{\partial y}(D_y\frac{\partial c_z}{\partial y}) - B_zg_zc_z + \\ B_z(g_{az}c_{az} + g_{0z}c_{0z}) + f_0, \frac{\partial c_z(0,y,t)}{\partial x} = \frac{\partial c_z(x,0,t)}{\partial y}, \\ D_x\frac{\partial c_z(L_x,y,t)}{\partial x} + \alpha_x(c_z(L_x,y,t) - c_{ax}) = 0, \\ D_y\frac{\partial c_z(x,L_y,t)}{\partial y} + \alpha_y(c_z(x,L_y,t) - c_{ay}) = 0, \\ c_z(x,y,0) = c_{z0}(x,y), \end{cases} \tag{10.27}$$

where $B_z = 2D_zk_z/L_z, c_{z0}(x,y) = \frac{1}{L_z}\int_0^{L_z} c_0(x,y,z)dz$.

10.5.2 The CAM in y-direction

For $c_z(x,y,t) = c_y(x,t) + m_y(x,t) f_{y1} + e_y(x,t) f_{y2}$,
we have $f_{y1} = \frac{0.5L_y \sinh(a_y(y-0.5L_y))}{\sinh(0.5a_yL_y)}$, $f_{y2} = \frac{\cosh(a_y(y-0.5L_y))-A_{0y}}{8\sinh^2(0.25a_yL_y)}$,
where $A_{0y} = \frac{sinh(0.5a_yL_y)}{0.5a_yL_y}, c_y(x,t) = \frac{1}{L_y}\int_0^{L_y} c_z(x,y,t)dy$ is the averaged value,
and for the parameter we chooze $a_y = \sqrt{B_z g_z / D_y}$.

Similarly, we determine the unknown functions $m_y(x,t), e_y(x,t)$, from boundary conditions by $y = 0, y = L_y$:
$e_y(x,t) = \frac{c_{ay}-c_y(x,t)}{g_y}, g_y = b_y + 0.5p_yL_y + 2k_yD_y/\alpha_y, m_y(x,t) = p_ye_y(x,t), p_y = k_y/d_y, d_y = 0.5a_yL_y \coth(0.5a_yL_y), k_y = 0.25a_y \coth(0.25a_yL_y)$,
$b_y = \frac{\cosh(0.5a_yL_y)-A_{0y}}{8\sinh^2(0.25a_yL_y)}$.
The boundary value 1-D problem is in following form

$$\begin{cases} \frac{\partial c_y}{\partial t} = \frac{\partial}{\partial x}(D_x \frac{\partial c_y}{\partial x}) + \frac{B_y}{g_y}(c_{ay} - c_y) \\ -B_z g_z c_y + B_z(g_{az}c_{az} + g_{0z}c_{0z}) + f_0, c_y(x,0) = c_{y0}(x), \\ \frac{\partial c_y(0,t)}{\partial x} = 0, D_x \frac{\partial c_y(L_x,t)}{\partial x} + \alpha_x(c_y(L_x,t) - c_{ax}) = 0, \end{cases} \quad (10.28)$$

where $B_y = 2D_yk_y/L_y, c_{y0}(x) = \frac{1}{L_y}\int_0^{L_y} c_{z0}(x,y)dy$.

10.5.3 The CAM in x-direction

For $c_y(x,t) = c_x(t) + m_x(t) f_{x1} + e_x(t) f_{x2}$,
we have $f_{x1} = \frac{0.5L_x \sinh(a_x(x-0.5L_x))}{\sinh(0.5a_xL_x)}$, $f_{x2} = \frac{\cosh(a_x(x-0.5L_x))-A_{0x}}{8\sinh^2(0.25a_xL_x)}$,
where $A_{0x} = \frac{sinh(0.5a_xL_x)}{0.5a_xL_x}, c_x(t) = \frac{1}{L_x}\int_0^{L_x} c_y(x,t))dx$ is the averaged value,
and for the parameter we chooze $a_x = \sqrt{(B_y/g_y + B_z g_z)/D_x}$.

Similarly, we determine the unknown constants $m_x(t), e_x(t)$, from boundary conditions by $y = 0, y = L_y$
and $e_x = \frac{c_{ax}-c_x}{g_x}, g_x = b_x + 0.5p_xL_x + 2k_xD_x/\alpha_x$,
$m_x = p_xe_x, p_x = k_x/d_x, d_x = 0.5a_xL_x \coth(0.5a_xL_x), k_x = 0.25a_x \coth(0.25a_xL_x)$,
$b_x = \frac{\cosh(0.5a_xL_x)-A_{0x}}{8\sinh^2(0.25a_xL_x)}$.
From problem (10.28) follows the initial-value problem of the first order ODEs in following form:

$$\begin{cases} \frac{\partial c_x(t)}{\partial t} = \frac{B_x}{g_x}(c_{ax} - c_x(t) + \frac{B_y}{g_y}(c_{ay} - c_x(t)) \\ -B_z g_z c_x(t) + B_z(g_{az}c_{az} + g_{0z}c_{0z}) + f_0, c_x(0) = c_{x0}, \end{cases} \tag{10.29}$$

where $B_x = 2D_x k_x / L_x, c_{x0} = \frac{1}{Lx}\int_0^{L_x} c_{y0}(x)dx$.
We can obtain the solution in following form:
$c_x(t) = (c_{x0} - \frac{Sk}{Sa})exp(-Sat) + \frac{Sk}{Sa}$, where
$Sk = f_0 + \frac{B_y}{g_y}c_{ay} + B_z(g_{az}c_{az} + g_{0z}c_{0z}) + \frac{B_x}{g_x}c_{ax}, Sa = \frac{B_x}{g_x} + \frac{B_y}{g_y} + B_z g_z$.
The stacionary solution is: $c_x = \frac{Sk}{Sa}$.

10.5.4 The CAM in y-direction and z- direction

For the parameter a_z we determine the iteration process with using also the CAM first in y-direction and then z-direction. In y-directon $c(x,y,z,t) = c_y(x,z,t) + m_y(x,z,t)f_{y1} + e_y(x,z,t)f_{y2}$,
where $c_y(x,z,t) = \frac{1}{L_y}\int_0^{L_y} c(x,y,z,t)dy$ is the averaged value and $a_y = \sqrt{B_z g_z / D_y}$ is the previous value. In z-directon $c_y(x,z,t) = c_z(x,t) + m_z(x,t)f_{z1} + e_z(x,t)f_{z2}$,
where $c_z(x,t) = \frac{1}{L_z}\int_0^{L_z} c_y(x,z,t)dz$ and $a_z = \sqrt{(B_y/g_y + B_z g_z)/D_z}$ is the new value for parameter a_z. We can used the iteration process for obtaining the parameters a_z, a_y, a_x.

10.5.5 The numerical approximations with ADI method for the 3-D problem

We use uniform grid in the space $((K+1)\times(\tilde{N}+1)\times(M+1))$:
$\{(z_k, y_i, x_j), z_k = (k-1)h_z, y_i = (i-1)h_y, x_j = (j-1)h_x, i = \overline{1,\tilde{N}+1}$,
$j = \overline{1,M+1}, k = \overline{1,K+1}$, $K h_z = L_z, \tilde{N} h_y = L_y, M h_x = L_x$.
For the time t we use the moments $t_n = n\tau, n = 0, 1, \cdots$.
Subscripts (k,i,j,n) refer to z,y,x,t indices with the mesh spacing and
for approximation the function $c(z,y,x,t)$ we have the grid function with values $U^n_{k,i,j} \approx c(z_k, y_i, x_j, t_n)$.

For solving 3-D problem (1.12) we use the discrete approximation $(U^{n+1}_{k,i,j} - U^n_{k,i,j})/\tau = (\Lambda_z + \Lambda_y + \Lambda_x)U^{n+1}_{k,i,j} + f^n_{k,i,j}, n \geq 0$,
$k = \overline{1,K+1}, i = \overline{1,\tilde{N}+1}, j = \overline{1,M+1}$ and ADI method by Douglas and Rachford [37]:

$$\begin{cases} (U^{n+1/3}_{k,i,j} - U^n_{k,i,j})/\tau = \Lambda_z U^{n+1/3}_{k,i,j} + \Lambda_y U^n_{k,i,j} + \Lambda_x U^n_{i,j} + f^n_{i,j}, \\ k = \overline{2,K}, i = \overline{2,\tilde{N}}, j = \overline{2,M}, \\ (U^{n+2/3}_{k,i,j} - U^{n+1/3}_{k,i,j})/\tau = \Lambda_y (U^{n+2/3}_{k,i,j} - U^n_{k,i,j}), \\ k = \overline{1,K+1}, i = \overline{2,\tilde{N}}, j = \overline{1,M+1}, \\ (U^{n+1}_{k,i,j} - U^{n+2/3}_{k,i,j})/\tau = \Lambda_x (U^{n+1}_{k,i,j} - U^n_{k,i,j}), \\ k = \overline{1,K+1}, i = \overline{1,\tilde{N}+1}, j = \overline{2,M}. \end{cases} \tag{10.30}$$

After elimating the fractional time moments $t_{n+1/3}, t_{n+2/3}$ we obtain the previous discrete problem with approximation error $O(\tau^2)$. Here $\Lambda_x, \Lambda_y, \Lambda_z$ are the discrete difference operators, approximated the expressions $\frac{\partial}{\partial x}(D_x \frac{\partial c(z,y,x,t)}{\partial x})$, $\frac{\partial}{\partial y}(D_y \frac{\partial c(z,y,x,t)}{\partial y})$, $\frac{\partial}{\partial z}(D_z \frac{\partial c(z,y,x,t)}{\partial z})$, respect to x, y, z and boundary conditions with central differences, $f^n_{k,i,j} = f_0$. For solving $U^{n+1/3}$, $U^{n+2/3}$ and U^{n+1} we use Thomas algorithm in z, y and x directions respectively.

10.5.6 Some numerical results

Using the ADI method for the initial condition the stationary averaged solution $c(x,y,z)$ is selected.
The numerical results are obtained for
$\tau = 1, t_f = 1000$ with the maximal error 10^{-7}, $c_{ax} = c_{ay} = 2, \alpha_x = \alpha_y = 20, \alpha_z = 2000, \beta_z = 10$,
$D_z = 10^(-3), D_x = D_y = 310^(-4)$.
The maximal value of $c(x,y,z) = 4.63$ for averaged and for ADI methods, the parameters of CAM $a_z = 1.75, a_y = 2.49, a_x = 3.20$ are obtained with 5 iterations (maximal dfference 10^{-5}, see Fig. 10.36).
In Figs. 10.29-10.35 are represented the comparision of numerical and averaged results depenrs on z by $x = L_x/2, y = L_y/2$. The number of the grid points $(K,\tilde{N},M)$ are $K = 19, M = \tilde{N} = 21$.

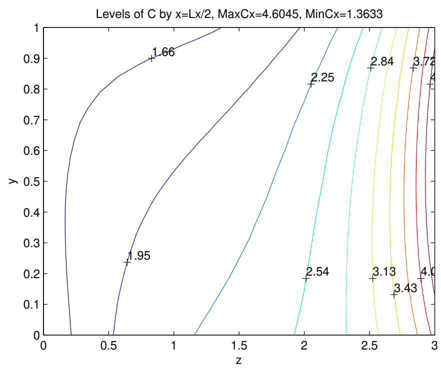


Fig. 10.29 Levels of averaged concentration $c(L_x/2, y, z)$

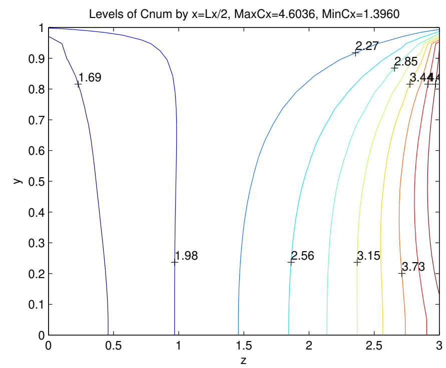


Fig. 10.30 Levels of numerical concentration $c(L_x/2, y, z)$

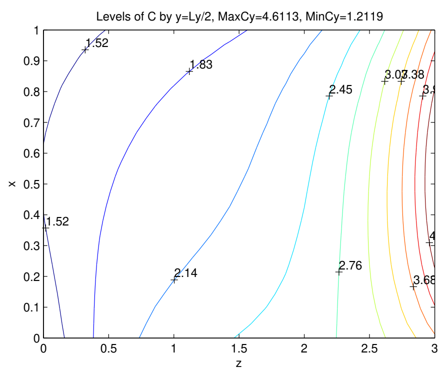


Fig. 10.31 Levels of averaged concentration $c(x, L_y/2, z)$

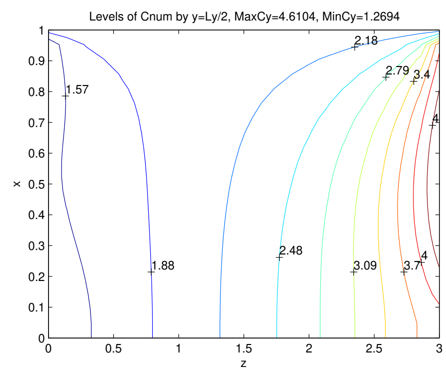


Fig. 10.32 Levels of numerical concentration $c(x, L_y/2, z)$

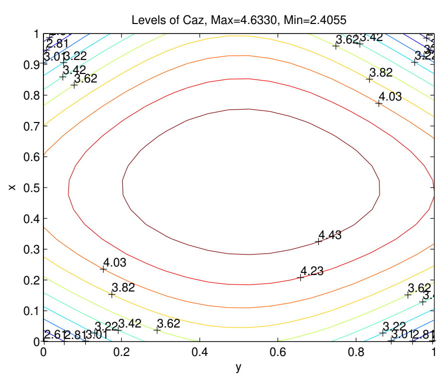


Fig. 10.33 Levels of concentration $c(x, y, L_z)$

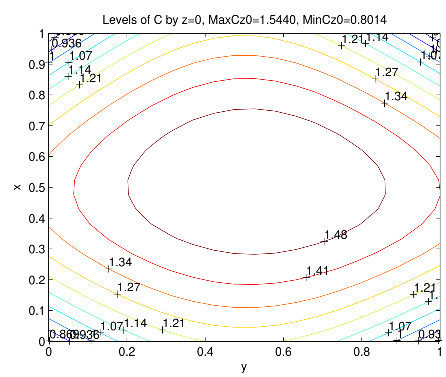


Fig. 10.34 Levels of l concentration $c(x, y, 0)$

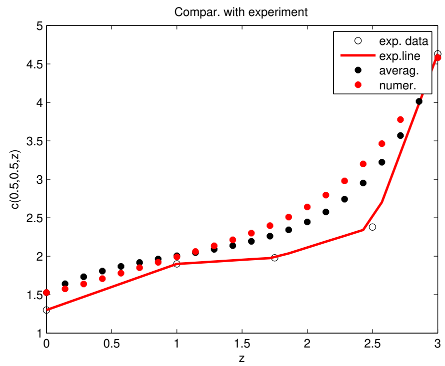


Fig. 10.35 Comparision with experiments by $c = (l_x/2, L_y/2, z)$

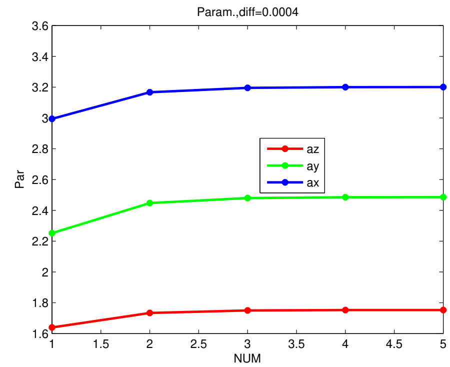


Fig. 10.36 5 iteration for parameters

Chapter 11
Some application of magnetic field influence on viscous incompresible liquid

The heating of buildings by ecologically clean and compact local devices is an interesting and actual problem. One of the modern areas of applications developed during last ten years is an effective usage of electrical energy by alternating current to produce heat energy. This work presents the mathematical model of one of such devices. It is a finite cylinder with viscous incompressible liquid and with metal conductors-electrods of the forms of bars placed parallel to the cylinder axis in the liquid. These conductors are connected to the alternating current.
We consider also 2-D stationary boundary value problem for the system of magnetohydrodynamic (MHD) equations along with the heat transfer equation. The viscous electrically conducting incompressible liquid moves between infinite cylinders placed periodically. We also examine similar 2-D MHD channel flow with periodically placed obstacles on the channel walls. We analyze the 2-D forced and free MHD convection flow as well as temperature around the cylinders and obstacles subject to homogeneous external magnetics field. The cylinders, obstacles and walls of channel with constant temperature are heated.

The goal of such investigation is to obtain the distributions of stream function, temperature, velocity and the vortex formation in the plane of the cylinders' cross-section and obstacles depending on the external magnetics field and on direction of the gravitation. For the numerical treatment we use finite difference method.

11.1 Introduction

In many physical experiments and technological applications it is important to stir and heat an electrically conducting liquid: liquid metals (steel, mercury, lithium), liquid magnetic materials, electrolyte, water, air. Liquid metals are considered to be the most promising coolants for high temperature applications, like nuclear fusion reactors because of the inherent high thermal diffusivity, thermal conductivity and hence excellent heat transfer characteristics. Lithium is the lightest of all metals and has the highest specific heat per unit mass.

In the developed mathematical models, vortex-type structures appear in liquid flows, as well as in problems related to energy conversion in new technological devices. MHD convection flow of a viscous incompressible fluid around cylinder with combined effects of heat and mass transfer is an important problem prevalent in many engineering applications [53], [56], [57]. These types of problems find applications in nuclear reactors, cooling systems and energy transfer systems.

Heat exchanger systems are employed in numerous industries. Steam generation in boiler, air cooling within the coil of an conditioner and automotive radiators represent just some of the conventional applications of this mechanical system. For the in-line arrangements of tube banks (cylinders), fluid at prescribed mass flow rate of velocity U_0 and an inlet ambient temperature T_0, much lower than the wall temperature T_w, enters the area from the left side and exits on the right. By taking the advantage of special geometrical features, such as the inherent repetitive nature of the flow behaviour, the computational fluid domain allows the possible exploitation of symmetric and periodic boundary conditions (PBCs) in speeding up the computations and in turn enhancing the computational accuracy of the simplified geometries. Using the conditions of symmetry and periodicity we can take into account only two cylinders. The heat transfer significant influence on the fluid flow behaviour with no impact of the magnetic field is investigated in [58].

In many technological applications it is important to mix an electroconductive liquid, using various magnetic fields.
In papers ([49],[50],[51], [52]) we had modelled cylinder form elec-

trical heat generators with six or nine circular conductors-electrodes. In this work we analyze different types of conductors, with the forms of bars and they are placed parallel to the cylinder axis in the electro-conductive liquid. It means that in distinction with the case of circular electrodes here we can't assume the axis summetry and we must consider full 3D mathematical model based on the system of Navier-Stokes equations.

Let the cylindrical domain $\Omega = \{(r,z) : 0 < r < R, 0 \leq \phi \leq 2\pi, 0 < z < Z\}$ contain conducting liquid-electrolyte, where R, Z are the radius and length of the cylinder. The alternating current is fed to N discrete conductors of forms of bars, which are placed parallel to the cylinder axis in the liquid.

The current creates in the weakly conductive liquid-electrolyte the radial B_r and the azimuthal B_ϕ components of the magnetic field as well the axial component of the induced electric field E_z, which, in its turn, creates the adial F_r and azimuthal F_ϕ components of the Lorentz' force.

For calculation of the electromagnetic fields outside the electrodes, the averaging method over the time interval $2\pi/\omega = 1/f$ is used. The averaged values of force $< F_r >$, $< F_\phi >$ give rise to a liquid motion, which can be described by the stationary Navier-Stokes equation.

11.2 Calculation of the electromagnetic field and force: A. Buikis, H. Kalis, 2002 [51]

Applying the Biot-Savar law we obtain the azimuthal component of the magnetic field B and axial compoent of vector potential A created by the current of density j from one infinite long circular conductor $L = \{r \leq a, 0 \leq \phi \leq 2\pi, -\infty \leq z \leq +\infty\}$ with radius a in following form [47]:

$$B(r,\phi) = \frac{\mu j a^2}{2\rho}, A(r,\phi) = -\frac{\mu j a^2}{2} \ln(\rho),$$

where $\rho = r > a, \mu = 4\pi 10^{-7} \frac{mkg}{s^2 A^2}$ is the magnetic permeability in vacuum.

For the limit case when the radius of the conductor tends to zero the magnitude $\frac{a^2}{2}$ must be by $\frac{1}{2\pi}$ replaced. If the bar type electrode is with finite length $z \in [C,D]$, then the azimuthal component of the magnetic field in point P out of electrods is in form

$$B(P) = \frac{\mu j}{4\pi\rho}(\cos(\alpha_1) + cos(\alpha_2)),$$

where $\alpha_1 = \angle PAB, \alpha_2 = \angle PBA$. If α_1 and α_2 tends to zero, then we obtain the previous expression. The magnetic field inside the electrode is not considered.
For the circular conductor $L_i = \{r - r_i \leq a, \phi_i - \alpha_i \leq \phi \leq \phi_i + \alpha_i, -\infty \leq z \leq +\infty\}$ in the polar coordinates (r,ϕ) follows that

$$B_i(r,\phi) = \frac{\mu j_i a^2}{2\rho_i}, A_i(r,\phi) = -\frac{\mu j_i a^2}{2}\ln(\rho_i),$$

where $\rho_i = \sqrt{(r_i^2 + r^2 - 2rr_i\cos(\phi - \phi_i))}, \alpha_i = \arcsin(a/r_i), (r_i, \phi_i)$ is the polar coordinate of the centers of circular wires L_i.
In the cases of alternating current

$$j_i = j_0\cos(\omega t) + (i-1)\theta), i = \overline{1,N}. \tag{11.1}$$

Here $j_0 = \frac{I}{\pi a^2}$ is the amplitude of density, $\omega = 2\pi f, f$ are the angular frequency and frequency of the alternating current, $\theta = const$ is the phase (usually $\theta = 120^0, f = 50Hz$), t is the time and I is the effective current intensity . We can consider that the azimuthal vector $B_i(r,\phi)$ as regards to the planes point (r_i, ϕ_i) can divide in the summ of two vector components $B_{r,i}, B_{\phi,i}$,where $B_{r,i} = B_i\, sin(\alpha_i), B_{\phi,i} = B_i\cos(\alpha_i)$, α_i —is the angle between the vectors B_i and $B_{\phi,i}$. Then

$$\cos(\alpha_i) = \frac{r_i\cos(\phi - \phi_i) - r}{\rho_i},\ \sin(\alpha_i) = \frac{r_i\sin(\phi - \phi_i)}{\rho_i}.$$

We can see that

$$div\mathbf{B}_i = \frac{1}{r}\frac{\partial}{\partial r}(rB_{r,i}) + \frac{1}{r}\frac{\partial B_{\phi,i}}{\partial \phi} = 0.$$

Therefore we obtain two components of the magnetic field induced by each current wire L_i in following form

$$\begin{cases} B_{r,i}(r,\phi,t) = \frac{\mu j_i a^2}{2\rho_i^2} r_i \sin(\phi-\phi_i), \\ B_{\phi,i}(r,\phi,t) = \frac{\mu j_i a^2}{2\rho_i^2}(r_i\cos(\phi-\phi_i)-r), i=\overline{1,N}. \end{cases} \tag{11.2}$$

Coresponding axial component of vector-potencial $\mathbf{A}(\mathbf{B} = rot\mathbf{A})$ is

$$A_{z,i}(r,\phi,t) = -\frac{\mu j_i a^2}{2}\ln(\rho_i), \tag{11.3}$$

because

$$B_{r,i} = \frac{1}{r}\frac{\partial A_{z,i}}{\partial \phi}, B_{\phi,i} = -\frac{\partial A_{z,i}}{\partial r}.$$

From Ohm's law follows that the axial components j_z of the vector of induced current density is

$$j_{z,i} = -\sigma \partial A_{z,i}/\partial t,$$

where σ is the electric conductivity.
From the vector of electromagnetic (Lorenz) force $\mathbf{F} = \mathbf{j}\times\mathbf{B}$ we can obtain the radial and azimuthal components $F_r = -B_\phi j_z,\ F_\phi = B_r j_z$ as the summ of all induced felds

$$B_\phi = \sum_{i=1}^{N} B_{\phi,i}, B_r = \sum_{i=1}^{N} B_{r,i}, j_z = \sum_{i=1}^{N} j_{z,i}. \tag{11.4}$$

Therefore , we obtain

$$\begin{cases} F_r(r,\phi,t) = K_0\sum_{i,j=1}^{N}\alpha_{i,j}cs(t), \\ F_\phi(r,\phi,t) = K_0\sum_{i,j=1}^{N}\beta_{i,j}cs(t), \end{cases} \tag{11.5}$$

where

$$K_0 = (\frac{a^2\mu j_0}{2})^2\sigma\omega, cs(t) = 0.5sin(2\omega t+(i+j-2)\theta)+0.5sin(\theta(j-i)),$$

$$\alpha_{i,j} = -\frac{\ln(\rho_i)(r_j\cos(\phi-\phi_j)-r)}{\rho_j^2}, \beta_{i,j} = \frac{\ln(\rho_i)r_j\sin(\phi-\phi_j)}{\rho_j^2}.$$

Similarly, the source term for heat transport equation has the form

$$j_z^2(r,\phi,t) = K_0\sigma\omega\sum_{i,j=1}^{N}\gamma_{i,j}ss(t), \tag{11.6}$$

where

$\gamma_{i,j} = \ln(\rho_i).\ln(\rho_j), ss(t) = -0.5\cos(2\omega t + (i+j-2)\theta) + 0.5\cos)\theta(j-i))$.

Denoting $A_i = \ln(\rho_i)$, gives

$$\alpha_{i,j} = A_i \frac{\partial A_j}{\partial r}, \beta_{i,j} = \frac{A_i}{r}\frac{\partial A_j}{\partial \phi}, \gamma_{i,j} = A_i A_j,$$

$$\frac{\partial A_j}{\partial \phi} = \frac{r_j \sin(\phi - \phi_j)}{\rho_j^2}, \frac{\partial A_j}{\partial r} = -\frac{r_j \cos(\phi - \phi_j) - r}{\rho_j^2}.$$

By averaging quantities in the time interval $\frac{2\pi}{\omega}$ we obtain

$$\begin{cases} < F_r(r,\phi) >= 0.5K_0 S_N^{\alpha}, \\ < F_\phi(r,\phi) >= 0.5K_0 S_N^{\beta}, \\ < j_z^2(r,\phi) >= 0.5K_0 \sigma\omega S_N^{\gamma}, \end{cases} \tag{11.7}$$

where

$$S_N^{\alpha} = \sum_{i,j=1}^{N} \sin((j-i)\theta)\alpha_{i,j}, S_N^{\beta} = \sum_{i,j=1}^{N} \sin((j-i)\theta)\beta_{i,j}, S_N^{\gamma} = \sum_{i,j=1}^{N} \cos((j-i)\theta)\gamma_{i,j}.$$

We can see that

$$S_N^{\alpha} = 2\sum_{k=1}^{N-1} \sin(k\theta) \sum_{i=1}^{N-k} \bar{\alpha}_{i,k+i}, S_N^{\beta} = 2\sum_{k=1}^{N-1} \sin(k\theta) \sum_{i=1}^{N-k} \bar{\beta}_{i,k+i},$$

$$S_N^{\gamma} = 2\sum_{k=1}^{N-1} \cos(k\theta) \sum_{i=1}^{N-k} \gamma_{i,k+i} + \sum_{i=1}^{N} \gamma_{i,i},$$

where

$$\bar{\alpha}_{i,j} = -0.5(A_i \frac{\partial A_j}{\partial r} - A_j \frac{\partial A_i}{\partial r}),$$

$$\bar{\beta}_{i,j} = -0.5\frac{1}{r}((A_i \frac{\partial A_j}{\partial \phi} - A_j \frac{\partial A_i}{\partial \phi}).$$

Having calculated the axial component of the curl of force vector

$$f = rot_z \mathbf{F} = \frac{1}{r}(\frac{\partial (rF_\phi)}{\partial r} - \frac{\partial F_r}{\partial \phi}) = B_r \frac{\partial j_z}{\partial r} + \frac{B_\phi}{r}\frac{\partial j_z}{\partial \phi},$$

we analogously obtain its average value as

$$< f(r,\phi) >= 0.5K_0 S_N^\delta, \tag{11.8}$$

where

$$S_N^\delta = \sum_{i,j}^{N} \sin((j-i)\theta)\delta_{i,j},$$

$$\delta_{i,j} = \frac{1}{r}[\frac{\partial}{\partial r}(r\beta_{i,j}) - \frac{\partial}{\partial \phi}(\alpha_{i,j})] = g_{i,j} - g_{j,i},$$

$$g_{i,j} = \frac{r_i \sin(\phi - \phi_i)(r_j \cos(\phi - \phi_j) - r)}{\rho_i^2 \rho_j^2}.$$

11.2.1 The mathematical model

The stationary flow of incompressible viscous liquid in a cylinder is described by the system of the Navier - Sokes equations, which are given in the cylindrical coordinates (r,ϕ,z) [48]:

$$\begin{cases} M(V_z) = -\tilde{\rho}^{-1}\frac{\partial p}{\partial z} + \nu\Delta V_z \\ M(V_r) - r^{-1}V_\phi^2 = -\tilde{\rho}^{-1}\frac{\partial p}{\partial r} + \\ \nu(\Delta V_r - r^{-2}V_r - 2r^{-2}\frac{\partial V_\phi}{\partial \phi}) + \tilde{\rho}^{-1} < F_r > \\ M(V_\phi) + r^{-1}V_r V\phi = -(\tilde{\rho}r)^{-1}\frac{\partial p}{\partial \phi} + \\ \nu(\Delta V_z - r^{-2}V_\phi + 2r^{-2}\frac{\partial V_r}{\partial \phi}) + \tilde{\rho}^{-1} < F_\phi > \\ \frac{\partial(rV_r)}{\partial r} + \frac{\partial(V_\phi)}{\partial \phi} + \frac{\partial(rV_z)}{\partial z} = 0, \end{cases} \tag{11.9}$$

Here V_r, V_z, V_ϕ are the radial, axial and azimuthal components of velocity vector $\mathbf{V}$, depending on the coordinates r,ϕ,z; and Δ is Laplace operator,

$$\Delta g = r^{-1}\frac{\partial}{\partial r}(r\frac{\partial g}{\partial r}) + r^{-2}\frac{\partial^2 g}{\partial \phi^2} + \frac{\partial^2 g}{\partial z^2},$$

$< F_r >, < F_\phi >$ are the components of the external avereged force $< \mathbf{F} >$,

$$M(g) = V_r\frac{\partial g}{\partial r} + r^{-1}V_\phi\frac{\partial g}{\partial \phi} + V_z\frac{\partial g}{\partial z}$$

are the convective parts of the equations , $\tilde{\rho}$, ν are the density and kinematic viscosity, p is the pressure, $g = V_r; V_\phi; V_z$.

The liquid has the following parameters:
kinematic viscosity $\nu \approx 10^{-5}\frac{m^2}{s}$, density of liquid $\tilde{\rho} \approx 1000\frac{kg}{m^3}$ and the electric conductivity $\sigma \approx 1000\Omega^{-1}m^{-1}$.
The parameter $K_0 = 1$, radius R of the cylinder is $0.035m$, the length Z of the cylinder is $0.35m$, the density of the current amplitude $j_0 \approx 10^8\frac{A}{m^2}$ and the radius a of the elektrodes is $0.005m$.
At the inlet of the cylinder we assume an uniform velocity $U_0 = 0.1\frac{m}{s}$.
On the walls (the surfaces of the cylinder and electrodes) we have the non-slipping conditions $\mathbf{V} = 0$.

In the cross-section $z = const$ we assume that $V_z = 0, \frac{\partial^j g}{\partial z^j} = 0$ for all $j \geq 1$ and therefore we can consider the 2D problem. In this case by the elimination of pressure from the second two equations of the system of PDEs (2.1) we obtain

$$M(\tilde{\omega}) = \nu(\Delta\tilde{\omega} + \tilde{\rho}^{-1} < f >, \tag{11.10}$$

where $\tilde{\omega} = r^{-1}(\partial(rV_\phi)/\partial r - \partial V_r/\partial\phi$ is the axial component of vector's $curl\mathbf{V}$ or the function of the vorticity, f is the axial component of the vector's $curl\mathbf{F}$.
The stream function ψ can be determined with formulas

$$V_r = r^{-1}\frac{\partial\psi}{\partial\phi}, V_\phi = -\frac{\partial\psi}{\partial r}. \tag{11.11}$$

Then from the equation of continuity and from vorticity function it follows, that

$$\tilde{\omega} = -\Delta\psi. \tag{11.12}$$

From (11.10, 11.12) follows the system of two PDEs for solving the vorticity function $\tilde{\omega}$ and stream function ψ:

$$\begin{cases} \Delta\psi = -\tilde{\omega} \\ r^{-1}J(\tilde{\omega},\psi) = \nu\Delta\tilde{\omega} + \tilde{\rho}^{-1} < f >, \end{cases} \tag{11.13}$$

where $J(\tilde{\omega},\psi) = (\partial\tilde{\omega}/\partial r)(\partial\psi/\partial\phi) - (\partial\tilde{\omega}/\partial\phi)(\partial\psi/\partial r)$ is the Jacobian of the functions ψ and $\tilde{\omega}$.

In the 2-D case we have the following boundary conditions:
1) the conditions of periodicity $g(r,0) = g(r,2\pi), \frac{\partial g(r,0)}{\partial \phi} = \frac{\partial g(r,2\pi)}{\partial \phi}$, where $g = \psi; \tilde{\omega}$
2) the non-slipping conditions on the walls $\psi = \partial \psi / \partial \mathbf{n} = 0$, and special conditions for vorticity function $\tilde{\omega} = \omega_w$, where ω_w is the value of the vorticity on the walls (the modificated Wood's conditions [49]) which characterizes the non-slip of the liquid on the wall, $\mathbf{n}$ is the external normal on the walls surfaces.

11.2.2 Some numerical experiments

Calculations and graphic visualization were made with the help of the computer programs MATLAB and FLUENT. We consider 3 conductors ($N = 3$) placed parallel to the cylinder axis creating the regular triangle with following coodinates of their center $(r_1, \phi_1) = (r_0, 0), (r_2, \phi_2) = (r_0, 120^0), (r_3, \phi_3) = (r_0, 240^0), r_0 = 0.02m$.
If $N = 2, \theta = \pi, (r_1, \phi_1) = (r_0, 0), (r_2, \phi_2) = (r_0, \pi)$ then $< F_r >=< F_\phi >= 0, < j_z^2 >= 0.5 K_0 \sigma \omega S_2^\gamma$, where $S_2^\gamma = \ln^2 \frac{r_0^2 + r^2 - 2rr_0 \cos(\phi)}{r_0^2 + r^2 + 2rr_0 \cos(\phi)}$. In this case the Lorenz' force is zero, but the heat source can be influence the temperature distribution in the liquid.
The phase is $2\pi/3$ and the frequency is $50Hz$.

In this case $\theta = \frac{2\pi}{3}, \sin(\theta) = \frac{\sqrt{3}}{2}, \sin(2\theta) = -\frac{\sqrt{3}}{2}, \cos(\theta) = \cos(2\theta) = -\frac{1}{2}$, and we corespondingly obtain

$$\begin{cases} S_3^\alpha = \frac{\sqrt{3}}{2}(\alpha_{1,2} + \alpha_{2,3} + \alpha_{3,1} - \alpha_{1,3} - \alpha_{2,1} - \alpha_{3,2}), \\ S_3^\beta = \frac{\sqrt{3}}{2}(\beta_{1,2} + \beta_{2,3} + \beta_{3,1} - \beta_{1,3} - \beta_{2,1} - \beta_{3,2}), \\ S_3^\gamma = \gamma_{1,1} + \gamma_{2,2} + \gamma_{3,3} - \gamma_{1,2} - \gamma_{2,3} - \gamma_{1,3}, \\ S_3^\delta = \sqrt{3}(\delta_{1,2} + \delta_{2,3} + \delta_{3,1}). \end{cases} \tag{11.14}$$

In the Figs. 11.1-11.3 we can see the results of calculations obtained by computer program MATLAB in cross-section $z = const$:
1) distribution of the averaged azimuthal Lorenz' force $< F_\phi >$ (Fig. 11.1),
2) distribution of the averaged radial Lorenz' force $< F_r >$(Fig. 11.2),
3) distribution of the averaged axial curl $< f >$ (Fig. 11.3).

These results are nondimensionalized by reffering all the lengths to r_0. Then we have the nondimensional radiuss of cylinder and electrods $R/r_0 = 1.75, a/r_0 = 0.25$. We can see the vortex formation in the cylinder.

11.3 2-D MHD convection around cylinders:H. Kalis, M. Marinaki, etc., 2012 [54]

Heat exchanger systems are employed in numerous industries. Steam generation in boiler, air cooling within the coil of an conditioner and automotive radiators represent just some of the conventional applications of this mechanical system. For the in-line arrangements of tube banks (cylinders), fluid at prescribed mass flow rate of velocity U_0 and an inlet ambient temperature T_0, much lower than the wall temperature T_w, enters the area from the left side and exits on the right. By taking the advantage of special geometrical features, such as the inherent repetitive nature of the flow behaviour, the computational fluid domain allows the possible exploitation of symmetric and periodic boundary conditions (PBCs) in speeding up the computations and in turn enhancing the computational accuracy of the simplified geometries. Using the conditions of symmetry and periodicity we can take into account only two cylinders. The heat transfer significant influence on the fluid flow behaviour with no impact of the magnetic field is investigated in [58].

The viscous electrically conducting incompressible liquid moves in the (x,y) plane in the Ox-axis direction between infinite cylinders placed periodically in the plane. The cross-section of cylinders is square in the plane. This process of the magnetohydrodynamics (MHD) is considered with the so-called inductionless approximation [53]. In [54] similar MHD problem without temperature and gravitation is considered.

We model the external magnetic field as the Lorentz force term, obtain the dimensionless stationary Navier-Stokes equations, set a computational domain and formulate the system of three equations involving stream, vorticity and temperature functions. The distribution of electromagnetic fields, forces, velocity and temperature around cylinders has been calculated using the finite difference method, Seidel iterations and specific boundary conditions for vorticity function. Some numerical results are analysed.

11.3.1 Mathematical model

The external homogeneous 2-D magnetic field has two components of the induction:
$B_x = B_0 \cos(\alpha), B_y = B_0 \sin(\alpha)$, where α is the angle between the Ox-axis and direction of the induction vector, B_0 is the magnitude of the magnetic field. The magnetic field creates the $F_x(t,x,y), F_y(t,x,y)$ components of the Lorentz' force $\mathbf{F}$.
Considering the vector of Lorenz force $\mathbf{F} = \mathbf{JxB}, \mathbf{J} = \sigma(\mathbf{E}+\mathbf{VxB})$, we obtain $J_z = \sigma(E_z + B_0(V_x \sin(\alpha) - V_y \cos(\alpha))), F_x = -B_y J_z, \quad F_y = B_x J_z$,
where $E_z = const, J_z$ are the azimuthal components of the electric field vector $\mathbf{E}$ and the density vector of the electric current $\mathbf{J}$, B_x, B_y are the components of the magnetic induction vector $\mathbf{B}$, σ is the electric conductivity, V_x, V_y are the components of velocity vector $\mathbf{V}$.

We analyze the flow depending on two settings of the homogeneous magnetic field: the field parallel to Ox-axis $(\alpha = 0)$ and the transverse field $(\alpha = \frac{\pi}{2})$. In the channel flow, incline magnetic fields for $(\alpha = \frac{\pi}{4}$ and $\alpha = \frac{3\pi}{4})$ are also examined.

The z-component $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$ of the vector $curl\mathbf{F} = \nabla \times \mathbf{F}$ affects the liquid motion, which can be described by Navier-Stokes equations in Boussinesq approximation and the heat transfer equation [54],[56]:

$$\begin{cases} \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{V} + \beta_t \mathbf{g}(T - T_0) + \frac{1}{\rho}\mathbf{JxB}, \\ \nabla.\mathbf{V} = 0, \\ \frac{\partial T}{\partial t} + (\mathbf{V}.\nabla)T = \frac{k}{\rho C_p}\Delta T + \frac{|J_z|^2}{\rho C_p \sigma}, \end{cases} \tag{11.15}$$

where Δ is the Laplacian, $p, T, \sigma, \rho, \nu, k, C_p, \beta_t$ are pressure, temperature, electrical conductivity, fluid density, kinematic viscosity, heat conductivity, specific heat, acceleration due to gravity, volumetric coefficient of thermal expansion, T_0 is the initial fluid temperature, $\mathbf{g} = g(\sin(\beta), -\cos(\beta), 0)$ is the gravitation, g is the gravitational acceleration, β is the angle between the Oy-axis and direction of the gravitation.

The equations (11.15) were made dimensionless by using characteristic values L_0 (side length of a square), U_0 (velocity), B_0 (magnetic field), $P_0 = U_0^2\rho$ (pressure), L_0/V_0 (time), $T_w - T_0$ (temperature), U_0B_0 (density vector of the electric current), where T_0, T_w are the initial temperature in the fluid and temperature on the cylinders.
Using the vorticity function $\zeta = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}$, one obtains

$$\begin{cases} \frac{\partial V_x}{\partial t} - \zeta V_y = -\frac{\partial \bar{p}}{\partial x} - \frac{1}{Re}\frac{\partial \zeta}{\partial y} + \frac{Gr}{Re^2} T\sin(\beta) - S\sin(\alpha) j_z, \\ \frac{\partial V_y}{\partial t} + \zeta V_x = -\frac{\partial \bar{p}}{\partial y} + \frac{1}{Re}\frac{\partial \zeta}{\partial x} - \frac{Gr}{Re^2} T\cos(\beta) + S\cos(\alpha) j_z, \\ \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0, \\ \frac{\partial T}{\partial t} + V_x\frac{\partial T}{\partial x} + V_y\frac{\partial T}{\partial y} = \frac{1}{Pe}\Delta T + \frac{K_T}{Pe} j_z^2, \end{cases} \tag{11.16}$$

where $j_z = e_z + V_x\sin(\alpha) - V_y\cos(\alpha)$, e_z is the dimensionless form of azimuthal component for electric current density and electric field, $\bar{p} = p + 0.5\mathbf{V^2}$,
$Re = \frac{U_0L_0}{\nu}, S = \frac{\sigma B_0^2 L_0}{\rho U_0}$, $Gr = \frac{\beta_t g(T_w - T_0)L_0^3}{\nu^2}$ are Reynolds, Stewart and Grashof numbers, $Pe = PrRe, Pr = \frac{\nu\rho C_p}{k}, K_T = \frac{\sigma B_0^2 L_0^2 U_0^2}{k(T_w - T_0)}$ are Prandtl number and heat source parameters.

The hydrodynamic stream function ψ can be determined by relations
$V_x = \frac{\partial \psi}{\partial y}, V_y = -\frac{\partial \psi}{\partial x}$. Eliminating the pressure from (11.16), one obtains

$$\begin{cases} \frac{\partial \zeta}{\partial t} - J(\psi,\zeta) = \frac{1}{Re}\Delta\zeta - \frac{Gr}{Re^2}\left(\frac{\partial T}{\partial x}\cos(\beta) + \frac{\partial T}{\partial y}\sin(\beta)\right) + Sf, \\ \zeta = -\Delta\psi, \\ \frac{\partial T}{\partial t} - J(\psi,T) = \frac{1}{Pe}\Delta T + \frac{K_T}{Pe} j_z^2, \end{cases} \tag{11.17}$$

where $f = \sin(2\alpha)\frac{\partial^2\psi}{\partial x\partial y} + \cos^2(\alpha)\frac{\partial^2\psi}{\partial x^2} + \sin^2(\alpha)\frac{\partial^2\psi}{\partial y^2}$
is the z-component of the vector $curl\mathbf{F}$,
$J(\psi,v) = \frac{\partial\psi}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial v}{\partial x}$ is the Jacobian of the functions ψ, v, $v = \zeta; T$,
$j_z = e_z + \frac{\partial\psi}{\partial l_B}$, is the derivative in the direction $l_B = (\cos(\alpha), \cos(\pi/2 - \alpha), 0)$.
Using the boundary conditions (BCs) of symmetry and periodicity, we can consider only the domain containing quarters of two cylinders [60].

For **in-line arrangement of cylinders (PC)** we consider the domain
$\Omega = \Omega_1 \bigcup \Omega_2$ (see Figs. 11.4, 11.5), where $\Omega_1 = \{(x,y) : l_1 \leq x \leq l_2, 0 \leq y \leq L_1\}$,
$\Omega_2 = \{(x,y) : 0 \leq x \leq l, L_1 \leq y \leq L\}, 0 < l_1 < l_2 < l, 0 < L_1 < L$.
Here $C_1 = \{(x,y) : 0 < x < l_1, 0 < y < L_1\}$ and
$C_2 = \{(x,y) : l_2 < x < l, 0 < y < L_1\}$ are the quarters of cylinders,
$L^1 = \{(x,L) : 0 \leq x \leq l\}, L^2 = \{(x,0) : l_1 \leq x \leq l_2\}$ are the plane of symmetry with BCs $V_y = 0, \frac{\partial T}{\partial y} = \zeta = 0, \psi = \psi_0$ on $L^1, \psi = 0$ on L^2,
$W^1 = \{(x,L_1) : 0 < x \leq l_1\}, W^2 = \{(x,L_1) : l_2 \leq x < l\}$,
$W^3 = \{(l_1,y) : 0 < y \leq L_1\}$ and $W^4 = \{(l_2,y) : 0 < y \leq L_1\}$ are the walls of the cylinders with the non-slip BCs $T = 1, V_x = V_y = \psi = 0$,
$I_n = \{(0,y) : L_1 < y \leq L\}$ is the inlet and $O_t = \{(l,y) : L_1 < y \leq L\}$ is the outlet with the periodical BCs for ψ, ζ, T, U_x, U_y.
In the case of free convection $\psi_0 = 0$.
For the **additional channel flow with symmetry (CFS)**,
$L^2 = W^5 = \{(x,0) : l_1 \leq x \leq l_2\}$ is the wall of the half-channel Ω with non-slip BCs $T = 1, V_x = V_y = \psi = 0$.

We consider also the additional **channel flow without symmetry (CF)** in the domain $\Omega = \Omega_1 \bigcup \Omega_2 \bigcup \Omega_3$ (Fig. 11.6), where
$\Omega_1 = \{(x,y) : l_1 \leq x \leq l_2, 0 \leq y \leq L_1\}$,
$\Omega_2 = \{(x,y) : 0 \leq x \leq l, L_1 \leq y \leq L2\}$,
$\Omega_3 = \{(x,y) : l_1 \leq x \leq l_2, L2 \leq y \leq L\}$,
$C_1 = \{(x,y) : 0 < x < l_1, L_2 < y < L\}$ and
$C_2 = \{(x,y) : l_2 < x < l, L-2 < y < L\}$ are the new quarters of cylinders,
$W^5 = \{(x,L_2) : 0 < x \leq l_1\}, W^6 = \{(x,L_2) : l_2 \leq x < l\}$,
$W^7 = \{(l_1,y) : L_2 < y \leq L\}$, $W^8 = \{(l_2,y) : L_2 < y \leq L\}$ and
$W^9 = \{(x,0) : l_1 < x \leq l_2\}, W^{10} = \{(x,L) : l_1 \leq x < l_2\}$ are the walls of the new cylinders and walls of the channel with the non-slip BCs:
$V_x = V_y = \psi = 2 * \psi_0, T = 1$, ($\psi = 0$ on W^9).
$I_n = \{(0,y) : L_1 < y \leq L_2\}$ is the inlet and $O_t = \{(l,y) : L_1 < y \leq L_2\}$ is the outlet with the periodical BCs.

The cylinders are electrically non-conducting and $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} j_z dxdy = 0$.
From $\int\limits_{\Omega}(\frac{\partial \psi}{\partial l_B} + e_z)dxdy = 0$, it follows that $e_z = -\frac{1}{mes\Omega}\int\limits_{\partial\Omega}\psi\cos(n,l_B)ds$ or $e_z = -\frac{l\sin(\alpha)}{L_1(l_2-l_1)+l(L-L_1)}$.

For $\psi_0 > 0$ we have the direct fluid flow in the direction of the positive Ox-axis with the fixed dimensionless fluid volume $Q = \psi_0$, but for $\psi_0 < 0$ we have the opposite flow in the direction of the negative Ox-axis with the fixed dimensionless fluid volume $Q = -\psi_0$. For the channel flow without symmetry, $\psi_0 = 2$.

On the walls we use the following BCs [55]: $\zeta^m = \gamma \frac{\partial \psi}{\partial n} + \zeta^{m-1}, m = 1, 2, \cdots,$
where m is the number of iterations with $\zeta^0 = 0$, $\gamma > 0$ is the parameter, n is the outer normal on the walls.

11.3.2 Physical parameters.

We consider following parameters for liquid metals (lithium, steel, mercury) and electrolyte [59] (Table 11.1).

Table 11.1 The physical parameters $\rho, C_p, \nu, k, \sigma, Pr$

Substance	$T_0 - T_w[K]$	$\rho[\frac{kg}{m^3}]$	$C_p[\frac{J}{kgK}]$	$\nu[\frac{m^2}{s}]$	$k[\frac{W}{mK}]$	$\sigma[\frac{1}{\Omega m}]$	Pr
Lithium	$500-700$	500	4000	$1.e-6$	45	$1.e+7$	0.04
Mercury	$300-500$	13000	140	$1.e-7$	10	$1.e+6$	0.02
Steel	$1600-1800$	7000	400	$1.e-6$	60	$7.e+5$	0.04
Electrolyte	$290-370$	1100	4000	$1.e-6$	1.1	100	4.0

The characteristic length scale is $L_0 = 4.e-3m$, the magnitude of the uniform flow velocity $U_0 = 1.e-2\frac{m}{s}$. The applied magnetic field is
$B_0 = \sqrt{S/40} \in [0, 0.8]$, $T, S \in [0, 25]$, $Re = 40$. To obtain dimensional values, we need to multiply the dimensionless values by the following scalar factors:
1) the values of velocity with $U_0 = 1.e-2$,
2) the vorticity with $U_0/L_0 = 2.5$,
3) the stream function with $U_0 L_0 = 4.e-5$,
4) the difference of the temperature $T - T_0$ with $T_w - T_0$.
For modelling, we chose following parameters: $\beta_t = 2.e-4, Re = 40, S = 0; 2.5; 25$, and $Gr = 25000$,
$Pr = 0.02, S = 0; 2.5; 25$ for liquid metals, $Gr = 11000, Pr = 4$ for electrolyte. The heat source parameter $K_T = 1.e-13S\frac{\sigma}{k} \in [1.e-12, 1.e-6]$ is negligible.

11.3.3 Numerical algorithm for solution of the problem

We consider an uniform Cartesian grid $((N+1)\times M)$.
For cylinders, arranged **in-line**
1) $\Omega_1^h = \{(x_i, y_j), x_i = (i-1)h, y_j = (j-1)h\}, i = \overline{N_1, N_2}, j = \overline{1, M_1}$, $(N_1-1)h = l_1, (M_1-1)h = L_1$,
2) $\Omega_2^h = \{(x_i, y_j), x_i = (i-1)h, y_j = (j-1)h\}, i = \overline{1, N+1}, j = \overline{M_1, M}$, $(N-1)h = l_1, (M-1)h = L$,
where $h = \frac{l_1}{N_1-1} = \frac{l_2}{N_2-1} = \frac{l}{N-1} = \frac{L}{M-1} = \frac{L_1}{M_1-1}$. Subscripts (i, j) refer to x, y indices with the mesh spacing h.
For the channel flow without symmetry, we use twice as much grid points in the y-direction.

Equations (11.17) in the uniform grid (x_i, y_j) are replaced by difference equations of second order approximation in a 5-point stencil and the numerical calculations are made using Seidel iterations with under-relaxation for vorticity and temperature (see Appendix, where $\Psi_{i,j} \approx \psi(x_i, y_j), \zeta_{i,j} \approx \zeta(x_i, y_j), T_{i,j} \approx T(x_i, y_j)$).

11.3.4 Some numerical results

Numerical results are obtained for
$l_1 = 0.5, l_2 = 1.5, l = 2, L = 2, L_2 = 1.5, L_1 = 0.5$ (channel flow without symmetry (CF)), $L = 1, L_1 = 0.5$ (channel flow with symmetry (CFS)), $Re = 40$,
$S = 0; 2.5; 25, Ha^2 = 0; 100; 1000, Pr = 0.02; 0.7; 7.0, Gr = 25000; 11000$,
$\beta = 0; \pm\frac{\pi}{2}, \alpha = 0; \frac{\pi}{2}; \pm\frac{\pi}{4}$, $\omega, \omega_1, \omega_2 \in [0.1, 0.5]$.
The calculations and their graphical visualization were made by means of the MATLAB software for 2 different grids:
1)$h = 0.0125, N_1 = 41, N_2 = 121, N = 161, M_1 = 41, M = 81$,
2)$h = 0.00625, N_1 = 81, N_2 = 241, N = 321, M_1 = 81, M = 161$.
For default calculation we use the grid no. 1 with $N = 161, M = 81$.

The iterative process with maximal errors $\leq 10^{-7}$ for Ψ and $\leq 10^{-4}$ for ζ and temperature (the number of iterations $\in [10000, 100000]$) depends on the parameters.

11.3.5 The flow with symmetry around infinite cylinders - PC.

In this case we consider $\beta = \pm\frac{\pi}{2}, \alpha = 0; \frac{\pi}{2}$.
If $Re = 40, S = Gr = 0, Pr = 0.02, \omega_1 = 0.4$ then depending on the number of iterations K we have corresponding minimal value of $\Psi = -0.048$
$(K = 10000), -0.051\ (K = 20000), -0.05237\ (K = 50000), -0.052384$
$(K = 100000)$.

In the corresponding figures for $Pr = 0.02$ we can see the levels of stream function for liquid metals $\Psi = const$, temperature $T = const$, velocity $V = const$ and vorticity $\zeta = const$. In Figs. 11.7-11.9 the distributions of temperature and the levels of the stream function are represented for $S = Gr = 0$, $S = 2.5$,
$Gr = 0, \alpha = \frac{\pi}{2}$. We can see that the vorticity between cylinders decreases.
In Fig. 11.10 we represent the levels of stream function by $S = 25, Gr = 0, \alpha = \frac{\pi}{2}$. We can see that the flow has no vortexes. Fig. 11.11 shows the levels of stream functions by $S = 25, Gr = 0, \alpha = 0$. The vortex between the cylinders has decreased. Figs. 11.12, 11.13 illustrate the levels of vorticity and temperature for $\alpha = \frac{\pi}{2}$,
$Gr = 25000, S = 2.5, \beta = \frac{\pi}{2}$.

For $Gr \in [0, 100000], \beta = \pm\pi/2$ one observes small change of flow and temperature.

For large value of Stewart (S) number on the walls the Hartmann boundary layers developed and the flow is vortex free.

In the Table 11.2, the maximal and minimal values of dimensionless V_x, V_y, ζ are presented, namely
$Mvx, mvx, Mvy, mvy, M\zeta, m\zeta$; minimal value of $\Psi : m\psi$ and maximal value of V for $\beta = \pi/2$.

The following Figs. 11.14, 11.15 represent the results of calculation in PC for electrolyte by $Pr = 4, \alpha = 0; \frac{\pi}{2}, S = 0; 2.5, Gr = 11000$.
If $\alpha = 0, S = 2.5$, then we have small velocity between the cylinders.

Table 11.3 contains the correspondent values for $Pr = 4, Gr = 11000, \beta = \pi/2$.

Table 11.2 The values of $Gr, S, \alpha, Mvx, mvx, Mvy, mvy, m\Psi, m\zeta, M\zeta, V$

Gr	S	α	Mvx	Mvy	mvx	mvy	$m\Psi$	$m\zeta$	$M\zeta$	V
0	0	$-$	2.92	0.27	$-.25$	$-.20$	$-.0524$	-55.7	6.5	2.92
0	2.5	$\frac{\pi}{2}$	2.45	1.43	$-.006$	$-.505$	$-.0014$	-166	8.3	2.45
0	25	$\frac{\pi}{2}$	2.61	2.93	0	-1.66	0	-411	9.5	3.44
$2.5e+4$	0	$-$	2.91	.27	$-.24$	$-.20$	$-.0520$	-55.8	6.5	2.92
$2.5e+4$	25	0	2.91	.16	$-.07$	$-.10$	$-.0148$	-46.7	6.75	2.91
$2.5e+4$	2.5	$\frac{\pi}{2}$	2.45	1.43	$-.006$	$-.506$	$-.0014$	-166	8.4	2.45

Table 11.3 The values of $S, \alpha, Mvx, mvx, Mvy, mvy, m\Psi, m\zeta, M\zeta, V$

S	α	Mvx	Mvy	mvx	mvy	$m\Psi$	$m\zeta$	$M\zeta$	V
0	$-$	2.84	0.35	$-.23$	$-.17$	$-.04778$	-62.6	6.8	2.84
2.5	$\frac{\pi}{2}$	2.42	1.51	$-.005$	$-.54$	$-.0012$	-176	8.7	2.42
.25	$\frac{\pi}{2}$	2.74	.51	$-.18$	$-.15$	$-.0329$	-73	7.25	2.74
2.5	0	2.84	.34	$-.15$	$-.12$	$-.0335$	-61.3	6.9	2.84

11.3.6 The channel flow with symmetry - CFS

For liquid metals and
$Pr = 0.02, Gr \in [0, 100000], \beta = \pi/2$, we have small change of flow and temperature.
If $S = 0, Gr = 0; 25000; 100000$, then we have following the minimal values of $\Psi = -0.03706; -0.03707; -0.03714$. In the Figs. 11.16-11.19 we can see the results by $Gr = 25000$ and $S = 0; 25, \alpha = \pi/2; 0, Pr = 0.02$.

11.3.7 The flow without symmetry in the channel - CF

In the Figs. 11.20-11.25 we can see the results of calculations in the channel flow without symmetry for $Pr = 0.02, Gr = 25000$ and $S = 0; 25, \alpha = \pi/2; \pi/4; 3\pi/4; 0. \beta = 0; \pm\pi/2$.

Table 11.4 contains the values for $Pr = 0.02, Gr = 25000$.

Table 11.4 The values of $\beta, S, \alpha, Mvx, mvx, Mvy, mvy, m\Psi, m\zeta, M\zeta, V$ in CF for $Pr = 0.02, Gr = 25000$

β	S	α	Mvx	Mvy	mvx	mvy	$m\Psi$	$m\zeta$	$M\zeta$	V
0	0	–	2.93	0.338	−.21	−.23	−.0278	−58.8	51.0	2.92
0	25	$\frac{\pi}{2}$	2.63	2.84	0	−2.89	0	−383	420	3.54
0	25	$\frac{\pi}{4}$	3.54	2.42	−.042	−2.02	$-7.6e-4$	−400	245	3.83
0	25	$\frac{3\pi}{4}$	3.54	2.02	−.041	−2.42	$-5.5e-3$	−244	405	3.89
$-\frac{\pi}{2}$	25	$\frac{\pi}{4}$	3.55	2.43	−.041	−2.03	$-5.5e-3$	−427	259	3.84
$\frac{\pi}{2}$	25	$\frac{3\pi}{4}$	3.55	2.03	−.041	−2.43	$-5.5e-3$	−245	407	3.89
$\frac{\pi}{2}$	25	$\frac{\pi}{2}$	2.63	2.93	0	−2.93	0	−406	423	3.57
$-\frac{\pi}{2}$	25	$\frac{\pi}{2}$	2.57	2.81	0	−2.81	0	−381	392	3.44

11.3.8 The free convection flow in CF, CFS and PC

In the Figs. 11.26-11.43 we can see the results of calculations in the free convection channel flow without symmetry (CF) for $Pr = 0.02$, $Gr = 25000$, $S = 0; 25$, $\alpha = \pi/2; 0$, $\beta = 0; \pi/2$ and for free convection in PC, CFS by $S = 0$.

In the Table 11.5 the values of free convection are represented for $Pr = 0.02$,
$Gr = 25000$ for CF, PC and CFS.

Table 11.5 The values of $\beta, S, \alpha, Mvx, mvx, Mvy, mvy, m\Psi, m\zeta, M\zeta, V$ for free convection and $Pr = 0.02, Gr = 25000$

β	S	α	Mvx	Mvy	mvx	mvy	$m\Psi$	$m\zeta$	$M\zeta$	V
$CF, \frac{\pi}{2}$	0	–	.0031	.0034	−.0047	−.0034	±.0012	−.218	.218	.0047
$CF, -\frac{\pi}{2}$	0	–	.0047	.0034	−.0031	−.0034	±.0012	−.218	.218	.0047
$CF, 0$	25	$\frac{\pi}{2}$	$7.7e-4$	.0027	$-7.7e-4$	−.0017	$\pm 4.3e-4$	−.161	.161	.0027
$CF, \frac{\pi}{2}$	25	$\frac{\pi}{2}$	$3.2e-4$	$3.8e-4$	$-3.1e-4$	$-3.8e-4$	$\pm 9.8e-5$	−.037	.037	$3.9e-4$
$CF, 0$	25	0	$2.5e-4$	$1.9e-4$	$-2.5e-4$	$-1.9e-4$	$\pm 5.8e-5$	−.0264	.0264	$2.6e-4$
$CF, \frac{\pi}{2}$	25	0	.022	.006	−.029	−.006	±.0044	−1.44	1.44	.029
$CF, 0$	25	$\frac{\pi}{4}$	.0019	.0021	.0022	−.002	$\pm 6.8e-4$	−.178	.182	.0029
$CF, 0$	25	$\frac{3\pi}{4}$	.0022	.0021	−.0019	−.002	$\pm 6.8e-4$	−.182	.178	.0029
$PC, \frac{\pi}{2}$	0	–	.0046	.0055	−.0073	−.0055	.0019	−.245	.039	.0073
$PC, \frac{\pi}{2}$	2.5	$\frac{\pi}{2}$	.0021	.0023	−.0025	−.0023	$2.e-5$	−.119	.014	.0025
$PC, \frac{\pi}{2}$	2.5	0	.0031	.0028	−.0046	−.0028	.001	−.198	.022	.0046
$CFS, \frac{\pi}{2}$	0	–	.0029	.0033	−.0045	−.0033	.0011	−.144	.0251	.0045

The free convection velocity for electrolyte for $Pr = 4, S = \beta = 0, Gr = 11000$ is small ($V = 4.e-4, m\Psi = \pm 9.4e-5$).

11.3.9 Appendix

The difference equations of second order approximation in 5-point stencil are in following form:

$$\begin{cases} 4\Psi_{i,j} = \bar{S}\Psi_{i,j} + h^2\zeta_{i,j}, \\ \frac{Re}{4}J_{i,j}(\Psi,\zeta) - 4\zeta_{i,j} + \bar{S}\zeta_{i,j} = \\ Ha^2((\sin(\alpha))^2 d_y^2\Psi_{i,j} + (\cos(\alpha))^2 d_x^2\Psi_{i,j} - 0.25\sin(2\alpha)d_{x,y}^2\Psi_{i,j}) + \\ \frac{Grh}{Re}(\cos(\beta)d_xT_{i,j} + \sin(\beta)d_yT_{i,j}), \\ \frac{Pe}{4}J_{i,j}(\Psi,T) - 4T_{i,j} + \bar{S}T_{i,j} = -K_Th^2(e_z + \\ h^{-1}(\cos(\alpha)d_x\Psi_{i,j} + \sin(\alpha)d_y\Psi_{i,j})^2, \end{cases} \tag{11.18}$$

where $\bar{S}q_{i,j} = q_{i,j-1} + q_{i,j+1} + q_{i-1,j} + q_{i+1,j}, q = \Psi; T; \zeta,$
$d_x^2\Psi_{i,j} = 2\Psi_{i,j} - \Psi_{i+1,j} - \Psi_{i-1,j}, d_y^2\Psi_{i,j} = 2\Psi_{i,j} - \Psi_{i,j+1} - \Psi_{i,j-1},$
$d_{xy}^2\Psi_{i,j} = \Psi_{i+1,j+1} + \Psi_{i-1,j-1} - \Psi_{i-1,j+1} - \Psi_{i+1,j-1},$
$d_xq_{i,j} = 0.5(q_{i+1,j} - q_{i-1,j}),\ d_yq_{i,j} = 0.5(q_{i,j+1} - q_{i,j-1}),\ q = \Psi; T,$
$J_{i,j}(\Psi,q) = (\Psi_{i+1,j} - \Psi_{i-1,j})(q_{i,j+1} - q_{i,j-1}) - (q_{i+1,j} - q_{i-1,j})(\Psi_{i,j+1} - \Psi_{i,j-1}),$
$q = \zeta; T.$

The numerical calculation of (11.18) is evaluated using Seidel iterations with under-relaxation for ζ, T functions :
$\zeta_{i,j}^m = \omega_1\zeta_{i,j}^z + (1-\omega_1)\zeta_{i,j}^{m-1}, T_{i,j}^m = \omega_2T_{i,j}^z + (1-\omega_2)T_{i,j}^{m-1}, m = 1,2,\cdots,$
where $\zeta_{i,j}^z, T_{i,j}^z$ are the grid function value in central mesh points, obtained for m-th iteration, $\omega_1, \omega_2 \in (0,1)$ are the relaxation coefficients. The discrete BCs [55] with $O(h^2)$ on the walls w are computed in following form: $\zeta_w^m = \frac{\gamma}{2h}(-4\Psi_{w-1}^m + \Psi_{w-2}^m + 3\Psi_w) + \zeta_w^{m-1}$,
where $\Psi_{w-1}^m, \Psi_{w-2}^m$ are the value of $\Psi_{i,j}$ for one and two steps h distant from the wall in the inner normal direction. On the corner of wall the value of ζ is equal to average value of the two nearest ζ values of the wall.

The velocity components are obtained in following way:
$Vx_{i,j} = \frac{1}{h}(\Psi_{i,j+1} - \Psi_{i,j}), Vy_{i,j} = -\frac{1}{h}(\Psi_{i+1,j} - \Psi_{i,j}), V = \sqrt{(V_x)^2 + (V_y)^2}.$
The dimensionless fluid volumes between two section $x = 0, x = l_* = 0.5(l_1 + l_2)$ are $Q_1 = \int_L^{L_1} V_x(0,y)dy = q,$

$Q_2 = \int_0^L V_x(l_*,y)dy = q,\ q = 1$ (for channel flow without symmetry q=2, for opposite flow the fluid volume is equal -q).

11.4 2-D MHD between cylinders: A. Buikis, H. Kalis, 2014 [50]

The viscous electrically conducting incompressible liquid is located between two infinite coaxial cylinders (rings). The vortical electromagnetic force drive magnetohydrodynamic flow between the cylinders. The distribution of electromagnetic forces are induced by two different way.

1. In internal cylinder parallel to the axis are placed metal conductors-electrods of the forms of bars. Those conductors the alternating current is connected. The water is weakly electrically conducting liquid (electrolyte). This is the mathematical model of one devices for electrical energy produced by alternating current in production of heat energy.

2. The distribution of electromagnetic forces are induced by the external magnetic field (homogeneous, radial, axial, dipolar) imposed in the cross-section of the cylinder. The surfaces of the cylinders with different angular velocity rotate can.

The distribution of electromagnetic fields, forces, 2-D magnetohydrodynamic flow and temperature
induced by the system of the altenating electric current or of external mabnetic field in a conducting cylinder has been calculated using finite difference methods.
An original method was used to calculate the mean values of electromagnetic forces and circulant matrices.

It is important to mix an electrically conducting liquid, using various magnetic fields in many technological applications ([48], [56], [57], [63], [49], [51]).
Let the cylindrical domain between two infinite cylinders $\Omega = \{(r,\phi,z) : r_0 < r < R, 0 \leq \phi \leq 2\pi, -\infty < z < \infty\}$ contain viscous electrically conducting incompressible liquid, where r_0, R are the radiis of the coaxial cylinders.
In this paper we consider two way for obtaining the magnetic fields in this cylindrical domain.

1. The new type of heat generator is ecologically clean, compact and effective (see 2 patents [61], [62]). In papers ([49],[50],[51], [52]) are modelled cylinder form electrical heat generators with six or nine circular conductors-electrodes placed on the surfaces of the cylinder. In this work we analyze other type of conductors. They have forms of

six bars and are placed parallel to the cylinder axis in the central part of the cylinder. In the (Fig.11.44) we can see the real electrical heat generator. In [63] is considered the coresponding 3-D problem, using computer program FLUENT.

The alternating current is fed to $\tilde{N}$ infinite discrete conductors of forms of bars, which are placed parallel to the cylinder axis in the domain $r < r_0 < R$. In the Fig. 11.45 we can see the mathematical model with 6 conductors ($\tilde{N} = 6$.)
The current creates in the weakly conductive liquid-electrolyte the radial $B_r(r,\phi)$ and the azimuthal $B_\phi(r,\phi)$ components of the magnetic field as well the axial component of the induced electric field $E_z(r,\phi)$. For calculating the electromagnetic fields outside the electrodes, the averaging method over the time interval $2\pi/\omega$ is used (ω is the angular frequency of the alternating current).

2. For second way magnetic field we analyze the 2D viscous electrically conducting incompressible flow between two infinite coaxial cylinders by different type of the external magnetics fields and angular velocity which the surfaces of the cylinders rotating can. This process is considered with the so-called inductionless approximation [56].
The external 2D magnetic fields are added in following form:
1) uniform magnetic field with the radial $B_r(r,\phi) = B_0(1 - r^{-2}a_\mu)\sin(\phi)$ and the azimuthal $B_\phi(r,\phi) = B_0(1 + r^{-2}a_\mu)\cos(\phi)$ components of the induction for magnetic field, where $a_\mu = \frac{(\mu-1)r_0^2}{\mu+1}, \mu = \frac{\mu_1}{\mu_0}, \mu_1, \mu_0$ – are the corresponding magnetic permeabilitys in the liquid $(R > r > r_0)$ and in the internal cylinder $(r \le r_0)$ (if $\mu = 1$ then this field is homogeneous and parallel to Oy axis [57], if $\mu = 0, \mu_0 = \infty$ then the internal cylinder is ferromagnetic, if $\mu = \infty, \mu_1 = \infty$ then the liquid is ferromagnetic),
2) radial magnetic field with the radial $B_r(r) = B_0/r$ component ($B_\phi = 0$),
3) axial magnetic field with the azimuthal $B_\phi(r) = B_0/r$ component ($B_r = 0$), this field can be obtained when the direct current is fed to internal cylinder parallel to Oz axis,
4) bipolar magnetic field with the radial $B_r(r,\phi) = B_0 r^{-2}\sin(\phi)$ and the azimuthal $B_\phi(r,\phi) = -B_0 r^{-2}\cos(\phi)$ components,
5) the sum of axial and uniform magnetics field.
Here B_0 is the scale of the induction for magnetic field. These mag-

netic fields with the vector of induction **B** are solutions of the following homogenous Maxwell's equations $div\mathbf{B} = curl\mathbf{B} = 0$.

For the visualization of the magnetic fields the following magnetic stream function A_z is used (the component of the vector potential):
1) $A_z = r(1 + r^{-2}a_\mu)\cos(\phi)$ for the uniform magnetic field, see Fig. 11.46 ($\mu = 0$); Fig. 11.47 ($\mu = \infty$),
2) $A_z = -r^{-1}\cos(\phi)$ for the bipolar magnetic field, see Fig. 11.48
3) $A_z = r(1 + r^{-2}a_\mu)\cos(\phi) - \ln(r)$ for the sum of the uniform and axial magnetic fields, see Fig. 11.49 ($\mu = 0$)
4) $A_z = \phi$ for the radial magnetic field,
5)$A_z = -ln(r)$ for the axial magnetic field,
The surfaces of the cylinders can correspondingly with angular velocities Ω_0, Ω_1 rotate.

The both type of magnetic fields creates the radial $F_r(r,\phi)$ and azimuthal $F_\phi(r,\phi)$ components of the Lorentz' force **F**. The axial component of the vector's $curl\mathbf{F}$ give rise to a liquid motion. The stationary 2D flow of incompressible viscous liquid between cylinders is described by the system of the Navier-Stokes equations in the polar coordinates $(r,\phi), r_0 < r < R$.

11.4.1 The mathematical model

The stationary 2-D flow of incompressible viscous liquid in a cross-section of the cylinders is described by the system of the Navier-Sokes equations in following form [48]:

$$\begin{cases} M(V_r) - r^{-1}V_\phi^2 = -\tilde{\rho}^{-1}\frac{\partial p}{\partial r} + \nu(\Delta V_r - r^{-2}V_r - 2r^{-2}\frac{\partial V_\phi}{\partial \phi}) + \tilde{\rho}^{-1}F_r \\ M(V_\phi) + r^{-1}V_r V\phi = -(\tilde{\rho} r)^{-1}\frac{\partial p}{\partial \phi} + \\ \nu(\Delta V_\phi - r^{-2}V_\phi + 2r^{-2}\frac{\partial V_r}{\partial \phi}) + \tilde{\rho}^{-1}F_\phi \\ \frac{\partial (rV_r)}{\partial r} + \frac{\partial (V_\phi)}{\partial \phi} = 0. \end{cases} \tag{11.19}$$

Here V_r, V_ϕ are the radial and azimuthal components of velocity vector **V**, depending on the coordinates r,ϕ; Δ is Laplace operator, $\Delta g = r^{-1}\frac{\partial}{\partial r}(r\frac{\partial g}{\partial r}) + r^{-2}\frac{\partial^2 g}{\partial \phi^2}$, $M(g) = V_r\frac{\partial g}{\partial r} + r^{-1}V_\phi\frac{\partial g}{\partial \phi}$ is the convective parts of the equations,

$\tilde{\rho}$, ν are the density and kinematic viscosity, p is the pressure, $g = V_r; V_\phi$.

Determined the vorticity functions or the axial component of vector's $curl\mathbf{V}$ with formulas $\tilde{\omega} = r^{-1}(\partial(rV_\phi)/\partial r - \partial V_r/\partial\phi)$ we obtain

$$\begin{cases} -V_\phi\tilde{\omega} = -\tilde{\rho}^{-1}\frac{\partial\tilde{p}}{\partial r} - \nu r^{-1}\frac{\partial\tilde{\omega}}{\partial\phi} + \tilde{\rho}^{-1}F_r, \\ V_r\tilde{\omega} = -\tilde{\rho}^{-1}r^{-1}\frac{\partial\tilde{p}}{\partial\phi} + \nu\frac{\partial\tilde{\omega}}{\partial r} + \tilde{\rho}^{-1}F_\phi, \\ \frac{\partial(rV_r)}{\partial r} + \frac{\partial(V_\phi)}{\partial\phi} = 0, \end{cases} \tag{11.20}$$

where $\tilde{p} = p + 0.5\mathbf{V^2}$.

1. **For alternating current** the averaged values of Lorenz force F_r, F_ϕ are obtained, applying the Biot-Savar and Ohm's laws [63] in following form:

$$\begin{cases} F_r(r,\phi) = 0.5K_0 S^\alpha_{\tilde{N}}, \\ F_\phi(r,\phi) = 0.5K_0 S^\beta_{\tilde{N}}, \end{cases} \tag{11.21}$$

where

$$S^\alpha_{\tilde{N}} = \sum_{i,j=1}^{\tilde{N}} \sin((j-i)\theta)\alpha_{i,j}, S^\beta_{\tilde{N}} = \sum_{i,j=1}^{\tilde{N}} \sin((j-i)\theta)\beta_{i,j},$$

$$\alpha_{i,j} = -\frac{\ln(\rho_i)(r_j\cos(\phi-\phi_j)-r)}{\rho_j^2}, \beta_{i,j} = \frac{\ln(\rho_i)r_j\sin(\phi-\phi_j)}{\rho_j^2},$$

$K_0 = (\frac{a^2\mu j_0}{2})^2\sigma\omega$, $\mu = 4\pi 10^{-7}\frac{mkg}{s^2A^2}$ is the magnetic permeability in vacuum, σ is the electric conductivity,
j_0 is the amplitude of alterating current density

$$j_i = j_0\cos(\omega t) + (i-1)\theta), i = \overline{1,\tilde{N}}, \tag{11.22}$$

$\theta = const$ is the phase (usually $\theta = 120^0$ and the frequency of the alternating current is $50Hz$)
(r_i, ϕ_i) is the polar coordinate of the center for wires
$L_i = \{r - r_i \le a < r_0, \phi_i - \alpha_i \le \phi \le \phi_i + \alpha_i, -\infty \le z \le +\infty\}$ with radius a,
$\rho_i = \sqrt{(r_i^2 + r^2 - 2rr_i\cos(\phi-\phi_i))}, \alpha_i = \arcsin(a/r_i)$.

Similarly for averaged values of source term in heat transport equation [63]

$$j_z^2(r,\phi) = 0.5K_0\sigma\omega S_{\tilde{N}}^{\gamma}, \tag{11.23}$$

where $S_{\tilde{N}}^{\gamma} = \sum_{i,j=1}^{\tilde{N}} \cos((j-i)\theta)\gamma_{i,j}$, $\gamma_{i,j} = \ln(\rho_i).\ln(\rho_j)$.

Having calculated the axial component of the curl for force vector $f = rot_z\mathbf{F}$,[63] the average value is

$$f(r,\phi) = 0.5K_0 S_{\tilde{N}}^{\delta}, \tag{11.24}$$

where $S_{\tilde{N}}^{\delta} = \sum_{i,j}^{\tilde{N}} \sin((j-i)\theta)\delta_{i,j}$,
$\delta_{i,j} = \frac{1}{r}[\frac{\partial}{\partial r}(r\beta_{i,j}) - \frac{\partial}{\partial\phi}(\alpha_{i,j})] = g_{i,j} - g_{j,i}$,
$g_{i,j} = \frac{r_i\sin(\phi-\phi_i)}{\rho_i^2}\frac{(r_j\cos(\phi-\phi_j)-r)}{\rho_j^2}$.
On the walls (the surfaces of the cylinders $r = R$ and $r = r_0$) we have the non-slipping conditions $\mathbf{V} = 0$.

2. In the case of 2-D **external magnetic field** with components $B_r(r,\phi), B_\phi(r,\phi)$ from the vector of Lorenz force $\mathbf{F} = \sigma(\mathbf{E} + \mathbf{VxB})\mathbf{xB})$ [56] follows

$$F_r = -\sigma B_\phi(V_r B_\phi - V_\phi B_r + E_z), \quad F_\phi = \sigma B_r(V_r B_\phi - V_\phi B_r + E_z),$$

where $E_z = const$ is the azimuthal component of the electric field $\mathbf{E}$. The walls (the surfaces) of the cylinders $r = R$ and $r = r_0$ can be rotated with the velocitys $V_\phi = r_0\Omega_0$ and $V_\phi = R\Omega_1$ corresponding $(V_r = 0)$.

For the 2-D problems the eliminating pressure $\tilde{p}$ from the first two equations of the system of PDEs (11.22) one obtains

$$M(\tilde{\omega}) = \nu\Delta\tilde{\omega} + \tilde{\rho}^{-1}f, \tag{11.25}$$

where f is the axial component of the vector's $curl\mathbf{F}$.
The hydrodynamic stream function ψ can be determined with formulas
$V_r = r^{-1}\frac{\partial\psi}{\partial\phi}, V_\phi = -\frac{\partial\psi}{\partial r}$.
Then from the equation of continuity and from vorticity function it follows, that $\tilde{\omega} = -\Delta\psi$.
From (11.25) follows the system of two PDEs for solving the vorticity function $\tilde{\omega}$ and stream function ψ :

$$\begin{cases} \Delta\psi = -\tilde{\omega} \\ r^{-1}J(\tilde{\omega},\psi) = \nu\Delta\tilde{\omega} + \tilde{\rho}^{-1}f, \end{cases} \tag{11.26}$$

where $J(\tilde{\omega},\psi) = (\partial\tilde{\omega}/\partial r)(\partial\psi/\partial\phi) - (\partial\tilde{\omega}/\partial\phi)(\partial\psi/\partial r)$ is the Jacobian of the functions ψ and $\tilde{\omega}$. Eliminated the functions $\tilde{\omega}$ from (11.26), we obtain the PDEs of the fourth order

$$r^{-1}J(\Delta\psi,\psi) = \nu\Delta^2\psi - \tilde{\rho}^{-1}f, \tag{11.27}$$

where $J(\Delta\psi,\psi)$ is the Jacobian of the functions ψ and $\Delta\psi$.
In the azimuthal direction we have the conditions of periodically $\psi(r,0) = \psi(r,2\pi), \frac{\partial\psi(r,0)}{\partial\phi} = \frac{\partial\psi(r,2\pi)}{\partial\phi}$.
On the walls $r = r_0, r = R$ we have the non-slipping conditions:
$\psi = \frac{\partial\psi}{\partial r} = 0$ (for alternating current),
and the non-slipping and rotation conditions:
$\psi = c_0, \frac{\partial\psi}{\partial r} = -r_0\Omega_0$, by $r = r_0$; $\psi = 0, \frac{\partial\psi}{\partial r} = -R\Omega_1$, by $r = R$ (for external fields). The constant c_0 can be determined from the second equation (11.26) using the conditions for uniqueness of pressure p by $\phi = 0$ and $\phi = 2\pi$.

The steady energy equation reduces to the heat transport equation for incompressible flow with source terms and with constant properties. The stationary distribution of temperature field $T(r,\phi)$ in a conducting ring is described by the following boundary-value problem for the heat transport equation:

$$\begin{cases} \tilde{\rho}cr^{-1}J(T,\psi) = k\Delta T + \sigma^{-1}j_z^2, \\ \frac{\partial T(R,\phi)}{\partial r} = 0, T(r_0,\phi) = T_a, \end{cases} \tag{11.28}$$

where c,k are a corresponding constants of specific thermal capacity $(c = 4000\frac{J}{kgK})$ and coefficient of heat conductivity $(k = 0.6\frac{W}{mK})$, j_z^2 is the source term, T_a is the given constant fixed temperature.

The equations (11.27, 11.28) were put in the dimensionless form scaling all the lengths to $L = R$ (the radius of the tube), the velocitys V_r, V_ϕ to U_0, stream function ψ to $\psi_0 = U_0R$, the induction B_r, B_ϕ of magnetics field to B_0, the pressure p to $P_0 = U_0^2\tilde{\rho}$ and temperature$T$ to T_a, where $U_0 = \nu/R$ (for alternating current), $U_0 = R\max(\Omega_0,\Omega_1)$ (for external fields). Further the denotes of all variables are unchangeable. Then we have non-dimensional equations for the following iterations procedure:

$$\begin{cases} \Delta^2\psi^{(m+1)} = a_0 r^{-1} J(\Delta\psi^{(m)}, \psi^{(m)}) + b_0 f, \\ \Delta T^{(m+1)} = Pr\, r^{-1} J(T^{(m)}, \psi^{(m+1)}) - K_T j_z^2, \end{cases} \tag{11.29}$$

where $a_0 = 1, b_0 = K_H = K_0 \frac{L^2}{\tilde{\rho}\nu^2}$ (K_H– the electrodinamics force number) for alternating fields,
$a_0 = Re, b_0 = Ha^2 = ReS$ ($Re = U_0R/\nu, S = \sigma B_0^2 R/(\tilde{\rho}U_0), Ha = \sqrt{ReS}$– the Reinolds, Stjuart and Hartman numbers) for external fields,
$K_T = K_0 \frac{\omega L^2}{kT_a}$ – the heat sources parameter,
$Pr = \frac{c\tilde{\rho}\nu}{k}$ – the Prandtl number,
$m = 0, 1, 2, ...M_{it}$ – the iterations numbers.
The constant c_0 for external field can be determined from following integral condition:

$$\int_0^{2\pi} \frac{\partial\tilde{\omega}}{\partial r} d\phi = -Ha^2 \int_0^{2\pi} F_\phi d\phi = 2\pi Ha^2 \eta\omega_0 C, \tag{11.30}$$

where $F_\phi = -B_r^2\eta\omega_0$, $\eta = r_0/R, \omega_k = \Omega_k R/U_0$. The constant C is equal to $\frac{1}{\mu+1}$(for uniform magnetic field), η^{-2} (for radial field), 0 (for axial field), $0.5\eta^{-4}$ (for bipolar field).

The source function f for the external magnetic field is in following form:
$f(r,\phi) = B_r^2 \frac{\partial^2\psi}{\partial r^2} + B_\phi^2 \frac{\partial^2\psi}{r^2\partial\phi^2} + 2B_rB_\phi \frac{\partial^2\psi}{r\partial r\partial\phi} + (B_\phi \frac{\partial B_r}{r\partial\phi} + B_r \frac{\partial B_r}{\partial r})\frac{\partial\psi}{\partial r} + (B_r \frac{\partial B_\phi}{\partial r} + \frac{B_\phi}{r}(\frac{\partial B_\phi}{\partial\phi} - B_r))\frac{\partial\psi}{r\partial\phi}$.
We have following formulas for function f in every case of the magnetic fields:
1) $f = s_3 \sin(2\phi)\frac{\partial^2\psi}{r\partial r\partial\phi} + s_2\cos^2(\phi)\frac{\partial^2\psi}{r^2\partial\phi^2} + s_1\sin^2(\phi)\frac{\partial^2\psi}{\partial r^2} + (s_4\cos^2(\phi) + s_5\sin^2(\phi))\frac{\partial\psi}{\partial r} - s_6\sin(2\phi)\frac{\partial\psi}{r\partial\phi}$ for the uniform magnetic field, where $s_1 = p_2^2, s_2 = p_1^2, s_3 = p_1p_2, s_4 = s_3/r, s_5 = 2p_2r^{-3}a_\mu, s_6 = 0.5s_5 + p_1/r$,
$p_1 = 1 + r^{-2}a_\mu, p_2 = 1 - r^{-2}a_\mu$,
2) $f = \frac{\partial}{r\partial r}(\frac{\partial\psi}{r\partial r})$ for the radial magnetic field,
3) $f = \frac{\partial^2\psi}{r^4\partial\phi^2}$ for the axial magnetic field,
4) $f = r^{-4}(-\sin(2\phi)\frac{\partial^2\psi}{r\partial r\partial\phi} + \cos^2(\phi)\frac{\partial^2\psi}{r^2\partial\phi^2} + \sin^2(\phi)\frac{\partial^2\psi}{\partial r^2}$

$-r^{-1}(\cos^2(\phi)+2\sin^2(\phi))\frac{\partial\psi}{\partial r}+r^{-1}\sin(2\phi)\frac{\partial\psi}{r\partial\phi})$ for the dipolar magnetic field.

The procedure of iterations (11.29) together with the boundary conditions is realized using finite difference approximation with central differences.
In the linear case ($J=0$) we have only one iteration. If the nonlinear terms (convective terms J) is dominant $a_0 >> 1$, then the method of underrelaxation is used with the parameter $\omega_r < 1$.

11.4.2 The finite-difference approximations and numerical method

We consider an uniform grid $(N+1\times M)$:
$\omega_h=\{(r_i,\phi_j),\ r_i=\eta+(i-1)h_1,\ \phi_j=jh_2, i=\overline{1,N+1},\ j=\overline{1,M}$, $\eta+Nh_1=1, Mh_2=2\pi$. Subscripts (i,j) refer to r,ϕ indices, the mesh spacing in the i,j directions are h_1 and h_2.
The stream function equation (11.27) in the uniform grid (r_i,ϕ_j) is replaced by vector difference equations of second order approximation on 5- point stencil:

$$A_i\Psi_{i-2}+B_i\Psi_{i-1}+C_i\Psi_i+D_i\Psi_{i+1}+E_i\Psi_{i+2}+F_i^H=0, \qquad (11.31)$$

where Ψ_i are vectors -column of M-order with components $\psi_{i,j}\approx\psi^{(m+1)}(r_i,\phi_j), j=\overline{1,M}, i=\overline{3,N-1}$,
A_i,B_i,C_i,D_i,E_i are the cirular symmetric matrix of M-order.
This matrix can to give with the first rows in following form:
$A_i=[a_{i,1},0,0,...,0], B_i=[b_{i,1},b_{i,2},0,0,...,b_{i,2}]$,
$C_i=[c_{i,1},c_{i,2},c_{i,3},0,0,....,c_{i,3},c_{i,2}], D_i=[d_{i,1},d_{i,2},0,0,...,d_{i,2}]$,
$E_i=[e_{i,1},0,0,...,0]$,
where $a_{i,1}=\frac{r_{i-1.5}r_{i-0.5}}{r_ir_{i-1}h_1^4}$, $e_{i,1}=\frac{r_{i+1.5}r_{i+0.5}}{r_ir_{i+1}h_1^4}$,
$b_{i,1}=-\frac{r_{i-0.5}}{r_ih_1^2}(4/h_1^2+2d_1/h_2^2)$, $b_{i,2}=d_1\frac{r_{i-0.5}}{r_ih_1^2h_2^2}$,
$d_{i,1}=-\frac{r_{i+0.5}}{r_ih_1^2}(4/h_1^2+2d2/h_2^2)$, $d_{i,2}=d_2\frac{r_{i+0.5}}{r_ih_1^2h_2^2}$,
$c_{i,3}=r_i^{-4}h_2^{-4}, c_{i,2}=-2d_3/(r_i^2h_2^2)$,
$c_{i,1}=d_3^2+2c_{i,3}+1/(r_ih_1^4)(r_{i+0.5}^2/r_{i+1}+r_{i-0.5}^2/r_{i-1})$,
$d_1=1/r_i^2+1/r_{i-1}^2, d_2=1/r_i^2+1/r_{i+1}^2, d_3=2/h_1^2+2/(r_i^2h_2^2)$.
F_i^H is the vector-column with components $f_{i,j}^H=-(a_0r_i^{-1}J_{i,j}^{(m)}+b_0f_{i,j})$,

$j=\overline{1,M}$, where
$J_{i,j}=\frac{1}{4h_1h_2}((\Delta\psi_{i+1,j}-\Delta\psi_{i-1,j})(\psi_{i,j+1}-\psi_{i,j-1})-(\Delta\psi_{i,j+1}-\Delta\psi_{i,j-1})(\psi_{i+1,j}-\psi_{i-1,j}))$, is the approximation for J with central differences,
$\Delta\psi_{i,j}=\frac{1}{r_ih_1^2}(r_{i+0.5}(\psi_{i+1,j}-\psi_{i,j})-r_{i-0.5}(\psi_{i,j}-\psi_{i-1,j}))+\frac{1}{r_i^2h_2^2}(\psi_{i,j+1}-2\psi_{i,j}+\psi_{i,j-1})$, $f_{i,j}=f(r_i,\phi_j)$.

For the external radial field $f_{i,j}=0$ and by the elements of matrix $b_{i,1},c_{i,1},d_{i,1}$ are added respectively
$-Ha^2d_4r_{i-0.5}^{-1}, Ha^2d_4(r_{i+0.5}^{-1}+r_{i-0.5}^{-1}), -Ha^2d_4r_{i+0.5}^{-1}$, where $d_4=1/(r_ih_1^2)$. Similarly for the axial field by the elements of matrix $c_{i,1},c_{i,2}$ are added $Ha^2\frac{2}{h_2^2r_i^4}, -Ha^2\frac{1}{h_2^2r_i^4}$.
For the uniform magnetic field we have following approximation of $f_{i,j}$:
$s_2\frac{\cos^2(\phi_j)}{h_2^2r_i^2}(\psi_{i,j+1}-2\psi_{i,j}+\psi_{i,j-1})+s_1\frac{\sin^2(\phi_j)}{h_1^2}(\psi_{i+1,j}-2\psi_{i,j}+\psi_{i-1,j})+s_3\frac{\sin(2\phi_j)}{4h_1h_2r_i}(\psi_{i+1,j+1}-\psi_{i-1,j+1}-\psi_{i+1,j-1}+\psi_{i-1,j-1})+(s_4\frac{\cos^2(\phi_j)}{2h_1}+s_5\frac{\sin^2(\phi_j)}{2h_1})\,(\psi_{i+1,j}-\psi_{i-1,j})-s_6\frac{\sin(2\phi_j)}{2h_2r_i}(\psi_{i,j+1}-\psi_{i,j-1})$.

The heat transport equation (11.28) is replaced by vector difference equations of second order approximation in 3-point stencil:

$$A1_iT_{i-1}-C1_iT_i+B1_iT_{i+1}+F_i^T=0, \tag{11.32}$$

where T_i are vectors -column with components $T_{i,j}\approx T^{(m+1)}(r_i,\phi_j)$, $j=\overline{1,M}$,
$A1_i,B1_i,C1_i$ are the cirular symmetric matrix of M-order in following form:
$A1_i=[a1_{i,1},0,0,...,0], B1_i=[b1_{i,1},0,0,...,0], C1_i=[c1_{i,1},c1_{i,2},0,0,....,c1_{i,2}]$, where
$a1_{i,1}=\frac{r_{i-0.5,j}}{r_ih_1^2}, b1_{i,1}=\frac{r_{i+0.5,j}}{r_ih_1^2}, c1_{i,2}=\frac{1}{r_i^2h_2^2}, c1_{i,1}=a1_{i,1}+b1_{i,1}+2c1_{i,2}$.
F_i^T is the vector-column with components
$f_{i,j}^T=-(Pr\,r_i^{-1}J1_{i,j}-K_Tf1_{i,j}), j=\overline{1,M}$,
where
$J1_{i,j}=\frac{1}{4h_1h_2}((T_{i+1,j}-T_{i-1,j})(\psi_{i,j+1}-\psi_{i,j-1})-(T_{i,j+1}-T_{i,j-1})(\psi_{i+1,j}-\psi_{i-1,j}))$,
$f1_{i,j}=j_z^2(r_i,\phi_j)$.

The boundary conditions for altenating field are replaced by difference equations of second or third order approximation. For the vector

function Ψ_i using the 3-point (r_1, r_2, r_3) or 4-point (r_1, r_2, r_3, r_4) stencils by $r = \eta, r = 1$
we have following second or third orders approximations:

$$\begin{cases} \Psi_1 = 0, \Psi_2 = 0.25\Psi_3, \\ \Psi_{N+1} = 0, \Psi_N = 0.25\Psi_{N-1}, \\ ************************* \\ \Psi_1 = 0, \Psi_2 = 0.5\Psi_3 - 1/9\Psi_4, \\ \Psi_{N+1} = 0, \Psi_N = 0.5\Psi_{N-1} - 1/9\Psi_{N-2}. \end{cases} \tag{11.33}$$

Similarly for the vector- function T_i using the 3-point (r_1, r_2, r_3) or 4-point (r_1, r_2, r_3, r_4) stencils by $r = 1$ we have following second or third orders approximations:

$$\begin{cases} T_N - 0.25T_{N-1} = 0.75T_{N+1}, \\ T_N - 0.25T_{N-1} + 1/9T_{N-2} = 11/18T_{N+1}. \end{cases} \tag{11.34}$$

The boundary conditions for external magnetic fields are replaced by difference equations from second order of approximation. For the vector function Ψ_i using the 3-point $(r_1, r_2, r_3), (r_{N+1}, r_N, r_{N-1})$ stencils by $r = \eta, r = 1$ we obtain following second orders approximations:

$$\begin{cases} \Psi_1 = c_0^{(m)}, \Psi_2 = 0.25(\Psi_3 + 2c_0 - 2\omega\eta h_1), \\ \Psi_{N+1} = 0, \Psi_N = 0.25(\Psi_{N-1} + 2h_1\omega_1). \end{cases} \tag{11.35}$$

For the approximations of integrals by $r = \eta$ using 4-point (r_1, r_2, r_3, r_4) stencil we obtain:
$-\frac{\partial\tilde{\omega}}{\partial r} = \frac{\partial^3\psi}{\partial r^3} + \frac{\partial^2\psi}{\eta\partial r^2} + \omega_0/\eta$,
$(\frac{\partial^3\psi}{\partial r^3})_{1,j} = (10\Psi_{1,j} - 15\Psi_{2,j} + 6\Psi_{3,j} - \Psi_{4,j} - 6\eta\,\omega_0 h_1)/h_1^3 + O(h_1^2)$,
$(\frac{\partial^2\psi}{\partial r^2})_{1,j} = (-3.5\Psi_{1,j} + 4\Psi_{2,j} - 0.5\Psi_{3,j} + 3\eta\,\omega_0 h_1)/h_1^2 + O(h_1^2)$,
$c_0 = M_h(\Psi_1) = (h_1^3\eta\,\omega_0 Ha^2 C - (15-4hr)M_h(\Psi_2) - (0.5hr-6)M_h(\Psi_3) - M_h(\Psi_4) - h_1\eta\,\omega_0(6-3hr-hr^2))/(3.5hr-10)$, where $hr = h_1/\eta, M_h(\Psi_k) = M^{-1}\sum_{j=1}^{M}\Psi_{k,j}, k = 1;2;3;4$.

The vector difference schemes(11.33,11.34) are solved by the Gauss elimation method using the calculations of circulant matrix.

The calculations of circulant matrix
$A = [a_1, a_2, ..., a_M], B = [b_1, b_2, ..., b_M], C = [c_1, c_2, ..., c_M]$ and vectors-column

$b = (b_1, b_2, ..., b_M)^T, c = (c_1, c_2, ..., c_M)^T$ can be carried out using following formulae: 1) the matrix A invertion

$$B = A^{-1}, b_k = \frac{1}{M} \sum_{j=0}^{M-1} (\alpha_j \cos \frac{2\pi k j}{M} - \beta_j \sin \frac{2\pi k j}{M})/(\alpha_j^2 + \beta_j^2), k = \overline{1, M},$$

where $\alpha_j = \sum_{i=0}^{M-1} a_{i+1} \cos \frac{2\pi i j}{M}, \beta_j = \sum_{i=0}^{M-1} a_{i+1} \sin \frac{2\pi i j}{M}$,
2) the matrix A and B multiplication

$$C = A.B, c_s = \sum_{k=1}^{s} a_k b_{s-k+1} + \sum_{k=s+1}^{M} a_k b_{M+s-k+1}, s = \overline{1, M},$$

3) the matrix A multiplication with vector b

$$c = A.b, c_s = \sum_{k=1}^{s-1} a_{M-s+k+1} b_k + \sum_{k=s}^{M} a_{k-s+1} b_k, s = \overline{1, M}.$$

This algorithmus can be easily realized by MATLAB.

11.4.3 Approbation of numerical algorithmus

We consider for alternating field following linear test for the approbation of the flow calculations by $J = 0, f_{i,j} = \sin(\phi_j)$:

$$\Delta^2 \psi = K_H \sin(\phi),\ \psi(\eta, \phi) = \psi(1, \phi) = \frac{\partial \psi(\eta, \phi)}{\partial r} = \frac{\partial \psi(1, \phi)}{\partial r} = 0, \tag{11.36}$$

where $\Delta \psi = r^{-1} \frac{\partial}{\partial r}(r \frac{\partial \psi}{\partial r}) + r^{-2} \frac{\partial^2 \psi}{\partial \phi^2}$, $K_H = 300$.
The solution of the test is in following form:

$$\psi(r, \phi) = 100 \sin(\phi)(1/8 C_1 r^3 + 1/2 C_2 r \ln r + C_3 r + C_4 r^{-1} + 1/15 r^4),$$

where
$C_1 = -2/15(\eta^4(6\ln(\eta) - 5) + (\eta + 1)(5 + 6\ln(\eta) + 6\eta^2 \ln(\eta)) - 5\eta^3)/d$,
$C_2 = 1/15((\eta^2 + 3\eta + 1)(\eta - 1)^3(\eta + 1))/d$,
$C_3 = -1/60(\eta^5(1+\eta) + \eta^4(5 - 12\ln(\eta)) - 2\eta^3 \ln(\eta) - \eta^2(5 + 2\ln(\eta)) - (\eta + 1)(1 + 2\ln(\eta)))/d$,

$C_4 = 1/60(\eta^2(\eta^4+\eta^3-6\eta^2\ln(\eta)-\eta-1))/d;$
$d = (\eta+1)(\eta^2\ln(\eta)-\eta^2+\ln(\eta)+1).$
The linear test for the approbation of the heat calculations by $Pr = 0, f1_{i,j} = \sin(\phi_j)$ is in following form:

$$\Delta T = -K_T\sin(\phi),\ T(\eta,\phi) = T_a \geq 0, \frac{\partial T(1,\phi)}{\partial r} = 0. \qquad (11.37)$$

The solution of this test for $K_T = 3$ is

$$T(r,\phi) = \sin(\phi)(C_1 r + C_2 r^{-1} - r^2),$$

where
$C_1 = (2+\eta^3+T_a\eta))/(1+\eta^2), C_2 = (-2\eta^2+\eta^3+T_a\eta)/(1+\eta^2).$
We obtain $T(1,\pi/2) = ((1-\eta)^2(2\eta+1)+2\eta T_a)/(1+\eta^2) > 0.$

If $\eta = 0.2, T_a = 0$ then, using finite differece schemes (11.36-11.37), is calculated maximal values $(\max|\psi|, \max|T|)$ by different order K of approximation and different N, M which are compared with the tested values $\max|\psi_*| = 0.4361, \max|T_*| = T(1,\pi/2) = 0.86154.$
We have following results:
1)$\max|\psi| = 0.4156, \max|T| = 0.8645$ for N=M=20 and K=2,
2)$\max|\psi| = 0.4266, \max|T| = 0.8640$ for N=M=20 and K=3,
3)$\max|\psi| = 0.4355, \max|T| = 0.8618$ for N=M=40 and K=2,
4)$\max|\psi| = 0.4360, \max|T| = 0.8616$ for N=M=40 and K=3.
Using first order of approximation for heat calculations by $(N,M) = (40,40)$ we have $max|T| = 0.8242$.

For the external magnetic fields we have two test for approbation of numerical algorithmus. The analytical solutions with radial symmetry by $J = S = 0$ in the following form can be used for initial conditions of iteration $(m = 0)$:
$\psi(r) = 0.25r^2ln(r)C_1 + C_2 + ln(r)C_3 + r^2C_4$, where
$C_1 = -\frac{4}{d_0}(-\eta^2(\omega_0+\omega_1)+2\eta^2ln(\eta)(\omega_1-\omega_0)+\eta^4\omega_0+\omega_1),$
$C_2 = -\frac{\eta^2 ln(\eta)}{d_0}(2ln(\eta)\omega_1+(\eta^2-1)\omega_0),$
$C_3 = -\frac{\eta^2}{d_0}(2\eta^2ln(\eta)\omega_0+(\eta^2-1)(\omega_1-\omega_0)-2ln(\eta)\omega_1),$
$C_4 = -C_2, d_0 = 1+\eta^4-2\eta^2-4\eta_2 ln^2(\eta).$

For the radial magnetic field we have the exact solutions of the differential equations $\Delta^2\psi = Ha^2\frac{\partial}{r\partial r}(\frac{\partial\psi}{r\partial r})$ in the following form
$\psi(r) = -0.5r^2/Ha^2C_1 + C_2 + r^{g_2}C_3 + r^{g_3}C_4,$

where $g_2 = 1+\sqrt{1+Ha^2}, g_3 = 1-\sqrt{1+Ha^2}$. The constants C_1, C_2, C_3, C_4 can be determined from boundary conditions.

11.4.4 Some numerical experiments

Calculation and their graphic visualization were made by means of the computer programs MATLAB with $\eta = 0.2, N = M = 80$.
The stability and convergence analysis of considered algorithm in this paper are not presented. The exactness of numerical results are testing with different numbers of grid points N, M. For numerical experiment follows that by N=M=40 and N=M=80 the numerical results are coincident with 4 decimal place.
The number M_{it} of iterations and the underrelaxation parameter ω_r are depending on the parameters Re, S, Ha, K^H, K^T. This values are determined for fulfilment the following inequalitys:

$$\psi_{er} = \frac{\max|\psi^{m+1} - \psi^m|}{\max|\psi^{m+1}|} < 10^{-4}, T_{er} = \frac{\max|T^{m+1} - T^m|}{\max|T^{m+1}|} < 10^{-4}.$$

11.4.5 Alternating current

For altenating field induced by six electrodes are used $\theta = \frac{2\pi}{3}$ and $\theta = \frac{\pi}{3}$.
The liquid have following parameters:
kinematic viscosity $\nu \approx 10^{-6}\frac{m^2}{s}$, density of liquid $\tilde{\rho} \approx 1000\frac{kg}{m^3}$ and the electric conductivity $\sigma \approx 100\Omega^{-1}m^{-1}$. The parameter $K_0 = \pi 10^{-10} I^2$, radius R of the cylinder is $0.10m$, the density of the current amplitude $j_0 \approx 10^4 I \frac{A}{m^2}$, the radius a of the electrodes is $0.005m$, where $I = 100A$.
We have following parameters: $K^H \approx 31, Pr = 6.7, K_T \approx 50$. For given values of parameters we have $\psi_{er} < 10^{-4}, T_{er} < 10^{-4}$ by $M_{it} = 200, \omega_r = 0.5$.

We consider different connections of the conductors $[L_1, L_2, L_3, L_4, L_5, L_6]$. This connections were denoted with $[1,2,3,4,5,6]$.

By $\theta = \pi/\mathbf{3}$ and with following coodinates of conductors centers
$L_1(r_1, \phi_1) = (r_*, 0^0), L_2(r_2, \phi_2) = (r_*, 60^0), L_3(r_3, \phi_3) = (r_*, 120^0)$,
$L_4(r_4, \phi_4) = (r_*, 180^0), L_5(r_5, \phi_5) = (r_*, 240^0), L_6(r_6, \phi_6) = (r_*, 300^0), r_* =$

$0.015m$,
follows
$\sin(\theta) = \sin(2\theta) = \frac{\sqrt{3}}{2}, \sin(3\theta) = 0, \sin(4\theta) = \sin(5\theta) = -\frac{\sqrt{3}}{2},$
$\cos(\theta) = \cos(5\theta) = \frac{1}{2}, \cos(2\theta) = \cos(4\theta) = -\frac{1}{2}, \cos(3\theta) = -1,$

$$\begin{cases} S_6^{\gamma} = \sum_{i=1}^{6} \gamma_{i,i} + \sum_{i=1}^{5} \gamma_{i,i+1} - \sum_{i=1}^{4} \gamma_{i,i+2} - 2\sum_{i=1}^{3} \gamma_{i,i+3} - \\ \gamma_{1,5} - \gamma_{2,6} + \gamma_{1,6}, \\ S_6^{\delta} = \sqrt{3}(\delta_{1,2} + \delta_{2,3} + \delta_{3,4} + \delta_{4,5} + \delta_{5,6} + \delta_{1,3} + \delta_{2,4} + \delta_{3,5} + \delta_{4,6} - \\ \delta_{1,5} - \delta_{2,6} - \delta_{1,6}). \end{cases} \quad (11.38)$$

In Figs. 11.50, 11.51 are represented the distributions of stream function and temperature for connections $[1,2,3,4,5,6]$ of conductors by maximal temperature $\max(T) = \mathbf{10.56}$.
In the Figs. 11.52, 11.53 are the results for connections $[2,3,5,6,1,4]$ by $\max(T) = 6.57$ and by two vortices.
In the Table 11.6 are results for different connections of electrods.

Table 11.6 The results by $\theta = \pi/3$

Connections	$\max(T)$	$[\min V_\phi, \max V_\phi]$	$[\min V_r, \max V_r]$	$[\min \psi, \max \psi]$
$[1,2,3,4,5,6]$	**10.56**	(-1.174,1.833)	(-0.085, 0.090)	(-0.37, 0.00)
$[2,3,5,6,1,4]$	6.57	(-0.586,0.969)	(-0.218, 0.248)	(-0.19, 0.04)
$[2,4,6,1,3,5]$	2.97	(-0.193,0.244)	(-0.160, 0.100)	(-0.045, 0.03)
$[1,4,5,2,3,6]$	3.70	(-0.347,0.240)	(-0.108, 0.118)	(0,00, 0.074)
$[1,6,2,5,3,4]$	5.55	(-0.750,0.434)	(-0.263, 0.363)	(-0.070, 0.14)
$[1,4,5,6,2,3]$	6.36	(-0.618,1.017)	(-0.253, 0.312)	(-0.200, 0.043)
$[2,6,5,3,1,4]$	2.95	(-0.382,0.217)	(-0.157,0.189)	(-0.0183,0.071
$[3,4,5,6,2,1]$	8.29	(-0.959,1.502)	(-0.216,0.228)	(-0.300, 0.00)
$[1,3,5,2,4,6]$	2.90	(-0.414, 0.710)	(-0.236,0.135)	(-0.134, 0.003)
$[1,4,2,5,3,6]$	3.06	(-0.383,0.658)	(-0.206,0.310)	(-0.124,0.049)
$[4,5,6,1,3,2]$	8.28	(-0.950,1.502)	(-0.217,0.228)	(-0.300,0.00)

We have
2 vortexes by connections $[2,3,5,6,1,4], [1,3,5,2,4,6], [1,4,5,6,2,3]$, 3 vortexes by connections $[1,6,2,5,3,4], [1,4,2,5,3,6]$ and 4 vortexes by connections $[2,4,6,1,3,5], [2,6,5,3,1,4]$.

For $\boldsymbol{\theta = 2\pi/3}$
$\sin(\theta) = \sin(4\theta) = \frac{\sqrt{3}}{2}, \sin(3\theta) = 0, \sin(2\theta) = \sin(5\theta) = -\frac{\sqrt{3}}{2},$
$\cos(\theta) = \cos(2\theta) = \cos(4\theta) = \cos(5\theta) = -\frac{1}{2}, \cos(3\theta) = 1,$
follows

$$\begin{cases} S_6^\gamma = \sum_{i=1}^6 \gamma_{i,i} - \sum_{i=1}^5 \gamma_{i,i+1} - \sum_{i=1}^4 \gamma_{i,i+2} + 2\sum_{i=1}^3 \gamma_{i,i+3} - \\ \gamma_{1,5} - \gamma_{2,6} + \gamma_{1,6}, \\ S_6^\delta = \sqrt{3}(\delta_{1,2} + \delta_{2,3} + \delta_{3,4} + \delta_{4,5} + \delta_{5,6} - \delta_{1,3} - \delta_{2,4} - \delta_{3,5} - \delta_{4,6} + \\ \delta_{1,5} + \delta_{2,6} - \delta_{1,6}). \end{cases} \tag{11.39}$$

In the Figs. 11.54, 11.55 are the results for connections $[1,3,5,2,4,6]$ of conductors by maximal temperature $\max(T) = \mathbf{8.22}$.
This results remained also by connections
$[1,3,5,6,2,4],[2,4,6,1,3,5],[4,6,2,3,5,1],[6,4,2,5,3,1]$.
In the Figs. 11.56, 11.57 are the results for connections $[6,3,1,4,2,5]$ by $\max(T) = 5.76$ (the results remained by connection
$[1,3,6,5,2,4],[2,3,4,1,5,6],[2,3,5,6,1,4],[6,3,1,2,4,5]$ by opposite directions the vortices).
The maximal temperature $\max(T) = 6.93$ are obtained by connection
$[6,5,4,1,2,3],[6,2,4,1,5,3],[1,2,6,4,3,5],[3,4,6,2,1,5],[3,6,1,2,5,4]$.
In the Table 11.7 are results for different connections of electrods.

Table 11.7 The results by $\theta = 2\pi/3$

Connections	$\max(T)$	$[\min V_\phi, \max V_\phi]$	$[\min V_r, \max V_r]$	$[\min \psi, \max \psi]$
$[1,2,3,4,5,6]$	1.31	(-0.186,0.349)	(-0.005, 0.005)	(-0.062, 0.00)
$[2,3,5,6,1,4]$	5.76	(-0.497,0.828)	(-0.286, 0.286)	(-0.16, 0.00)
$[2,4,6,1,3,5]$	**8.22**	(-0.861,1.342)	(-0.049, 0.049)	(-0.27, 0.00)
$[6,5,4,1,2,3]$	6.93	(-0.772,0.772)	(-0.247, 0.247)	(-0.143, 0.143)
$[6,3,1,4,2,5]$	5.76	(-0.828, 0.497)	(-0.286,0.286)	(0.00, 0.16)
$[1,6,5,4,3,2]$	1.31	(-0.349,0.186)	(-0.005,0.005)	(0.062, 0.00)

For the connections $[6,3,1,4,2,5],[1,3,6,5,2,4]$ and for $[3,4,6,2,1,5],[3,6,1,2,5,4]$ we have 2 symetrical vortices which rotate in the opposite direction,
but for connections $[2,3,4,5,6,1],[1,6,5,4,3,2]$ the fluid between the cylinder rotate also in the opposite direction.

11.4.6 External magnetic field

Calculation and their graphic visualization were made for
$Re \in [100,1000]; S \in [0.1,10], \eta = 0.2, \omega_0 \in [-5,5], \omega_1 \in [-1,1], \omega_r \in [0.3,0.8], M_{it} \in [80,400]$.
For $Re = 100, S = 1 (Ha = 10)$ we have $\psi_{er} < 10^{-4}$ by $M_{it} = 80, \omega_r =$

0.8, for $Re = 100, S = 10$ — $M_{it} = 200, \omega_r = 0.5$, but for $Re = 1000, S = 1$ — $M_{it} = 400, \omega_r = 0.3$.

For the **radial magnetic field** the distributions of stream function and axial velocity depend only on r. For large value of Ha number by the rotated walls the Hartman boundary layers developed.
In the Figs. 11.58, 11.59 we can see the distributions of axial velocity by $Ha = 0$ and $Ha^2 = 1000, \omega_0 = -5; 0, \omega_1 = 1$.

For the **axial magnetic field** the distributions of stream function and axial velocity are non depending of Ha number and we have the 1-D solution of radial symmetry.

By the **uniform magnetic field** with different values of μ in the rotated flows the different form of vortexes developed. In the Figs. 11.60, 11.61 are represented the distributions of stream function by $Re = 100, S = 10, \omega_0 = -5; 0, \omega_1 = 1, \mu = 1; 0$. In the Table 11.8 we can see the extremal value of axial velocity and of stream functions for $Re \in [100, 1000], S \in [1, 10], \omega_0 = 0, \omega_1 = 1$. The markers $[\mathbf{0}, \mathbf{1}, \infty)]^*$ denoted the values obtained from sum of uniform and axial magnetic fields.

Table 11.8 Max. and min. values for uniform magnetic fields by $\omega_0 = 0, \omega_1 = 1, Ha^2 \geq 1000$

μ	Re	S	$\min(V_\phi)$	$\min(V_r)$	$\max(V_r)$	$\max(\Psi)$
0	100	10	-0.1610	-0.0805	0.0754	0.0710
1	100	10	-0.1701	-0.0698	0.0646	0.0711
∞	100	10	-0.1746	-0.0605	0.0534	0.0714
$\mathbf{0}^*$	100	10	-0.1427	-0.0400	0.0384	0.0587
$\mathbf{1}^*$	100	10	-0.1324	-0.0376	0.0364	0.0600
∞^*	100	10	-0.1277	-0.0346	0.0322	0.0610
0	400	3	-0.1447	-0.0725	0.0676	0.0627
1	400	3	-0.1586	-0.0653	0.0590	0.0630
0	1000	1	-0.1325	-0.0485	0.0543	0.0559
1	1000	1	-0.1561	-0.0509	0.0506	0.0574

For the **bipolar radial magnetic field** in the Table 11.9 are the extremal values for axial velocity and stream function. The following distributions of the stream functions we can see in Figs. 11.62, 11.63.

Table 11.9 Max. and min. values for bipolar magnetic field by $Re = 100$

ω_0	ω_1	S	min(V_ϕ)	min(V_r)	max(V_r)	max(Ψ)	min(Ψ)
0	1	0.1	-0.3229	-0.0203	0.0224	0.1049	0.
0	1	0.5	-0.2939	-0.0298	0.0314	0.0822	0.
0	1	1.0	-0.2839	-0.0321	0.0325	0.0728	0.
5	0	1	-0.0539	-0.0345	0.0347	0.0	-0.0145
5	1	1	-0.3044	-0.0340	0.0349	0.0704	-0.0145
-5	1	1	-1.0	-0.0352	0.0353	0.0754	0.

11.5 Conclusions

1) The distribution of electromagnetic fields and forces induced by a three- phase system of the alternating electric current in the conducting liquid in the cylinder of finite length has been calculated. An original method was used to calculate the mean values of magnetic field and electromagnetic forces. The 2-D averaged magnetic field, source terms for the tempeature and Lorenz' forces, induced by alternating current with three bar type electrodes is calculated in cross-section of cylinder by computer program MATLAB, but 3-D magnetohydrodynamics flow of the liquid is calculated with the help of the computer programs FLUENT.
In future it will be interesting to obtain with the help of the finite difference method the distributions of magnetohydrodynamical and termodinamical fields for 2-D problem in the fixed cross-section of cylinder depending on the electromagnetic and thermodinamical forces.

2)

- Using the finite difference method the distributions of magnetohydrodynamic flows and temperature are obtained.
- The distribution of stream function and velocity field depends on external magnetic field and direction of gravitation has been calculated.
- In the channel with periodically placed obstacles, free convection flows are investigated.
- From the numerical results one can conclude that the vortex formation and MHD flow are depending on the form of external magnetic fields, direction of gravitation and on the values of Stewart number.
- In the strong transverse magnetic field the vortexes are deleted and on the walls of the cylinders
  Hartmann boundary layers are developed.

The distribution of 2-D electromagnetic fields and forces induced by the alternating electric current and by external magnetic fields in the electro conducting liquid on the cross-section between two infinite cylinder has been calculated. An original method was used to calculate the mean values of electromagnetic forces and circulant matrices. For the altenating current the 2-D averaged magnetic field, source terms for the tempeature and Lorenz' forces, induced by alternating current with 6 bar type electrodes is calculated.
With the finite difference method the distributions of magnetohydrodynamics flows and maximal temperature depending of the connections of electrods are obtained.
From numerical results for the external magnetic fields follows that the distributions of MHD flow is depending of the velocity of walls rotation and of the form of the external magnetic fields:
1) for the radial magnetic field we have the radial symmetry of the flow and on the walls of the cylinder the Hartman boundary layers developed,
2) for the uniform magnetic field we can see different form of vortex formation in the fluid depending of the level for ferromagnetics.,

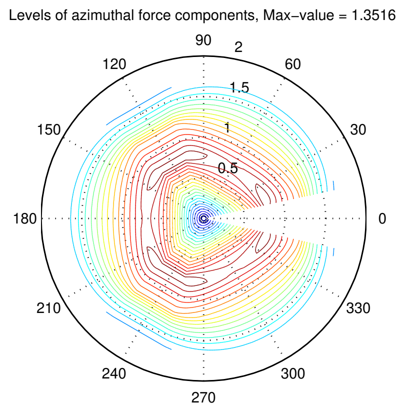


Fig. 11.1 Azimuthal Lorenz force

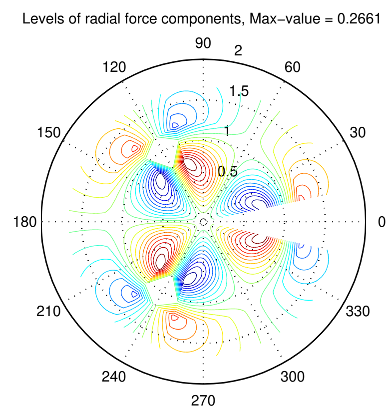


Fig. 11.2 Radial Lorenz force

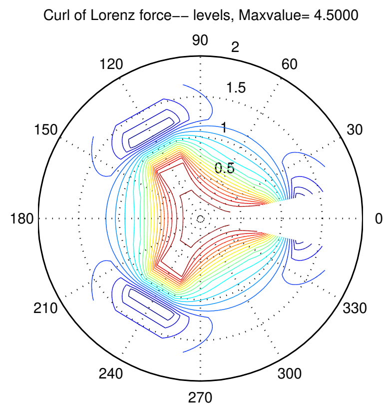


Fig. 11.3 Curl of Lorenz force

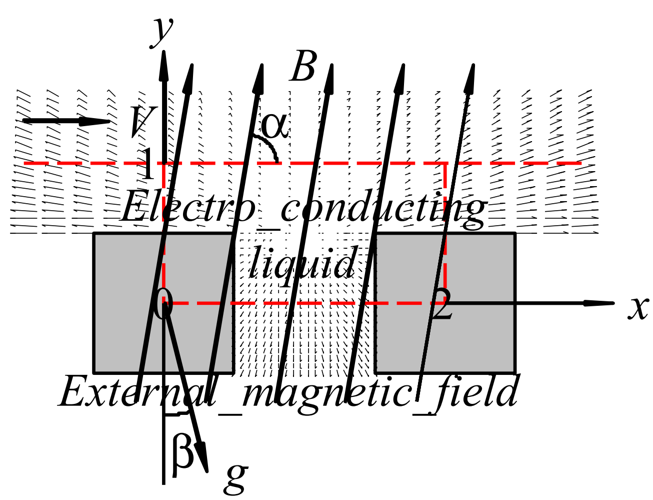


Fig. 11.4 Domain for in-line placed cylinders(2 cylinders, $L_1 = 0.5, L = 1, l_1 = 0.5, l_2 = 1.5, l = 2$)

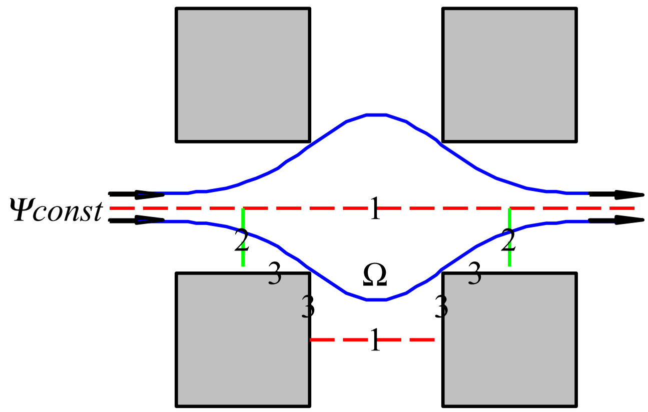


Fig. 11.5 Domain for in-line placed cylinders(4 cylinders, BCs: 1-symetry,2- periodic, 3-walls)

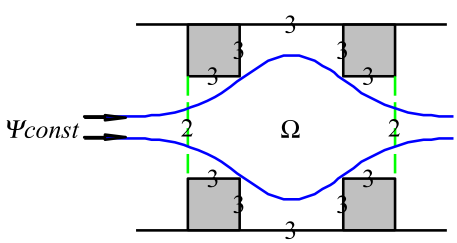


Fig. 11.6 Domain for channel flow (quarter of 4 cylinders, $L_1 = 0.5, L_2 = 1.5, L = 2, l_1 = 0.5, l_2 = 1.5, l = 2$)

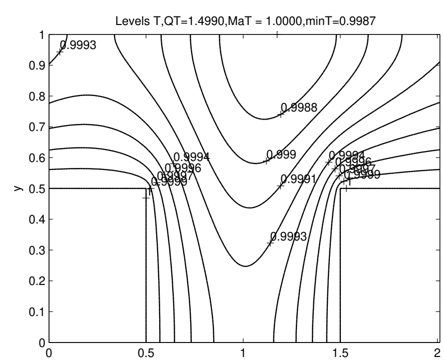


Fig. 11.7 Levels of temperature for PC by $Gr = 0, S = 0$

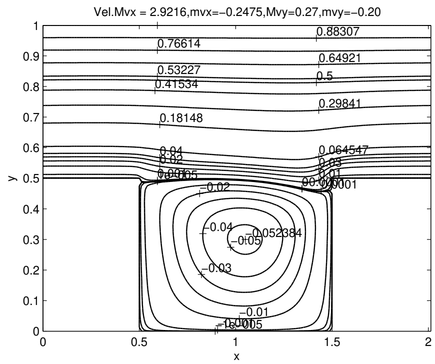


Fig. 11.8 Levels of stream function in PC for $Gr = S = 0$

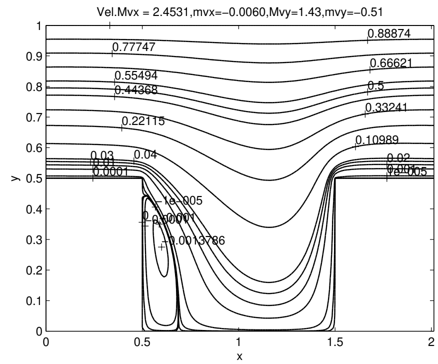


Fig. 11.9 Levels of stream function in PC for $\alpha = \frac{\pi}{2}, Gr = 0, S = 2.5$

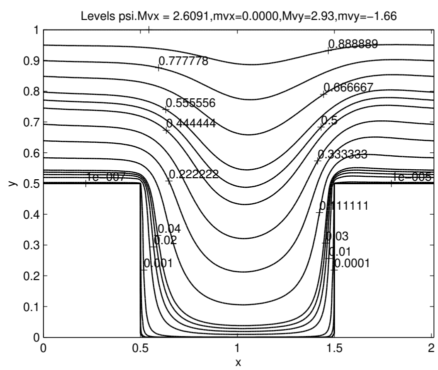


Fig. 11.10 Levels of stream function in PC for $\alpha = \frac{\pi}{2}, Gr = 0, S = 25$

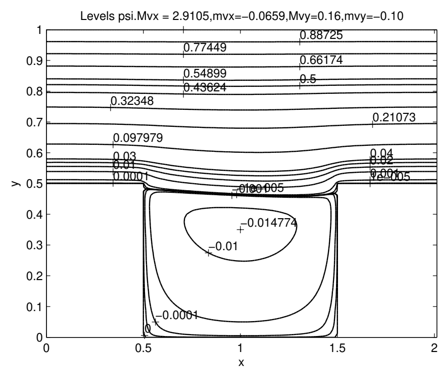


Fig. 11.11 Levels of stream function in PC for $\alpha = 0, Gr = 0, S = 25$

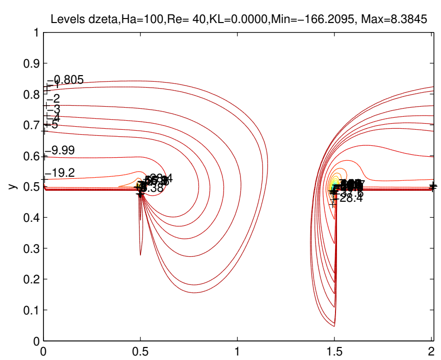


Fig. 11.12 Levels of vorticity function in PC for $\alpha = \frac{\pi}{2}, Gr = 25000, S = 2.5, \beta = \frac{\pi}{2}$

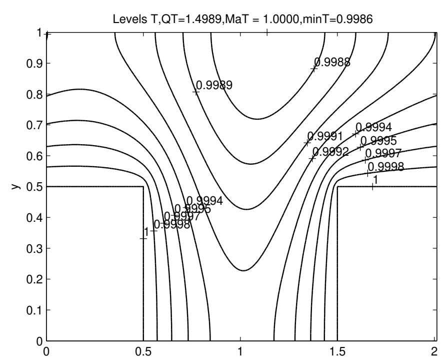
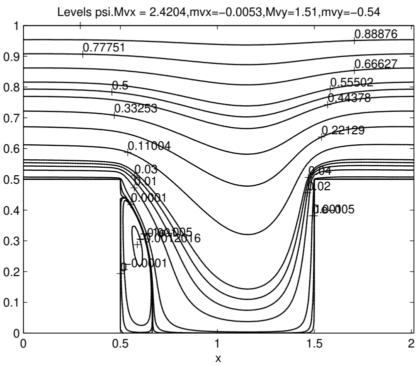


Fig. 11.13 Levels of temperature in PC for $\alpha = \frac{\pi}{2}, Gr = 25000, S = 2.5, \beta = \frac{\pi}{2}$



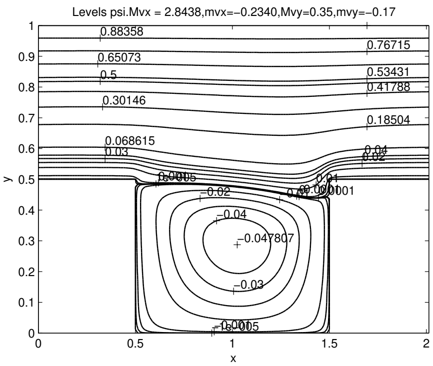
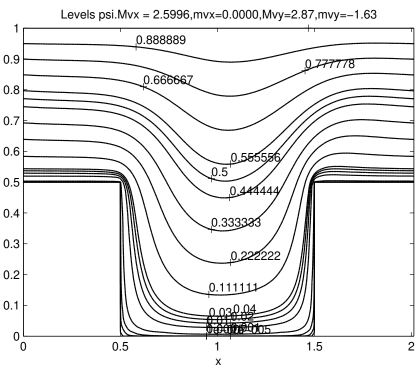


Fig. 11.14 Levels of stream function in PC by $\alpha = \frac{\pi}{2}, Gr = 11000, S = 2.5$

Fig. 11.15 Levels of stream function in PC by $S = 0, Gr = 11000$



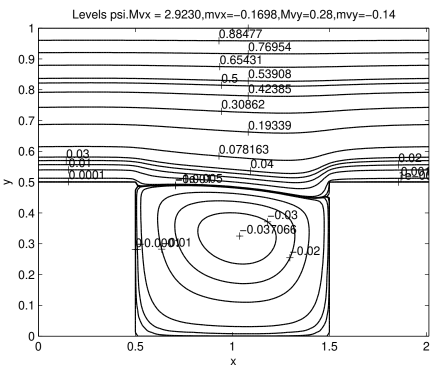
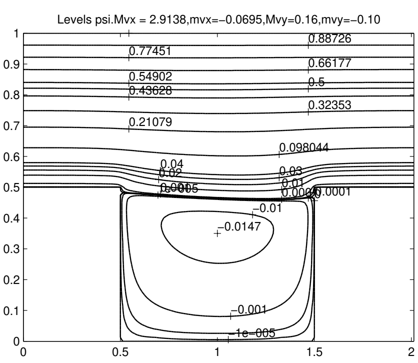


Fig. 11.16 Levels of stream function in CFS for $\alpha = \frac{\pi}{2}, S = 25, Gr = 25000, \beta = \frac{\pi}{2}$

Fig. 11.17 Levels of stream function in CFS for $S = 0, Gr = 25000, \beta = \frac{\pi}{2}$



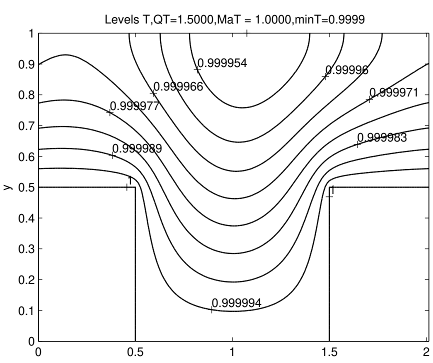


Fig. 11.18 Levels of stream function in CFS for $\alpha = 0, S = 25, Gr = 25000, \beta = \frac{\pi}{2}$

Fig. 11.19 Levels of temperature in CFS for $\alpha = 0, S = 25, Gr = 25000, \beta = \frac{\pi}{2}$

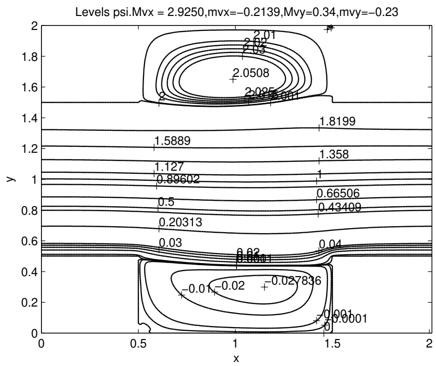


Fig. 11.20 Levels of stream function in CF for $S = 0, Gr = 25000, \beta = 0$

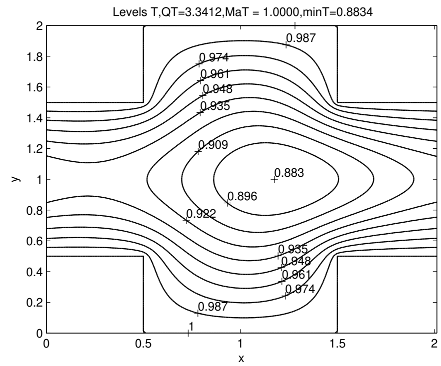


Fig. 11.21 Levels of temperature in CF for $S = 0, Gr = 25000, \beta = 0$

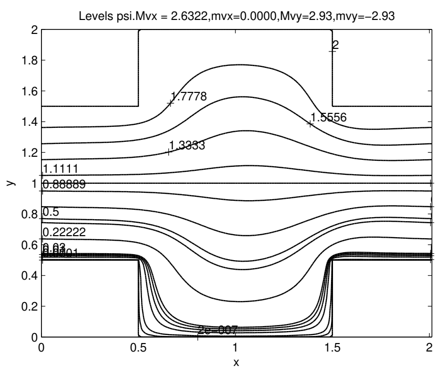


Fig. 11.22 Levels of stream function in CF for $\alpha = \frac{\pi}{2}, S = 25, Gr = 25000, \beta = \frac{\pi}{2}$

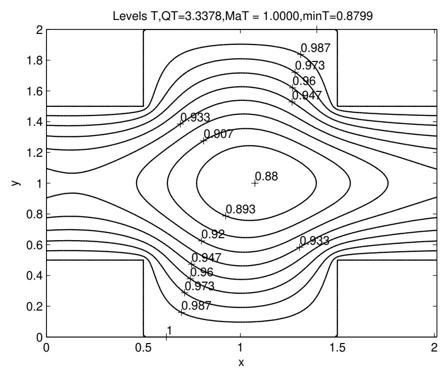


Fig. 11.23 Levels of temperature in CF for $\alpha = \frac{\pi}{2}, S = 25, Gr = 25000, \beta = \frac{\pi}{2}$

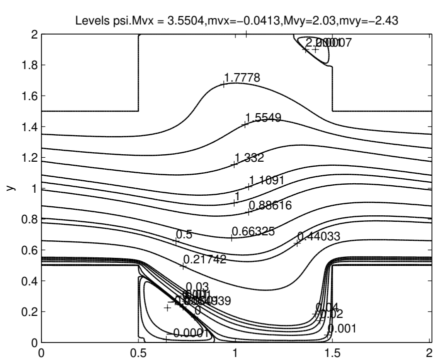


Fig. 11.24 Levels of stream function in CF for $\alpha = \frac{3\pi}{4}, S = 25, Gr = 25000, \beta = \frac{\pi}{2}$

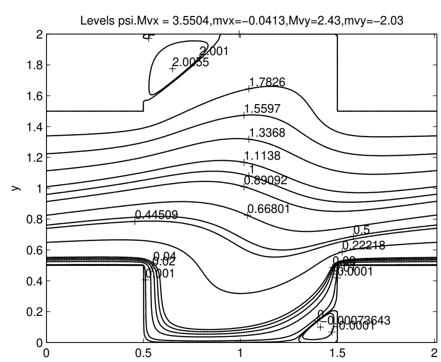


Fig. 11.25 Levels of stream function in CF for $\alpha = \frac{\pi}{4}, S = 25, Gr = 25000, \beta = \frac{\pi}{2}$

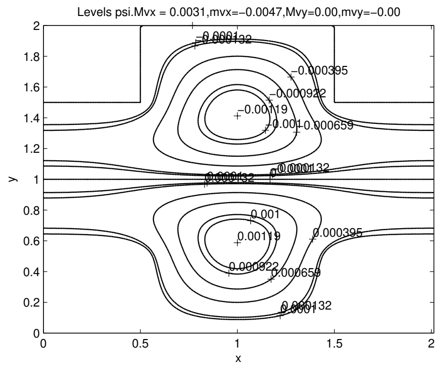


Fig. 11.26 Levels of stream function in free convection CF for $S = 0, Gr = 25000, \beta = \frac{\pi}{2}$

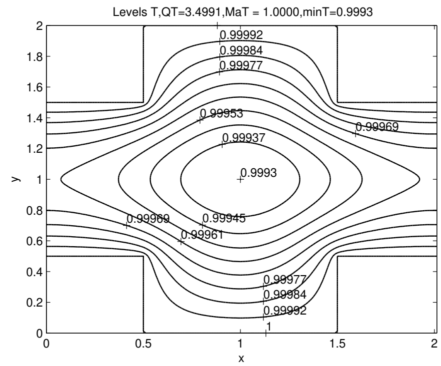


Fig. 11.27 Levels of temperature in in free convection CF for $S = 0, Gr = 25000, \beta = \frac{\pi}{2}$

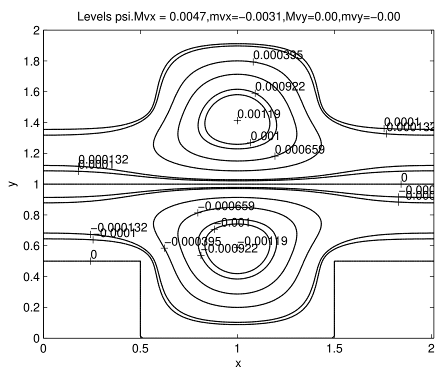


Fig. 11.28 Levels of stream function in in free convection CF for $S = 0, Gr = 25000, \beta = -\frac{\pi}{2}$

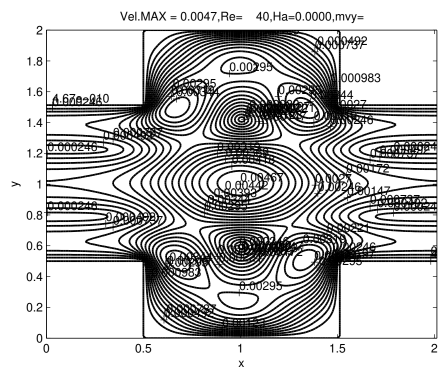


Fig. 11.29 Levels of velocity in in free convection CF for $S = 0, Gr = 25000, \beta = -\frac{\pi}{2}$

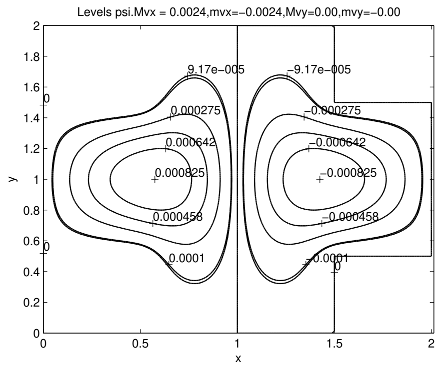


Fig. 11.30 Levels of stream function in in free convection CF for $S = 0, Gr = 25000, \beta = 0$

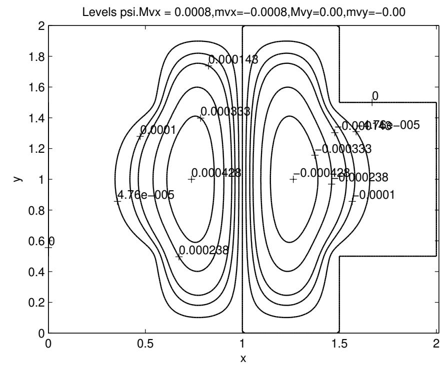


Fig. 11.31 Levels of stream function in in free convection CF for $S = 25, \alpha = \frac{\pi}{2}, Gr = 25000, \beta = 0$

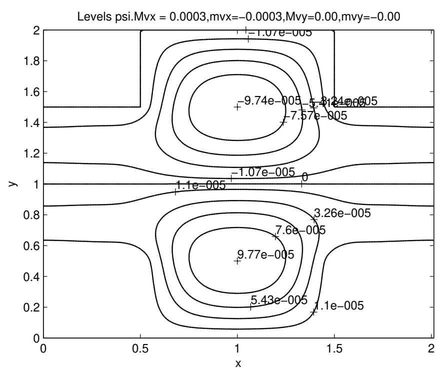


Fig. 11.32 Levels of stream function in in free convection CF for $S = 25, \alpha = \frac{\pi}{2}, Gr = 25000, \beta = \frac{\pi}{2}$

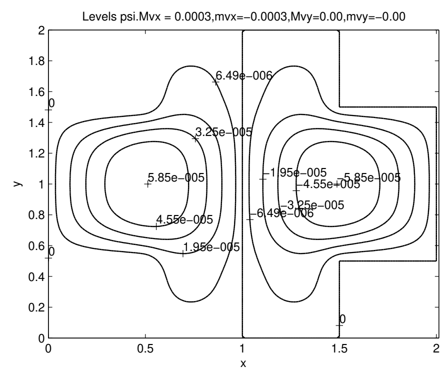
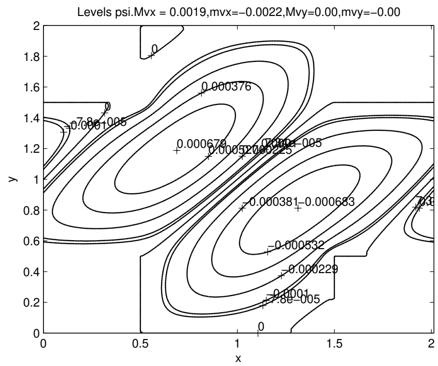


Fig. 11.33 Levels of stream function in in free convection CF for $S = 25, \alpha = 0, Gr = 25000, \beta = 0$



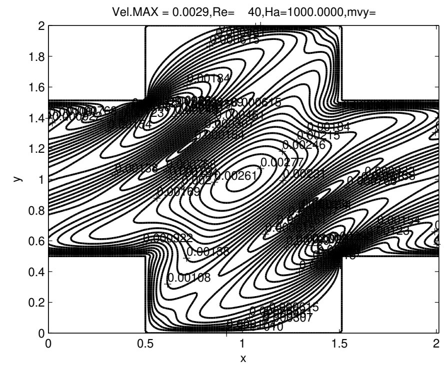
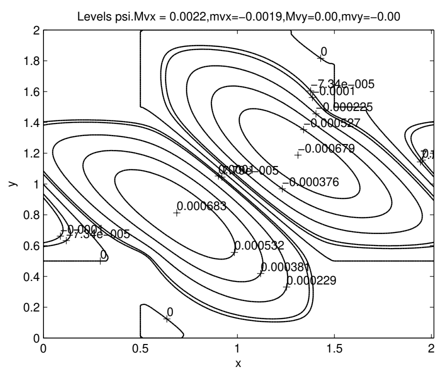


Fig. 11.34 Levels of stream function in free convection CF for $S = 25, \alpha = \frac{\pi}{4}, Gr = 25000, \beta = 0$

Fig. 11.35 Levels of velocity in free convection CF for $S = 25, \alpha = \frac{\pi}{4}, Gr = 25000, \beta = 0$



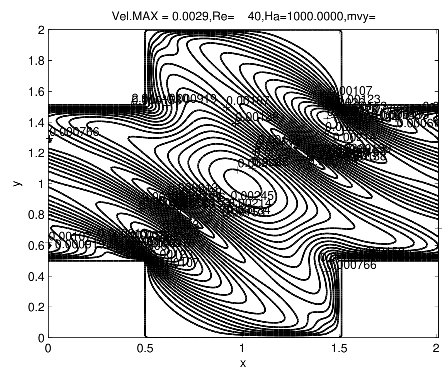


Fig. 11.36 Levels of stream function in free convection CF for $S = 25, \alpha = \frac{3\pi}{4}, Gr = 25000, \beta = 0$

Fig. 11.37 Levels of velocity in free convection CF for $S = 25, \alpha = \frac{3\pi}{4}, Gr = 25000, \beta = 0$

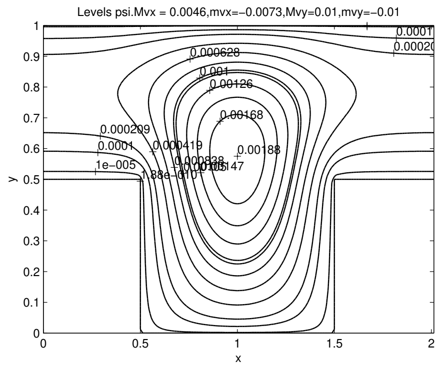


Fig. 11.38 Levels of stream function in free convection PC flow for $S = 0, Gr = 25000, \beta = \frac{\pi}{2}$

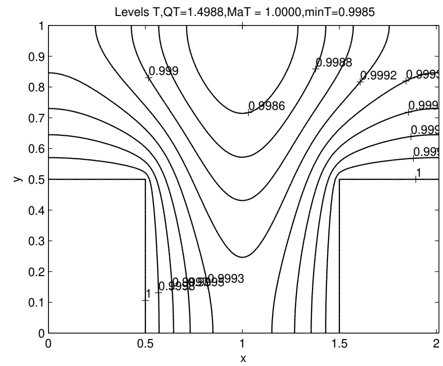


Fig. 11.39 Levels of temperature in free convection PC flow for $S = 0, Gr = 25000, \beta = \frac{\pi}{2}$

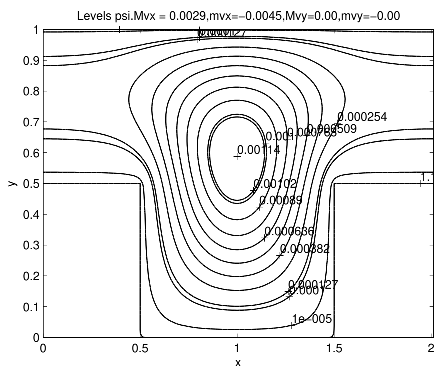


Fig. 11.40 Levels of stream function in free convection CFS flow for $S = 0, Gr = 25000, \beta = \frac{\pi}{2}$

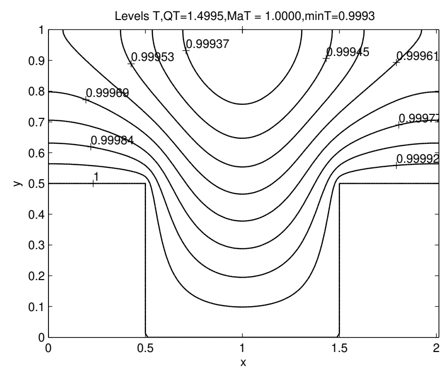


Fig. 11.41 Levels of temperature in free convection CFS flow for $S = 0, Gr = 25000, \beta = \frac{\pi}{2}$

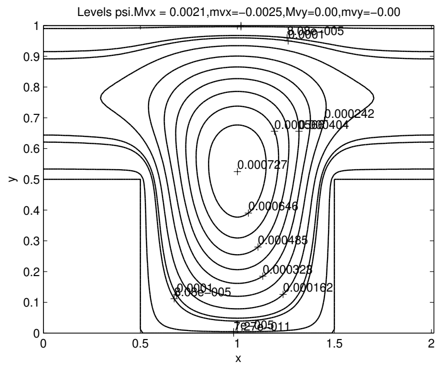


Fig. 11.42 Levels of stream function in free convection PC flow for $S = 2.5, Gr = 25000, \beta = \frac{\pi}{2}, \alpha = \frac{\pi}{2}$

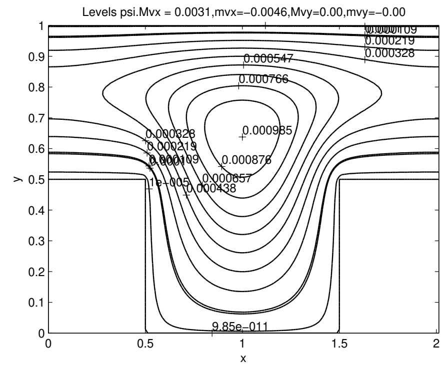


Fig. 11.43 Levels of stream function in free convection PC flow for $S = 2.5, Gr = 25000, \beta = \frac{\pi}{2}, \alpha = 0$

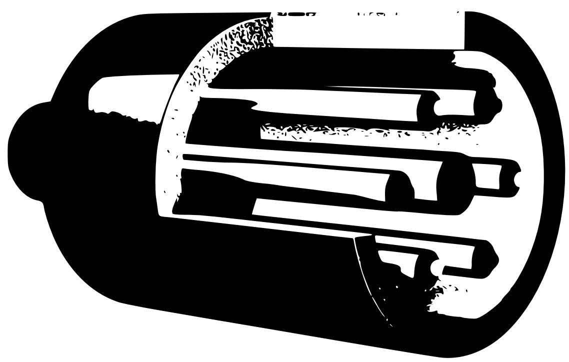


Fig. 11.44 The real heat generator

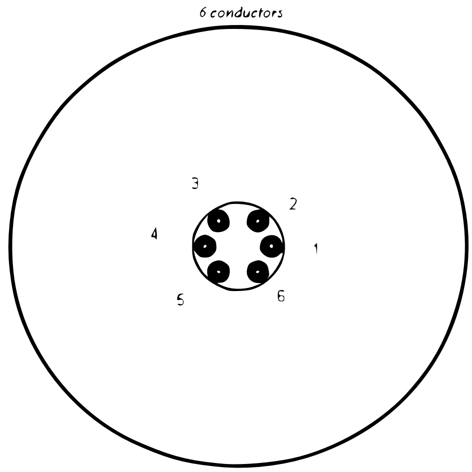


Fig. 11.45 The 2-D mathematical model of the heat generator

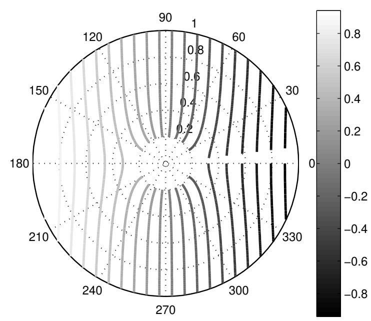


Fig. 11.46 Uniform magnetic field by $\mu = 0$

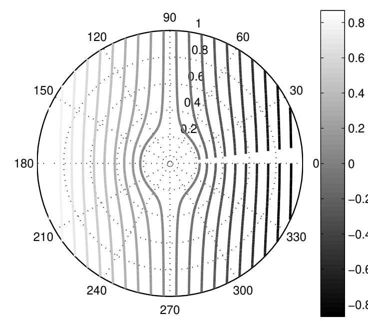


Fig. 11.47 Uniform magnetic field by $\mu = \infty$

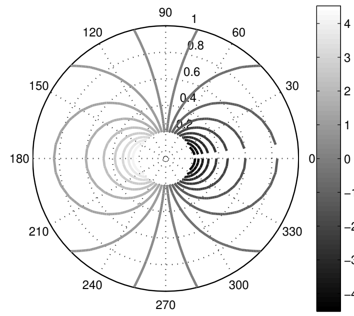


Fig. 11.48 Bipolar magnetic field

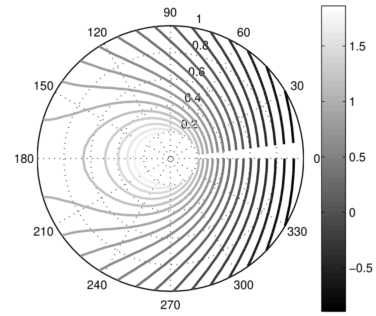


Fig. 11.49 Uniform and axial magnetic fields $\mu = 0$

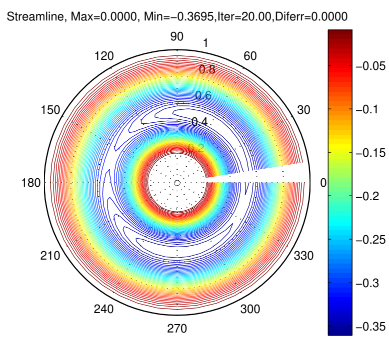


Fig. 11.50 Stream function for [1,2,3,4,5,6], $\theta = \pi/3$

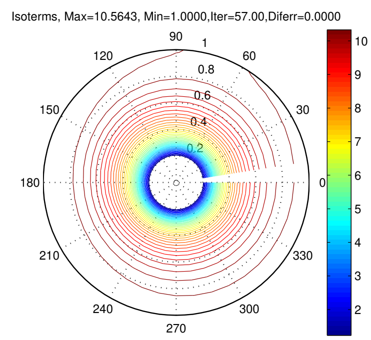


Fig. 11.51 Temperature for [1,2,3,4,5,6],$\theta = \pi/3, T_{max} = 10.56$

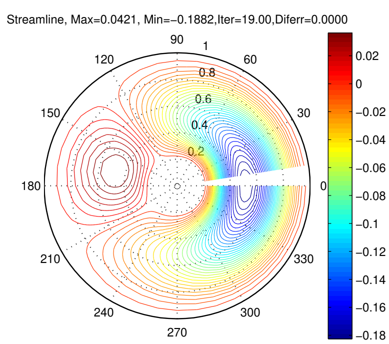


Fig. 11.52 Stream function for [2,3,5,6,1,4], $\theta = \pi/3$

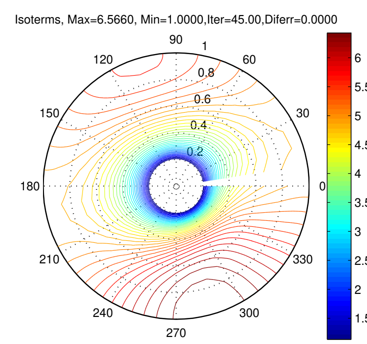


Fig. 11.53 Temperature for [2,3,5,6,1,4],$\theta = \pi/3, T_{max} = 6.57$

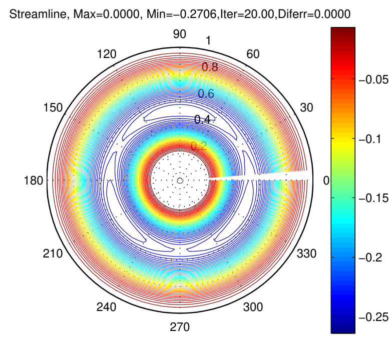


Fig. 11.54 Stream function for [1,3,5,2,4,6], $\theta = 2\pi/3$

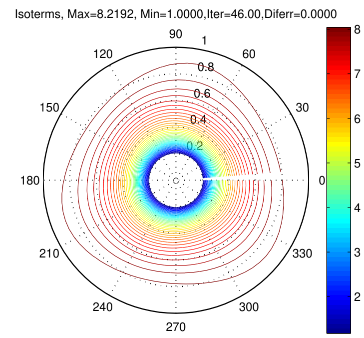


Fig. 11.55 Temperature for [1,3,5,2,4,6],$\theta = 2\pi/3, T_{max} = 8.22$

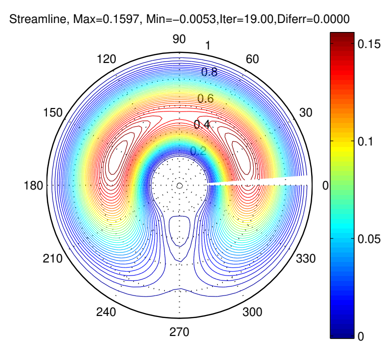


Fig. 11.56 Stream function for [6,3,1,4,2,5], $\theta = 2\pi/3$

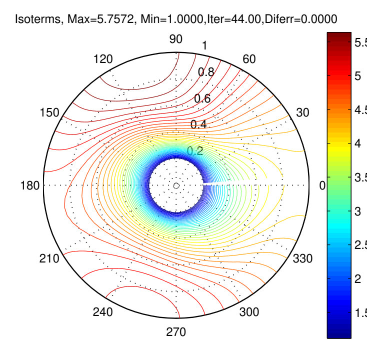


Fig. 11.57 Temperature for [6,3,1,4,2,5],$\theta = 2\pi/3, T_{max} = 5.76$

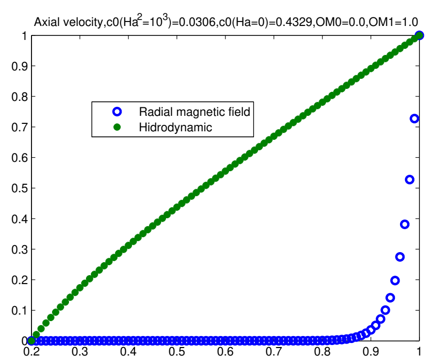


Fig. 11.58 Axial velocity for radial field by $S = 10, \omega_0 = 0, \omega_1 = 1$

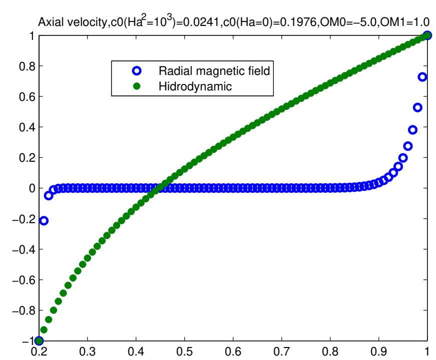


Fig. 11.59 Axial velocity for radial field by $S = 10, \omega_0 = -5, \omega_1 = 1$

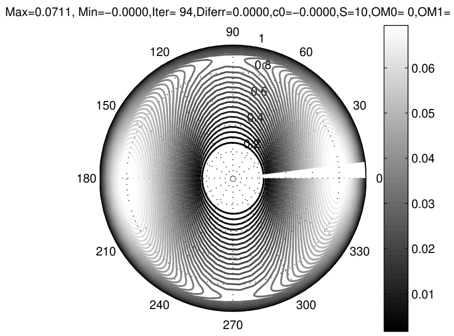


Fig. 11.60 Stream function for uniform magnetic field by $\mu = 1, S = 10, Re = 100, \omega_0 = 0, \omega_1 = 1$

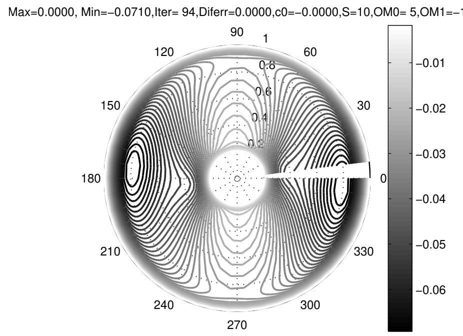


Fig. 11.61 Stream function for uniform magnetic field by $\mu = 0, S = 10, Re = 100, \omega_0 = -5, \omega_1 = 1$

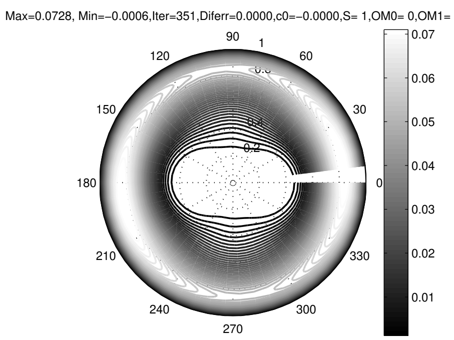


Fig. 11.62 Stream function for bipolar magnetic field by $S = 1, Re = 100, \omega_0 = 0, \omega_1 = 1$

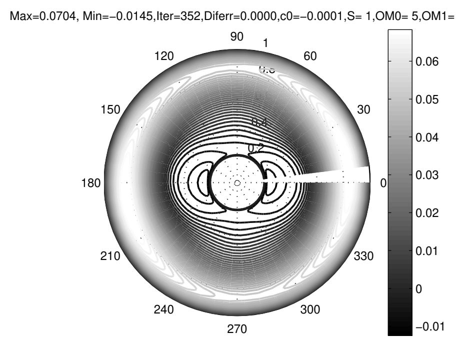


Fig. 11.63 Stream function for bipolar magnetic field by $S = 1, Re = 100, \omega_0 = 5, \omega_1 = 1$

Chapter 12
Velocity induced by the vortexes: H. Kalis, J. G. Schatz, 2008 [82]

In new technological applications it is important to use vortex distributions in area for obtaining large velocity fields [64]. In this paper was calculated the distribution of velocity field for ideal incompressible fluid, induced by a different system of finite number of vortex threads:
1) circular vortex lines in a finite cylinder, positioned on its inner,
2) spiral vortex threads, positioned on the inner surface in the finite cylinder or conus,
3) linear vortex lines in the plane channel, positioned on its boundary.
An original method was used to calculate the components of the velocity vectors. Such kind of procedure allows calculating the velocity fields inside the domain depending on the arrangement, of the intensity and on the radii of vortex lines.

12.1 The introduction

The effective use of vortex energy in production of strong velocity fields by different devise is one of the modern areas of applications, dveloped during last years [65].This work presents three mathematical models of such devices. It are
1) a finite cylinder with finite number of circular vortex lines positioned on its inner surface with a fixed distance between each other,
2) a finite cylinder or conus with finite number of spiral vortex threads positioned on its inner surface,
3) a plane channel with finite number of linear vortex lines positioned

on its boundary.
It is well known that the vortex theory to begin from the Decart papers. First of all was investigated the behaviour the discreate N lineary vortex lines with equal intensity Γ, which are in the vertices of regular restangle (authors are Helmholc, Kelvin, Kirhof, see [48], [47]).The investigation of contemporary are write in the book [66]: completely are investigated linear vortex lines, vortex cheets, vortex wakes, votexes of Karman, but difficulties cause the curves vortex lines.

12.2 The mathematical model

Let the cylindrical domain (conus) $\Omega_{r,z}(\varepsilon) = \{(r,z,\varphi) : 0 < r < a - \varepsilon z, 0 < z < Z, 0 < \varphi < 2\pi(M+1)\}$ $(0 \leq \varepsilon Z < a)$ contain ideal incompressible fluid, where a, Z are the maximal radius and legth of the cylinder, M is the number of circulation periods. If $\varepsilon = 0$, then we have the circular cylinder with the radius a.
Consider the situation when the N discreate circular vortex lines $L_i = \{(r,z), r = a_i, z = z_i\},\ 0 < z_i < Z,\ 0 < a_i < a, \}\ i = \overline{1,N}$, with intensity $\Gamma_i(\frac{m^2}{s})$ and radii $a_i(m)$ are placed in the cylinder. The vortex creates in the ideal compressible liquid the radial v_r and axial v_z components of the velocity field, which rise to the liquid motion.
Similar can be consider N discreate spiral vortex threads
$S_i = \{(r,z,\varphi), r = a - \varepsilon t, z = bt, \varphi = t + i\delta\},\ i = \overline{1,N}$, with parameter $\delta = \frac{2\pi}{N},\ \tau = \frac{Z}{2\pi a M}, \frac{2\pi}{N} \leq \varphi \leq 2\pi(M+1), b = a\tau, t \in [0, 2\pi M]$.
Here τ is the rise of the vortex threads , the spiral vortex with $Z = 2\pi, a = 1, N = 6, M = 1, \tau = 1, \varepsilon = 0; 1$, and in the Fig. 12.2 the circular vortex lines).

The spiral vortexes creates in the ideal compressible liquid the radial v_r, axial v_z and azimuthal v_φ components of the velocity field.
The linear vortex lines creates in the plane domain-channel $\Omega_{x,y} = \{(x,y) : x \in [0,L], y \in [0,2], z \in (-\infty,\infty)\}$ the v_x, v_y components of the velocity field.
The main aim of this work is to analyze the diversity of connection schemes of vortex curves influence maximal value of velocity.

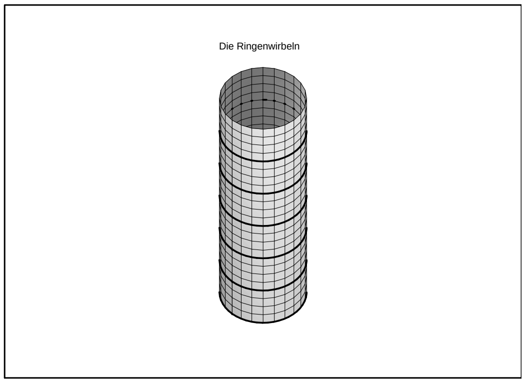


Fig. 12.1 The surface of the cylinder with circular vortexes lines

12.3 Calculation of the velocity field for the spiral vortexes

The vector potential A is determined from the equations of vortex motion of ideal incompressible fluid

$$div\, v = 0, \quad rot\, v = \Omega,$$

in the following form:

$$\Delta A = -\Omega,$$

where $v = rotA$ and v, Ω the vectors of velocity and vortex fields are, Δ is the Laplace operator.

Applying the Biot-Savar [48] law we receive the following form of the vector potential created by the vortex thread W_i ($W_i = S_i$ or $W_i = L_i$):
$A(P)_i = \frac{\Gamma_i}{4\pi} \int_{W_i} \frac{dl}{R(QP)_i}$
where dl is an element of the curves, $P = P(x,y,z)$ is the fixed point in the liquid, $Q = Q(\xi, \eta, \zeta)$ is the changeable point in the integral, $R(QP)_i = \sqrt{((z-\zeta)^2 + (x-\xi_i)^2 + (y-\eta_i)^2)}$.

From cylindrical coordinates $x = r\cos\varphi, y = r\sin\varphi$.

For the spiral vortexes S_i:
$\xi_i = a_*(t)\cos(t+i\delta), \eta_i = a_*(t)\sin(t+i\delta),\ \zeta = bt, (b = a\tau)$, $t \in [0, 2\pi M]$ $(a_*(t) = a - \varepsilon t)$ and we have following components of the vector potential:

$$A_{x,i} = \frac{\Gamma_i}{4\pi} \int\limits_{S_i} \frac{d\xi}{R_i}, A_{y,i} = \frac{\Gamma_i}{4\pi} \int\limits_{S_i} \frac{d\eta}{R_i},$$

$$A_{z,i} = \frac{\Gamma_i}{4\pi} \int\limits_{S_i} \frac{d\zeta}{R_i},$$

where $R_i = R(QP)_i$.
Therefore
$d\xi = (-a_*(t)\sin(t+i\delta) - \varepsilon\cos(t+i\delta))dt$, $d\eta = (a_*(t)\cos(t+i\delta) - \varepsilon\sin(t+i\delta))dt$, $d\zeta = bdt$,
$R_i = \sqrt{r^2 + a_*(t)^2 - 2a_*(t)r\cos(\varphi - t - i\delta) + (z-bt)^2}$
and

$$A_{x,i} = -\frac{\Gamma_i}{4\pi} \int\limits_0^{2\pi M} \frac{(a_*(t)\sin(t+i\delta) + \varepsilon\cos(t+i\delta))dt}{R_i},$$

$$A_{y,i} = \frac{\Gamma_i}{4\pi} \int\limits_0^{2\pi M} \frac{(a_*(t)\cos(t+i\delta) - \varepsilon\sin(t+i\delta))dt}{R_i},$$

$$A_{z,i} = \frac{\Gamma_i b}{4\pi} \int\limits_0^{2\pi M} \frac{dt}{R_i}.$$

The vector components of the velocity field (radial, axial, azimuthal) induced by the spiral vortex curves are in the form

$$\begin{cases} v_{r,i} = -\frac{\partial A_{\varphi,i}}{\partial z} + \frac{\partial A_{z,i}}{r\partial\varphi}, \\ v_{z,i} = \frac{1}{r}\frac{\partial}{\partial r}(rA_{\varphi,i}) - \frac{1}{r}\frac{\partial A_{r,i}}{\partial\varphi} \\ v_{\varphi,i} = \frac{\partial A_{r,i}}{\partial z} - \frac{\partial A_{z,i}}{\partial r}, \end{cases} \tag{12.1}$$

where
$A_{r,i} = A_{x,i}\cos(\varphi) + A_{y,i}\sin(\varphi) = \frac{\Gamma_i}{4\pi}\int_0^{2\pi M} \frac{(a_*(t)\sin(\psi(t)) - \varepsilon\cos(\psi(t)))dt}{R_i}$,
$A_{\varphi,i} = -A_{x,i}\sin(\varphi) + A_{y,i}\cos(\varphi) = \frac{\Gamma_i}{4\pi}\int_0^{2\pi M} \frac{(a_*(t)\cos(\psi(t)) + \varepsilon\sin(\psi(t)))dt}{R_i}$,
$(\psi = \varphi - t - i\delta)$
are the radial and azimuthal components of vector potentials.
Then from the partial derivatives

$\frac{\partial R_i}{\partial r} = \frac{r - a_*(t)\cos(\psi(t))}{R_i}$, $\frac{\partial R_i}{\partial z} = \frac{z-bt}{R_i}$, $\frac{\partial R_i}{\partial \varphi} = \frac{a_*(t) r \sin(\psi(t))}{R_i}$,
follows

$$v_{r,i} = \frac{\Gamma_i}{4\pi} \int_0^{2\pi M} \frac{1}{R_i^3} [(z-bt)(a_*(t)\cos(\psi(t)) + \varepsilon \sin(\psi(t))) - b a_*(t) \sin(\psi(t))] dt, \tag{12.2}$$

$$v_{z,i} = \frac{\Gamma_i}{4\pi} \int_0^{2\pi M} \frac{1}{R_i^3} [a_*(t)(a_*(t) - r\cos(\psi(t))) - \varepsilon r \sin(\psi(t))] dt, \tag{12.3}$$

$$v_{\varphi,i} = \frac{\Gamma_i}{4\pi} \int_0^{2\pi M} \frac{1}{R_i^3} [b(r - a_*(t)\cos(\psi(t))) - (z-bt)(a_*(t)\sin(\psi(t)) + \varepsilon \cos(\psi(t)))] dt.$$

For $\varepsilon = 0$ and for the symmetrical properties respect to $z = Z/2$ follows that for the all components of velocity $\mathbf{v}_i(r, Z/2 - z, \varphi) = \mathbf{v}_i(r, Z/2 + z, \varphi)$.

If $r = 0$, then

$$v_{z,i}(0,z) = \frac{\Gamma_i}{4\pi} \int_0^{2\pi M} \frac{a_*(t)^2 dt}{(a_*(t)^2 + (z-bt)^2)^{1.5}} \tag{12.4}$$

or

$$v_{z,i}(0,z) = \frac{\Gamma_i \varepsilon^2}{4\pi} \int_{a-2\pi M \varepsilon}^{a} \frac{q^2 dq}{R(q)^3},$$

where
$R(q) = \sqrt{a_1 + b_1 q + c_1 q^2}$, $a_1 = b^2 z_0^2, b_1 = -2b^2 z_0, c_1 = \varepsilon^2 + b^2$,
$z_0 = a - \frac{z\varepsilon}{b}$.

Therefore, from [67]:

$$\begin{cases} v_{z,i}(0,z) = \frac{\Gamma_i}{4 c_1 \pi} [\frac{d_2 a_2 - 2 a_1 b_1}{d_1 R(a_2)} - \frac{d_2 a - 2 a_1 b_1}{d_1 R(a)} - \\ \frac{\varepsilon^2}{\sqrt(c_1)} \ln \frac{\sqrt(c_1) R(a_2) + c_1 a_2 + b_1/2}{\sqrt(c_1) R(a) + c_1 a + b_1/2}], \end{cases} \tag{12.5}$$

where
$a_2 = a - 2\pi\varepsilon M$, $d_1 = 4b^2 z_0^2$, $d_2 = d_1(\varepsilon^2 - b^2)$.

If $\varepsilon = 0$, then

$$v_{z,i}(0,z) = \frac{\Gamma_i M}{2Z}\left[\frac{z}{\sqrt{a^2+z^2}} + \frac{Z-z}{\sqrt{a^2+(Z-z)^2}}\right], \tag{12.6}$$

and the maximal value of velocity is

$$v_{z,i}(0,Z/2) = \frac{\Gamma_i M}{2a\sqrt{1+(Z/(2a))^2}} \tag{12.7}$$

by $z = Z/2$.

The minimal value we have in the form

$$v_{z,i}(0,0) = v_{z,i}(0,Z) = \frac{\Gamma_i M}{2a\sqrt{1+(Z/a)^2}} \tag{12.8}$$

by $z = 0$ and $z = Z$.

The averaged value of the axial component of velocity field in the axes of the cylinder $(r = 0)$ is

$$v_{av,i} = \frac{1}{Z}\int_0^Z v_{z,i}(0,z)dz. \tag{12.9}$$

The averaged value for $\varepsilon = 0, r = 0$ is

$$v_{av,i} = \frac{\Gamma_i M}{2a}\frac{2}{1+\sqrt{1+(Z/a)^2}}. \tag{12.10}$$

From I. Rechenberg [65] $(\varepsilon = 0)$ in the middle point of finite vortex spool $(z = Z/2)$ with the length Z the axial component of one vortex thread is

$$v_{max} = \frac{\Gamma_i}{\pi D}ctg(\beta)\sin(\arctan(\frac{Z}{D})) \tag{12.11}$$

where
β is the rise of vortex thread angles $(\beta = \arctan(\tau))$ and $D = 2a$ is the diameter of the vortex spool.
For the minimal value of velocty (in the points $z = 0$ und $z = Z$) [65]:

$$v_{min} = \frac{\Gamma_i}{2\pi D}ctg(\beta)\sin(\arctan(\frac{Z}{a})). \tag{12.12}$$

We have equal values of v_{max} from (12.11) and from (12.7) using $\sin(\arctan(y)) = \frac{y}{\sqrt{1+y^2}}, y = \frac{Z}{D}$, $ctg(\beta) = \tau^{-1} = \frac{\pi DM}{Z}$.

The averaged value (12.10) for $\varepsilon = 0$ is in the following form

$$v_{av} = \frac{\Gamma_i}{\pi D} ctg(\beta) \frac{\alpha}{\alpha a/Z + 1}, \tag{12.13}$$

where $\alpha = \sin(\arctan(\frac{Z}{a}))$.
In the formulas parameters M and Z are depending:

$$M = \frac{Z}{\tau \pi D},\ \tau = \tan(\beta).$$

Therefore from (12.4)ā(12.13) for the velocity components (v_r, v_z, v_φ) and for azimuthal component of vector potential A_φ
induced by N discrete vortex are

$$v_r = \sum_{i=1}^{N} v_{r,i}, v_z = \sum_{i=1}^{N} v_{z,i}, v_\varphi = \sum_{i=1}^{N} v_{\varphi,i}, A_\varphi = \sum_{i=1}^{N} A_{\varphi,i}. \tag{12.14}$$

Integrals are with the trapezoid formulas calculated.

If the intensity Γ_i of $N-$ spiral vortex S_i is equal Γ, than from (12.6) - (12.12) follows:

$$v_z(0, Z/2) = \frac{\Gamma N M}{D} \frac{1}{\sqrt{1 + (Z/D)^2}}, \tag{12.15}$$

$$v_z(0,0) = v_z(0,Z) = \frac{\Gamma N M}{D} \frac{1}{\sqrt{1 + (Z/a)^2}}, \tag{12.16}$$

$$v_{max} = \frac{\Gamma N}{\pi D} ctg(\beta) \sin(\arctan(\frac{H}{D})), \tag{12.17}$$

$$v_{min} = \frac{\Gamma N}{2\pi D} ctg(\beta) \sin(\arctan(\frac{H}{a})), \tag{12.18}$$

where $N-$ number of vortex threads, $H = Z-$ the height of the vortex spool (in building synonym of the length) are.
For averaged value of velocity $(\varepsilon = 0)$ we have the formula

$$v_{av} = \frac{\Gamma N M}{D} \frac{2}{1 + \sqrt{1 + (Z/a)^2}}, \tag{12.19}$$

or

$$v_{av} = \frac{\Gamma N}{\pi D} \frac{\alpha}{\alpha a/H + 1} ctg(\beta), \tag{12.20}$$

where $\alpha = \sin(\arctan(\frac{H}{a}))$.

If the averaged value v_{av} is known, then can be calculated from (12.19) also the dimensionless length $y = \frac{Z}{a}$ in following form $y = \frac{2\delta}{\delta^2-1}$, where $\delta = \Gamma N ctg(\beta)/(\pi D v_{av})$.
An example, if $\Gamma = 6.0319(\frac{m^2}{s}), \beta = 10^0(C), D = 0.25(m)$,
$N = 1, v_{av} = 30(\frac{m}{s})$, then $\delta = 1.452$ and $y = 2.62, Z = 0.3275(m)$.

The coresponding formulas (12.15, 12.17); (12.16,12.18) and (12.19, 12.20) are identical, but from (12.15),(12.16) and (12.19) follows, that the velocity depending of the parameter $M * N$ is, where $M = \frac{H}{\tau\pi D}$.

From (12.15, 12.16) and (12.19) we can the corespondings multiplicators by $\frac{\Gamma NM}{D}$ calculated (see Tab. 12.1):
$R_1 = \frac{1}{\sqrt{1+(Z/D)^2}}$, $R_2 = \frac{1}{\sqrt{1+(Z/a)^2}}$, and $R_3 = \frac{2}{1+\sqrt{1+(Z/a)^2}}$.

12.4 Calculation of the velocity field for the circular vortex lines

For the circular one vortex lines:
$\xi = a_i \cos\alpha, \eta = a_i \sin\alpha, \zeta = z_i, d\xi = -a_i \sin\alpha d\alpha, d\eta = a_i \cos\alpha d\alpha$, $d\zeta = 0$ and from axially-symmetric condition follows that by $\varphi = 0$ is $A_{x,i} = A_{z,i} = 0$ and

$$A_{y,i} = A_{\varphi,i} = A_i(r,z) = \frac{\Gamma_i a_i}{4\pi} I_i,$$

where $I_i = \int_0^{2\pi} \frac{\cos\alpha d\alpha}{\sqrt{(z-z_i)^2+a_i^2+r^2-2a_i r\cos\alpha}}$.
The integral I_i is equal [47]
$I_i = \int_0^{\pi/2} \frac{(1-2\sin^2 t)dt}{\sqrt{((z-z_i)^2+(r+a_i)^2)}\sqrt{1-k_i^2\sin^2 t}} = \frac{2}{\sqrt{ra_i}}[(\frac{2}{k_i} - k_i)K(k_i) - \frac{2}{k_i}E(k_i)]$,
where
$t = (\alpha - \pi)/2, k_i = 2\sqrt{ar}/c_i, \quad c_i = \sqrt{(a_i + r)^2 + (z - z_i)^2}$,
$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}$ is the total elliptical integral of first kind,
$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t} dt$ is the total elliptical integral of second kind.

Therefore the azimuthal component of vector potential A_i induced by a circular vortex line L_i with intensity Γ_i and with radius a_i is

$$A_i(r,z) = \frac{\Gamma_i}{2\pi}\sqrt{\frac{a_i}{r}}[(\frac{2}{k_i} - k_i)K(k_i) - \frac{2}{k_i}E(k_i)].$$

The vectorial components of velocity field (the radial and axial components) induced by vortex line L_i are

$$v_{r,i} = -\frac{\partial A_i}{\partial z}, v_{z,i} = \frac{1}{r}\frac{\partial}{\partial r}(rA_i). \qquad (12.21)$$

or

$$v_{r,i}(r,z) = \frac{\Gamma_i}{2\pi r}\frac{z-z_i}{c_i}[E(k_i)\frac{a_i^2+r^2+(z-z_i)^2}{(a_i-r)^2+(z-z_i)^2} - K(k_i)], \qquad (12.22)$$

$$v_{z,i}(r,z) = \frac{\Gamma_i}{2\pi c_i}[K(k_i) + \frac{a_i^2-r^2-(z-z_i)^2}{(a_i-r)^2+(z-z_i)^2}E(k_i)]. \qquad (12.23)$$

If $r=0$ then

$$v_{z,i}(0,z) = \frac{\Gamma_i}{2}\frac{a_i^2}{(a_i^2+(z-z_i)^2)^{1.5}}. \qquad (12.24)$$

This component of vectors have the maximal value $v_{z,i} = \frac{\Gamma_i}{2a}$ by $z = z_i, a_i = a$. By $z = z_i + Z/2$ we have $v_{z,i} = \frac{\Gamma_i}{2\sqrt{a^2+Z^2/4}}\frac{a^2}{a^2+Z^2/4} < \frac{\Gamma_i}{2\sqrt{a^2+Z^2/4}}$ – this is the value of the component of velocity induced by spiral vortex ($\varepsilon = 0$).

If $z = Z/2, a_i = a$, then from (12.24) follows

$$v_{z,i}(0,Z/2) = \frac{\Gamma_i}{D}\frac{1}{(1+((Z/2-z_i)/a)^2)^{1.5}}. \qquad (12.25)$$

For the averaged value of the velocity we have

$$v_{av,i} = \frac{\Gamma_i}{D}\frac{a}{Z}(\frac{(Z-z_i)/a}{\sqrt{1+((Z-z_i)/a)^2}} + \frac{z_i/a}{\sqrt{1+(z_i/a)^2}}). \qquad (12.26)$$

If $z_i = Z/2$, then $v_{av,i} = \frac{\Gamma_i}{D}\frac{1}{\sqrt{1+(Z/D)^2}}$.

The summary velocity field (v_r, v_z) and the vector potential A_φ induced by N discrete vortex lines we obtain in the form (12.14). The hydrodynamic stream function $\psi = \psi(r,z)$ for velocity components $v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}$, $v_r = \frac{1}{r}\frac{\partial\psi}{\partial r}$, from (12.21) is $\psi(r,z) = rA_\varphi(r,z)$.

The amount of flow through cross section $[z = z_0, 0 < r < a_0]$ is $Q(a_0,z_0) = \int_0^{a_0}\int_0^{2\pi} v_z(r,z_0) r dr\, d\varphi = 2\pi a_0 A_\varphi(a_0,z_0) = 2\pi\psi(a_0,z_0)$. The total amount of flow through cross cylindrical domain $[0 < z < Z, 0 < r < a_0]$ is $Q_t(a_0) = \int_0^Z Q(a_0,z)dz = 2\pi\int_0^Z \psi(a_0,z)dz$.

For the circular vortex line, if $z_i/a = 0.2i,\ i = \overline{1,N}$, $N \leq 6$, we can calculated following multiplicators by the factor $\frac{\Gamma}{D}$:
$R_4(Z) = \sum_{i=1}^N (1+((Z/2-z_i)/a)^2)^{-1.5} --$ for (12.25),
$R_5 = \frac{a}{Z}\sum_{i=1}^N (\frac{(Z-z_i)/a}{\sqrt{1+((Z-z_i)/a)^2}} + \frac{z_i/a}{\sqrt{1+(z_i/a)^2}}) --$ for (12.26).
An example, if $Z/a = 1.4$ then we can the multiplicators $R_4((0), R_4(Z/2), R_4(Z), R_5$ for the circular vortex lines and R_1, R_2, R_3 for the spiral vortexes by the factor $\frac{\Gamma M}{D}$ in the form $R_1 * N$, $R_2 * N$, $R_3 * N$ calculated (see Tab. 12.1).

Table 12.1 Multiplicators of the velocity for vortexes by $\frac{Z}{a} = 1.4$

N	$R_4(0)$	$R_4(Z/2)$	$R_4(Z)$	R_5	R_1	R_2	R_3
1	0.94	0.71	0.26	0.69	0.82	0.58	0.74
2	1.74	1.59	0.62	1.46	1.64	1.16	1.47
3	2.37	2.58	1.09	2.27	2.46	1.74	2.21
4	2.85	3.56	1.72	3.09	3.28	2.32	2.94
5	3.20	4.44	2.52	3.85	4.10	2.91	3.68
6	3.47	5.16	3.47	4.55	4.92	3.48	4.41

In the following calculations we use the dimensionless form scaling all the lengths to $r_0 = a$ (the inlet radius of the tube), the axial v_z and radial v_r velocity to $v_0 = \frac{\Gamma_0}{2\pi r_0}$, the azimuthal components of vector potential A_φ to $A_0 = \frac{\Gamma_0}{2\pi}$, the stream function ψ to $\psi_0 = A_0 r_0$ and the total amount of flow Q_t to $Q_0 = \psi_0 r_0$. Here Γ_0 is dimensional scaling of vortex intensity $\Gamma_i, i = \overline{1,N}$.

12.5 The flows field induced by linear vortex lines in a channel

For symmetry-conditions $\frac{\partial v_x}{\partial y)}|_{y=1}$ we consider half the plane channel $y \in [0,1]$. In the plane $y = 0$ we have the slip-conditions $v_x = v_y = 0$ for the velocity vectors of viscous incompressible liquid. The flow in the channel is given by fixed amount of flow through cross section of

the half channel $Q = \int_0^1 v_x|_{x=0} dy$. If $L = \infty$, then $v_x = u(y), v_y = 0$ and we have the Puaseil flow $u = Q(3y - 1.5y^2)$- the solution of Navier-Stokes equation in the channel $\Omega_{x,y}$.
An the wall $y = 0$ of the channel are placed linear chain of vortexes with the axis transver the (x,y) plane. The one linear vortex line in the point (x_k, y_k) create the following componemts of velocity:

$$v_x = -\frac{\Gamma_k}{2\pi}\frac{y - y_k}{R^2}, v_y = \frac{\Gamma_k}{2\pi}\frac{x - x_k}{R^2}, \tag{12.27}$$

where $R^2 = (x - x_k)^2 + (y - y_k)^2$.
In the center of this point wise vortex the velocity field is infinite therefore we consider the vortex line with finite cross section
the circle with radius a. In this case the expressions (12.27) are valid when $R \geq a$, but for $R < a$ we have

$$v_x = -\frac{\Gamma_k}{2\pi a^2}(y - y_k), v_y = \frac{\Gamma_k}{2\pi a^2}(x - x_k). \tag{12.28}$$

12.6 Some numerical results

12.6.1 The flow in the channel

We consider the channel with finite length $L = 2.5$, Puaseil flow with $Q = 3$ and three wise of the chain of vortexes:
1) the main chain with coordinates and radius of the linear vortex

$$x_k = 0.2 + (k-1)0.4, y_k = 2a, k = 1,2,3,4,5,6, a = 0.05, \tag{12.29}$$

rotate clockwise with the intensity Γ_1,
2)the second chain with coordinates and radius of the linear vortex

$$x_k = 0.4 + (k-1)0.4, y_k = 2a_1, k = 1,2,3,4,5, a_1 = 0.025, \tag{12.30}$$

rotate opposite clockwise with the intensity Γ_2,
3)the thread chain with coordinates and radius of the linear vortex

$$x_k = 0.3 + (k-1)0.4, y_k = 2a + a_1, k = 1,2,3,4,5, a = 0.05, a_1 = 0.025, \tag{12.31}$$

rotate opposite clockwise with the intensity Γ_3.
In following Table12.2 can see the amount (Q), maximal value of velocity $u,(mV)$ with the coordinates (mX,mY) depending of the vortex intensity Γ_1, Γ_2,Γ_3. For the pointwise vortexes line (12.29) outside

Table 12.2 The dependence of flow velocity from intensity of the vortexes

Γ_1	Γ_2	Γ_3	Q	mV	mX	mY
0	0	0	3.00	4.500	0.00	1.00
-6	3	3	3.97	18.19	2.20	0.15
-6	4	4	3.46	22.90	0.30	0.10
-6	3	0	4.62	18.36	0.20	0.15
-6	2	2	4.49	18.63	2.20	0.15
-6	1	1	5.00	19.08	2.20	o.15
-6	1	0	5.22	19.14	0.20	0.15
-6	0	1	5.30	19.47	2.20	0.15
-6	0	0	5.52	19.86	1.00	0.15

the channel $(y_k = -0.025)$ with $\Gamma_1 = -6$ we have following results: $mV = 5.9895, mX = 1.00, mY = 0$. For the Karman chain [47] of vortexes (preliminary vortexes line and (12.30) with $y_k = -0.05, \Gamma_2 = 6$) we haveā$mV = 3.9790, mX = 0.20, mY = 0$.

12.6.2 The circular vortexes lines

As the basis for the calculations of N circular vortex lines $L_i, i = \overline{1,N}$ are $N \leq 6$ chosen, which are arranged in the axial direction at the points with following dimensionless coordinates $(z_i = 0.2i, r_i = a_i), i = \overline{1,N}$.
The dimensionless radius of the circular vortex lines a_i are considered in three forms (the sequence $a = [a_1, a_2, a_3, a_4, a_5, a_6])$:
1)the constant sequence(radius of the cylinder) $a_c = [1,1,1,1,1,1]$,
2)the monotonous increasing sequence $a_{in} = [.75, .80, .85, .90, .95, 1.0]$,
3)the monotonous decreasing sequence $a_d = [1.0, .95, .90, .85, .80, .75]$.
The results of numerical experiments for dimensionless values
v_r, v_z, ψ, Q_t was obtained of different dimensionless intensity of vortex lines $\tilde{\Gamma_i} = \frac{\Gamma_i}{2\pi\Gamma_0} = \pm 6; \pm 3; \pm 2; 1; 0.5$, and $l = Z/r_0 = 2$, $a_0 = 0.7$.
The summary intensity of absolute values is equal to 6.
The velocity field is calculated an the uniform grid $(n_r \times n_z)$ by the steps $h_1 = h_2 = 0.1$ in the r,z directions.

The numerical results show that the velocity field induced by circular vortex lines are concentrated inside the cylinder. The results depend on the arrangement and on the radius of vortex lines a_i.

Typical results of calculations are: the dimensionless velocity field and the distribution of stream function in the cylinder. We can see the velocity formation depending on the arrangement of vortices lines with coordinates $zj = [z_1, z_2, z_3, z_4, z_5, z_6]$, and of the radii a_i.
If $\tilde{\Gamma_i} > 0$ then the all vortices move in the positive direction of Oz axis $(v_z > 0)$, but the radii of vortex lines to stay a different way (for $v_r < 0$ the radius is decreasing and for $v_r > 0$ the radius is increasing).

We obtain for the dimensionless values of
$v_r \in [v_{r.min}, v_{r.max}]$, $v_{z.max}, \psi_{max}, Q_t$
for $zj = [0.2, 0.4, 0.6, 0.8, 1.0, 1.2]$ and for different radius of vortex lines a_i and sequence of intensity $gj = [g_1, g_2, g_3, g_4, g_5, g_6]$ the following results:

1. The radii are constant $a_c = [1, 1, 1, 1, 1, 1]$
1.1 The intensity of the one vortex lines L_3 is $\tilde{\Gamma_3} = 6$, $N = 1$:
$v_r \in (-5.9, 5.9)$, $vz_{max} = 18.85$, $\psi_{max} = 3.25$,
$v_r = 0$ if $z = z_3 = 0.6$ and $v_r > 0$ if $z > z_3$, therefore the radius of vortex increased [48];
1.2 The intensity of the one vortex lines L_3 is $\tilde{\Gamma_3} = -6$, $N = 1$ (the opposite direction):
$v_r \in (-5.9, 5.9)$, $vz_{max} = -18.85$, $\psi_{max} = -3.25$,
the vortex move in the negative direction of Oz axes $(v_z < 0)$, $v_r = 0$ if $z = z_3 = 0.6$ and $v_r > 0$ if $z < z_3$, therefore the radius of vortex also increased [48];
1.3 The intensity of the two vortex lines L_3, L_4 are $\tilde{\Gamma_3} = 3, \tilde{\Gamma_4} = 3$, $N = 2$:
$v_r \in (-5.7, 5.7)$, $vz_{max} = 18.57$, $\psi_{max} = 3.17$,
the vortexes move in the positive direction of Oz axes $(v_z > 0)$, $v_r = 0$ if $z = (z_3 + z_4)/2 = 0.7$ and $v_r(a_0, z_3) = -2.46$, $v_r(a_0, z_4) = 4.37$, therefore the radius of the first vortex lines L_3 decreased, but for the second vortex lines L_4 increased and the first vortex can be move through the second vortex [48];
1.4 The intensity of the two vortex lines L_3, L_4 are $\tilde{\Gamma_3} = -3, \tilde{\Gamma_4} = 3$, $N = 2$:
$v_r \in (-2.9, 0.64)$, $v_z \in (-3.0, 3.0)$, $\psi \in (-0.32, 0.32)$,
$v_z = 0$ if $z = 0.7$ and $v_z(a_0, z_3) = -1.72$, $v_z(a_0, z_4) = 2.76$, therefore

the first vortex move to the negative direction, but the second to the positive direction of Oz axes and the radii of the vortexes decreased (this case is'n in [48] considered);
1.5 The intensity of the two vortex lines L_3, L_4 are $\tilde{\Gamma_3} = 3, \tilde{\Gamma_4} = -3$, $N = 2:$
$v_r \in (-0.64, 2.9)$, $v_z \in (-3.0, 3.0)$, $\psi \in (-0.32, 0.32)$,
$v_z = 0$ if $z = 0.7$ and $v_z(a_0, z_3) = 1.72$, $v_z(a_0, z_4) = -2.76$, the first vortex move to the positive direction, but the second to the negative direction of Oz axes and the radiis of the vortexes increased [48];
1.6 The intensity of the three vortex lines L_1, L_3, L_5 are $\tilde{\Gamma_1} = 2, \tilde{\Gamma_3} = 2, \tilde{\Gamma_5} = 2$, $N = 3:$
$v_r \in (-4.1, 4.1)$, $vz_{max} = 16.34$, $\psi_{max} = 2.63$,
$v_r = 0$ if $z = z_3 = 0.6$ and $v_z(a_0, z_1) = 15.92$, $v_z(a_0, z_3) = 16.16$, $v_z(a_0, z_5) = 15.92$, $v_r(a_0, z_1) = -3.8$, $v_r(a_0, z_5) = 1.6$, the vortexes move in the positive direction of Oz axis and the radius of the first vortex decreased, but of the third vortex increased;
1.7 The intensity of the three vortex lines L_1, L_3, L_5 are $\tilde{\Gamma_1} = -2, \tilde{\Gamma_3} = 2, \tilde{\Gamma_5} = -2$, $N = 3:$
$v_r \in (-1.6, 1.6)$, $vz_{min} = -5.83$, $\psi_{min} = -0.74$,
$v_r = 0$ if $z = z_3 = 0.6, z = 0.1, z = 1.1$ and $v_z(a_0, z_1) = -5.67$, $v_z(a_0, z_3) = -2.42$, $v_z(a_0, z_5) = -3.56$, $v_r(a_0, z_1) = -0.77$, $v_r(a_0, z_5) = 0.77$, the vortexes move in the negative direction of Oz axis and the radius of the first vortex decreased, but of the third vortex increased;
1.8 The intensity of the three vortex lines L_1, L_3, L_5 are $\tilde{\Gamma_1} = 2, \tilde{\Gamma_3} = -2, \tilde{\Gamma_5} = 2$, $N = 3:$
$v_r \in (-1.6, 1.6)$, $v_{z.max} = 5.83$, $\psi_{max} = 0.74$,
$v_r = 0$ if $z = z_3 = 0.6$ and $v_z(a_0, z_1) = 5.67$, $v_z(a_0, z_3) = 1.97$, $v_z(a_0, z_5) = 5.67$, $v_r(a_0, z_1) = 0.77$, $v_r(a_0, z_5) = -0.77$, the vortexes move in the positive direction of Oz axis and the radius of the first vortex increased, but of the third vortex decreased;
1.9 The intensity of the three vortex lines L_1, L_3, L_5 are $\tilde{\Gamma_1} = -2, \tilde{\Gamma_3} = 2, \tilde{\Gamma_5} = 2$, $N = 3:$
$v_r \in (-4.9, 2.6)$, $v_z \in (-1.75, 11.1)$, $\psi \in (-0.10, 1.45)$,
$v_r = 0$ if $z = 0.9$ and $v_z(a_0, z_1) = -0.64$, $v_z(a_0, z_3) = 8.28$, $v_z(a_0, z_5) = 10.89$, $v_r(a_0, z_1) = -3.17$, $v_r(a_0, z_3) = -3.95$, $v_r(a_0, z_5) = 0.77$, the two vortexes L_3, L_5 move in the positive direction, but the first in the negative direction of Oz axis and the radii of the two vortexes L_1, L_3 are decreased, but of the third vortex increased;
1.10 The intensity of the three vortex lines L_1, L_3, L_5 are $\tilde{\Gamma_1} = 2, \tilde{\Gamma_3} =$

$2, \tilde{\Gamma}_5 = -2,\ N = 3:$
$v_r \in (-2.6, 4.9)$, $v_z \in (-1.75, 11.1)$, $\psi \in (-0.10, 1.45)$,
$v_r = 0$ if $z = 0.3$ and $v_z(a_0, z_1) = 10.89$, $v_z(a_0, z_3) = 8.28$, $v_z(a_0, z_5) = -0.64$, $v_r(a_0, z_1) = -0.77$, $v_r(a_0, z_3) = 3.95$, $v_r(a_0, z_5) = 3.17$, the two vortexes L_1, L_3 move in the positive direction, but the vortex L_5 in the negative direction of Oz axis and the radii of the two vortexes L_3, L_5 are increased, but of the third vortex L_1 decreased;
1.11 The intensity of the three vortex lines L_1, L_3, L_5 are $\tilde{\Gamma}_1 = -2, \tilde{\Gamma}_3 = -2, \tilde{\Gamma}_5 = 2,\ N = 3:$
$v_r \in (-4.9, 2.6)$, $v_z \in (-11.1, 1.75)$, $\psi \in (-1.45, 0.10)$,
$v_r = 0$ if $z = 0.3$ and $v_z(a_0, z_1) = -10.89$, $v_z(a_0, z_3) = -8.28$, $v_z(a_0, z_5) = 0.64$, $v_r(a_0, z_1) = 0.77$, $v_r(a_0, z_3) = -3.95$, $v_r(a_0, z_5) = -3.17$, the two vortexes L_1, L_3 move in the negative direction, but the third in the positive direction of Oz axis and the radii of the two vortexes L_3, L_5 are decreased, but of the first vortex increased.

2.The radii are increasing a_{in}
2.1 The non-uniform distribution of intensity $gj = [2, 2, 1, .5, .5, 0]$, $N = 5:$
$v_r \in (-12.7, 7.4)$, $v_{z.max} = 21.15$, $\psi_{max} = 4.6, Q_t = 28.34$,
$v_r = 0$ if $z = 0.3$, the radius of the first vortex decreased, but increased the radii of the last four vortexes ;
2.2 The distribution of intensity $gj = [2, 2, 2, 0, 0, 0]$, $N = 3:$
$v_r \in (-13.3, 10.02)$, $v_{z.max} = 22.23$, $\psi_{max} = 4.8, Q_t = 28.69$,
$v_r = 0$ if $z = 0.3$, the radius of the first vortex decreased, but increased the radii of the last vortex ;
2.3 The distribution of intensity $gj = [0, 0, 3, 3, 0, 0]$, $N = 2:$
$v_r \in (-10.2, 9.4)$, $v_{z.max} = 21.14$, $\psi_{max} = 4.64, Q_t = 29.20$,
$v_r = 0$ if $z = 0.7$, the radiis of the first vortex decreased, but increased the radii of the last vortex ;
2.4 The intensity of first vortex lines $gj = [6, 0, 0, 0, 0, 0]$, $N = 1:$
$v_r \in (-19.2, 19.2)$, $v_{z.max} = 25.13$, $\psi_{max} = 6.47, Q_t = 27.10$,
$v_r = 0$ if $z = 0.2$, the radius of the vortex increased;
2.5 The intensity of second vortex lines $gj = [0, 6, 0, 0, 0, 0]$, $N = 1:$
$v_r \in (-15.0, 15.0)$, $v_{z.max} = 23.56$, $\psi_{max} = 5.69, Q_t = 29.28$,
$v_r = 0$ if $z = 0.4$, the radius of the vortex increased;
2.6 The intensity of third vortex lines $gj = [0, 0, 6, 0, 0, 0]$, $N = 1:$
$v_r \in (-11.8, 11.8)$, $v_{z.max} = 22.18$, $\psi_{max} = 5.11, Q_t = 29.69$,
$v_r = 0$ if $z = 0.3$, the radius of the vortex increased;

2.7 The intensity of fourth vortex lines $gj = [0,0,0,6,0,0]$, $N = 1$:
$v_r \in (-9.6, 9.6)$, $v_{z.max} = 20.94$, $\psi_{max} = 4.66, Q_t = 28.72$,
$v_r = 0$ if $z = 0.3$, the radius of the vortex increased.

3. The uniform distribution of intensity $gj = [1,1,1,1,1,1]$

3.1 Radii of vortex lines are constant (the sequence a_c :
$v_r \in (-4.5, 4.5)$, $v_{z.max} = 16.21$, $\psi_{max} = 3.14, Q_t = 25.12$,
$v_r = 0$ if $z = 0.7$, the radii of the first three vortexes decreased, but of the least three vortexes increased ;
3.2 Radii of vortex lines are a_{in} :
$v_r \in (-8.4, 4.9)$, $v_{z.max} = 17.98$, $\psi_{max} = 3.52, Q_t = 27.36$,
$v_r = 0$ if $z = 0.8$, the radii of the first three vortexes decreased, but of the last two vortexes increased;
3.3 Radii of vortex lines are a_d :
$v_r \in (-4.9, 8.4)$, $v_{z.max} = 17.98$, $\psi_{max} = 3.52, Q_t = 27.36$,
$v_r = 0$ if $z = 0.5$, the radii of the first two vortexes decreased, but increased the radii of last four vortexes.

4. The distribution of intensity $gj = [2,2,.5,.5,.5,.5]$

4.1 Radii of vortex lines are a_{in} :
$v_r \in (-12.3, 6.9)$, $v_{z.max} = 20.19$, $\psi_{max} = 4.4, Q_t = 27.77$,
$v_r = 0$ if $z = 0.3$, the radius of the first vortex decreased, but increased the radii of the last five vortexes ;
4.2 Radii of vortex lines are a_d :
$v_r \in (-5.7, 5.6)$, $v_{z.max} = 17.30$, $\psi_{max} = 3.4, Q_t = 26.01$,
$v_r = 0$ if $z = 0.4$, the radius of the first vortex decreased, but increased the radii of the last four vortexes.

5. The distribution of intensity $gj = [.5,.5,.5,.5,2,2]$

5.1 Radii of vortex lines are a_{in} :
$v_r \in (-5.6, 5.8)$, $v_{z.max} = 17.30$, $\psi_{max} = 3.4, Q_t = 26.01$,
$v_r = 0$ if $z = 1.0$, the radii of the first four vortexes decreased, but increased the radius of the last vortex ;
5.2 Radii of vortex lines are a_d :
$v_r \in (-6.8, 12.3)$, $v_{z.max} = 20.19$, $\psi_{max} = 4.4, Q_t = 27.77$,
$v_r = 0$ if $z = 1.1$, the radii of the first five vortexes decreased, but of the last vortex increased.

6. The distribution of intensity $gj = [.5, .5, 2, 2, .5, .5]$
6.1 Radii vortex lines are a_{in} :
$v_r \in (-7.4, 6.6)$, $v_{z.max} = 19.47$, $\psi_{max} = 4.0, Q_t = 28.28$,
$v_r = 0$ if $z = 0.7$, the radii of the first two vortexes decreased, but increased the radii of the last four vortexes ;
6.2 Radii of vortex lines are a_d :
$v_r \in (-6.6, 7.4)$, $v_{z.max} = 19.47$, $\psi_{max} = 4.0, Q_t = 28.28$,
$v_r = 0$ if $z = 0.7$, the radii of the first two vortexes decreased, but increased the radii of the last four vortexes.

12.6.3 The spiral vortexes in the cylinder $(\varepsilon = 0)$

We consider $N \leq 6$ spiral vortexes $S_i, i = \overline{1,N}$, where started from the points $(a, 0, i2\pi/N)$ at the cylinder.
The dimensionless radius of the cylinder a is equal 1.

All results of the numerical experiments are for the dimensionless values $A_\varphi(a_0, z, \varphi), v_z(0, z), Q(z), Q_t$ and parametern $l = Z/a = 0.5; 1; 1.5; 2; 3,\ a_0 = 0.7$ obtain.
The summary intensity of absolute values is equal to 6.
The azimuthal components of the vector potential are in the uniform grid $(N_z \times N_\varphi)$ by the steps $h_z = l/N_z, h_\varphi = 2\pi/N_\varphi, (N_z = N_\varphi = 30)$ in the r, φ direction calculed.
The component $A_\varphi(z, \varphi), (r = a_0)$ using the trapezoid formula is calculated. Figures shows typical results of calculations: the dimensionless velocity field and the distribution of the azimuthal component of the velocity $(r = a_0)$ in the cylinder.
The velocity formation is depending on the length l of the cylinder.
The maximum of the azimutahl components of vector potentials A_{max} is depending of the intensity parameter $g_i = \tilde{\Gamma}_i$.
We obtain for the dimensionless values of $v_{z.max}, Q_{max}, A_{max}, Q_t$ and for different sequence of intensity $gj = [g_1, g_2, g_3, g_4, g_5, g_6]$ the following results:

1. The length is $l = 1.5$,
$(v_{z.max} = 15.08, Q_{max} = 24.98, Q_t = 33.20)$
1. The uniform distribution of the intensity

$gj = [1,1,1,1,1,1], N = 6 : A_{max} = 5.68$,
die Verteilung A_φ is uniform in the φ direction;
2.The distribution of the intensity is $gj = [2,2,1,.5,.5,0], N = 5 :$
$A_{max} = 5.68$, , the distibution of A_φ is nonuniform in the φ direction;
3. The distribution of the intensity is $gj = [2,2,2,0,0,0], N = 3 :$
$A_{max} = 5.75$, , the values of A_φ oscillate in the φ direction;
4. The distribution of the intensity is $gj = [2,1,1,1,1,0], N = 5 :$
$A_{max} = 6.14$, the distibution of A_φ is nonuniform in the φ direction; (the maximal value 6.14 is in the point $(0.75, 4.2)$);
5. The distribution of the intensity is $gj = [1.5,1.5,1.5,1.5,0,0]$,
$N = 4 :$
$A_{max} = 5.68$, the values of A_φ weakly oscillate in the φ direction;
6. The distribution of the intensity is $gj = [3,3,0,0,0,0], N = 2 :$
$A_{max} = 5.83$, the distibution of A_φ is nonuniform in the φ direction with 3 maximums;
7. The distribution of the intensity is $gj = [6,0,0,0,0,0], N = 1 :$
$A_{max} = 8.54$, the distibution of A_φ is nonuniform in the φ direction with one maximum;
8. The distribution of the intensity is $gj = [6,0,0,0,0,0], N = 1, b = 0 :$
$A_{max} = 8.40, v_{z.max} = 18.85, Q_{max} = 36.95, Q_t = 24.54$, the distribution A_φ is uniform in the φ direction (this is the velocity field induced by the circular vortex line $(z_i = 0)$).

2. Different lengths l

1. The distribution of the intensity is $gj = [6,0,0,0,0,0], N = 1 :$
$A_{max} = 8.42, v_{z.max} = 18.29, Q_{max} = 34.25, Q_t = 16.28$;
2. The distribution of the intensity is $gj = [2,2,1,.5,.5,0], N = 5, l = 3 :$
$A_{max} = 5.11, v_{z.max} = 10.46, Q_{max} = 16.36, Q_t = 43.03$;
3. The distribution of the intensity is $gj = [2,2,1,.5,.5,0], N = 5, l = 2 :$
$A_{max} = 6.0386, v_{z.max} = 13.3286, Q_{max} = 21.4252, Q_t = 37.6009$,
if $N_\varphi = N_z = M = 50$, then$A_{max} = 6.0388, v_{z.max} = 13.3286, Q_{max} = 21.4262, Q_t = 37.6017$;
4. The distribution of the intensity is $gj = [2,2,1,.5,.5,0], N = 5, l = 1 :$
$A_{max} = 7.35, v_{z.max} = 16.86, Q_{max} = 29.39, Q_t = 26.65$;
2. The distribution of the intensity is $gj = [2,2,1,.5,.5,0], N = 5, l =$

0.5 :
$A_{max} = 8.11, v_{z.max} = 18.29, Q_{max} = 34.25, Q_t = 16.28.$

12.6.4 The spiral vortexes in the cones $(\varepsilon \neq 0)$

In this case we have some results for behavior of spiral vortexes.
1. If $\Gamma = 6.0319(m^2/s), N = 1, \beta = 10^0(C)(\tau = tg(\beta) = 0.1763)$, $a = 0.125(m), Z \in [0.1, 1.0](m)$, then from the formulas (12.14, 12.18) can be the values $M; V_1(\varepsilon = 0)$; $V_2(\varepsilon = 0.001)$; $V_3(\varepsilon = 0.002)$; $V_4(\varepsilon = -0.002)(m/s)$ calculated (see the Tab.12.3).

Table 12.3 The velocity v_{av} by $a = 0.125$

Z	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
M	0.72	1.44	2.17	2.89	3.61	4.33	5.06	5.78	6.50	7.22
V_1	15.3	24.1	29.0	32.0	34.0	35.4	36.5	37.3	37.9	38.5
V_2	15.5	24.6	29.7	32.7	34.8	36.2	37.3	38.2	38.8	39.4
V_3	15.7	25.1	30.3	33.5	35.6	37.1	38.2	39.1	39.8	40.4
V_4	14.9	23.3	27.9	30.7	32.6	33.9	34.9	35.7	36.3	36.8

For V_2 and V_3 the radii by $Z = 1$ decreased from $a = 0.125(m)$ with $0.080(m)$ and $0.034(m)$, but for V_4 the radius increased with $0.216(m)$.

2. If $a = 0.25(m)$,then similar from the formulas (12.9, 12.20) can be the values $M; V_1(\varepsilon = 0)$; $V_2(\varepsilon = 0.004)$; $V_3(\varepsilon = 0.008)$; $V_4(\varepsilon = -0.008)(m/s)$ calculated (see the Tab.12.4).

Table 12.4 The velocity v_{av} by $a = 0.25$

Z	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
M	0.36	0.72	1.08	1.44	1.80	2.17	2.53	2.89	3.25	3.61
V_1	4.19	7.64	10.2	12.1	13.5	14.5	15.4	16.0	16.6	17.0
V_2	4.27	7.86	10.6	12.6	14.0	15.2	16.0	16.7	17.3	17.8
V_3	4.34	8.10	11.0	13.1	14.6	15.9	16.8	17.6	18.2	18.7
V_4	4.06	7.23	9.54	11.2	12.4	13.4	14.1	14.7	15.2	15.6

For V_2 and V_3 the radii by $Z = 1$ decreased from $a = 0.25(m)$ with $0.16(m)$ and $0.07(m)$, but for V_4 the radius increased with $0.43(m)$.

12.7 Conclusions

1. Velocity field of ideal compressible fluid influenced by curved vortex field in a finite cylinder is investigated.
2. Numerical results shows that the maximum of axial velocity and total amount of flow depends on the connection method of producers of vortex energy.
3. The maximal velocity is developed in the case of non-uniform destributions of vortexes intensity and of smaller radius of vortex lines.
4. The maximal value of the velocity induced by the spiral vortexes is in the middle of the cylinder.
5. The behaviour of vortex lines in the ideal uncompressible flow depends on the number and of the orientation of the vortex.

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